"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 30 (2020), No. 1, 135 - 150

A GENERAL FIXED POINT THEOREM FOR TWO PAIRS OF MAPPINGS SATISFYING A MIXED IMPLICIT RELATION IN G - METRIC SPACES

VALERIU POPA AND ALINA-MIHAELA PATRICIU

Abstract. In this paper we extend Theorem 3.2 [32] to G - metric space. As applications, we obtain new results for mappings satisfying contractive conditions of integral type and for mappings satisfying φ - contractive conditions.

1. Introduction

Let (X, d) be a metric space and S, T be two self mappings of X. Jungck [13] defined S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

This concept was frequently used to prove the existence theorems in fixed point theory.

A point $x \in X$ is a coincidence point of S and T if $w = Sx = Tx$ and w is said to be a point of coincidence for S and T. The set of coincidence points of S and T is denoted by $\mathcal{C}(S, T)$.

In [14], Jungck introduced the notion of weakly compatible mappings.

Definition 1.1 ([14])**.** Let f, g be self mappings of a nonempty set X. f and g are weakly compatible if $fgu = gfu$ for all $u \in \mathcal{C}(f, g)$.

Keywords and phrases: fixed point, G - metric space, $CLR_{(A,S)T}$ - property, integral type, mixed implicit relation.
(2010) Mathematics Subject Classification: 54H25, 47H10.

The study of common fixed points for noncompatible mappings is also interesting. The work in this regard has been initiated by Pant [21], [22].

Aamri and El - Moutawakil [1] introduced a generalization of noncompatible mappings.

Definition 1.2 ([1])**.** Let S and T be two self mappings of a metric space (X,d). We say that S and T satisfy $(E.A)$ - property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Remark 1.3. *Two self mappings S and T of a metric space (X,d) are noncompatible if there exists $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is nonzero or nonexistent. Therefore, two noncompatible self mappings of a metric space (X,d) satisfy property $(E.A)$.*

Definition 1.4 ([16])**.** Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) satisfy property $(E.A)$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$, for some $t \in X$.

In 2011, Sintunavarat and Kumam [36] introduced the notion of common limit range property.

Definition 1.5 ([36])**.** A pair (A,S) of self mappings of a metric space (X,d) is said to satisfy the common limit range property with respect to S (denoted $CLR_{(S)}$ - property) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X)$.

Thus we can infer that a pair (A,S) satisfying $(E.A)$ - property, along with the closedness of the subspace $S\,(X)$ always have $CLR_{(S)}$ - property with respect to S.

Recently, Imdad et al. [10] extended the notion of common limit range property to two pairs of self mappings.

Definition 1.6 ([10])**.** Two pairs (A,S) and (B,T) of self mappings of a metric space (X,d) are said to satisfy common limit range property with respect to (S,T) (denoted $CLR_{(S,T)}$ - property) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$, for some $t \in S(X) \cap T\,(X)$.

Some results for pairs of mappings satisfying $CLR_{(S)}$ - and $CLR_{(S,T)}$ - property are obtained in [11], [12] and in other papers.

Quite recently [25], the first present author introduced a new type of common limit range property.

Definition 1.7 ([25])**.** Let (A, S) and T be self mappings of a metric space (X, d). The pair (A, S) is said to satisfy a common limit range property with respect to T (denoted $CLR_{(A,S)T}$ - property) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X) \cap T(X)$.

Example 1.8. *Let* $(\mathbb{R}_+, d)$ *be the metric space endowed with the usual metric* d *and* $Ax = \frac{x^2+1}{2}$, $Sx = \frac{x+1}{2}$, $Tx = x + \frac{1}{4}$. *Then*

$$S(X) = \left[\frac{1}{2}, \infty\right), \ T(X) = \left[\frac{1}{4}, \infty\right), \ S(X) \cap T(X) = \left[\frac{1}{2}, \infty\right).$$

Let $\{x_n\}$ *be a sequence in* X *such that* $\lim_{n\to\infty} x_n = 0$. *Then*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} \in S(X) \cap T(X).$$

Hence (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property.*

Remark 1.9. *Let* A, B, S *and* T *be self mappings of a metric space. As in Example 1.8,* (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property and* $Bx = x^2 + \frac{1}{4}$.

Let $\{y_n\}$ *be a sequence in* X *such that* $\lim_{n\to\infty} y_n = 0$. *Then*

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \frac{1}{4} \neq \frac{1}{2}.$$

Hence, (A, S) *and* (B, T) *do not satisfy* $CLR_{(S,T)}$ *- property.*

In 1997, Alber and Guerre - Delabriere [3] defined the concept of weak contraction as a generalization of contraction and established the existence of fixed points for self mappings in Hilbert spaces. Rhoades [33] extended this concept in metric spaces. In [5], the authors studied the existence of fixed points for pairs of (ψ, φ) - weak contractive mappings. New results are obtained in [9] and in other papers. Also, some fixed point theorems for mappings with common limit range property satisfying (ψ, φ) - weak contractive conditions are proved in [11] and [12].

Quite recently, a generalization of (ψ, φ) - weak contractive conditions is determined in [32].

Definition 1.10 ([15]). A function $\psi : [0,\infty) \to [0,\infty)$ is an altering distance if:
$(\psi_1) : \psi$ is increasing and continuous,
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Some fixed point theorems involving altering distance have been studied in [27] and [35].

Definition 1.11 ([30]). A function $\psi : [0,\infty) \to [0,\infty)$ is an almost altering distance if:
$(\psi_1) : \psi$ is continuous,
$(\psi_2) : \psi(t) = 0$ if and only if $t = 0$.

Remark 1.12. *Every altering distance is an almost altering distance, but the converse is not true. For example,* $\psi(t) = \begin{cases} t, t \in [0,1] \\ \frac{1}{t}, t \in (1,\infty) \end{cases}$ *is an almost altering distance, which is not an altering distance.*

2. Preliminaries

In [7], [8], Dhage introduced a new class of generalized metric spaces, named D - metric space.

Mustafa and Sims [18], [19] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced the appropriate notion of generalized metric space, named G - metric space. In fact, Mustafa and Sims and other authors studied many fixed point results for self mappings under certain conditions in [18] - [20] and in other papers.

Definition 2.1 ([19]). Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:
$(G_1) : G(x,y,z) = 0$ if $x = y = z$,
$(G_2) : 0 < G(x,x,y)$ for all $x, y \in X$ with $x \neq y$,
$(G_3) : G(x,y,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
$(G_4) : G(x,y,z) = G(y,z,x) = ...$ (symmetry in all three variables),
$(G_5) : G(x,y,z) \leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

The function G is called a G - metric on X and (X,G) is called a G - metric space.

Remark 2.2. *Let (X,G) be a G - metric space. If $y = z$, then $G(x,y,y)$ is a quasi - metric on X. Hence, $Q(x,y) = G(x,y,y)$ is* *a quasi - metric and since every metric space is a particular case of quasi - metric space, it follows that the notion of* G *- metric space is a generalization of a metric space.*

Lemma 2.3 ([19]). *Let* (X, G) *be a* G *- metric space. The function* $G(x, y, z)$ *is jointly continuous in all three of its variables.*

Definition 2.4 ([19]). Let (X, G) be a G - metric space. A sequence $\{x_n\}$ in X is said to be G - convergent if for $\varepsilon > 0$, there exists $x \in X$ and $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $m, n \geq k$, $G(x, x_n, x_m) < \varepsilon$.

Lemma 2.5 ([19]). *Let* (X, G) *be a* G *- metric space. The following properties are equivalent:*
1) $\{x_n\}$ *is* G *- convergent to* x*;*
2) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$*;*
3) $G(x_n, x, x) \to 0$ *as* $n \to \infty$*;*
4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$*.*

The notion of new limit range property in G - metric spaces is similar to the notion from metric space (Definition 1.7).

3. Implicit relations

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function [23], [24] and other papers.

The study of fixed points for mappings satisfying implicit relations in G - metric spaces is initiated in [28], [29].

The study of fixed points for a pair of mappings with common limit range property in metric spaces satisfying an implicit relation is initiated in [11]. The study of fixed points for pairs of mappings with common limit range property in G - metric spaces is initiated in [30] and [31].

In 2008, Ali and Imdad [4] introduced a new type of implicit relation.

Let $\mathcal{F}$ be the family of lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:
$(F_1) : F(t, 0, t, 0, 0, t) > 0$, $\forall t > 0$.
$(F_2) : F(t, 0, 0, t, t, 0) > 0$, $\forall t > 0$,
$(F_3) : F(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

In [32] the present authors introduced two new types of implicit relations.

Let $\mathcal{F}^*$ be the set of all lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:
$(F_1^*) : F(t, 0, t, 0, 0, t) \geq 0, \ \forall t > 0.$
$(F_2^*) : F(t, 0, 0, t, t, 0) \geq 0, \ \forall t > 0,$
$(F_3^*) : F(t, t, 0, 0, t, t) \geq 0, \ \forall t > 0.$

Example 3.1. $F(t_1, ..., t_6) = t_1 - k \max \{t_2, t_3, t_4, t_5, t_6\}, \ k \in [0, 1].$

Example 3.2. $F(t_1, ..., t_6) = t_1 - k \max \left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}, \ k \in [0, 1].$

Example 3.3. $F(t_1, ..., t_6) = t_1 - k \max \left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}, \ k \in [0, 1].$

Example 3.4. $F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3, t_4\} - c \max\{t_5, t_6\}$, $a, b, c \geq 0$ *and* $a + b + c \leq 1$.

Example 3.5. $F(t_1, ..., t_6) = t_1 - \alpha \max \{t_2, t_3, t_4\} - (1-\alpha)(at_5 + bt_6)$, $\alpha \in (0, 1)$, $a, b \geq 0$ *and* $a + b \leq 1$.

Example 3.6. $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c \min\{t_5, t_6\}$, $a, b, c \geq 0$ *and* $a + b + c \leq 1$.

Example 3.7. $F(t_1, ..., t_6) = t_1 - at_2 - \frac{b(t_5+t_6)}{1+t_3+t_4}$, $a, b \geq 0$ *and* $a + 2b \leq 1$.

Example 3.8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, \ ct_3, \ ct_4, \ at_5 + bt_6\}$, $a, b, c \geq 0$ *and* $a + b + c \leq 1$.

Other examples will be presented in section *Applications*.

Let $\mathcal{H}^*$ be the set of all lower semi - continuous functions $H : \mathbb{R}_+^5 \to \mathbb{R}$ such that $G(s_1, s_2, ..., s_5) > 0$ if one of $s_1, s_2, ..., s_5 > 0$.

Example 3.9. $H(s_1, ..., s_5) = \max \{s_1, ..., s_5\}.$

Example 3.10. $H(s_1, ..., s_5) = \max \left\{s_1, \frac{s_2+s_3}{2}, \frac{s_4+s_5}{2}\right\}.$

Example 3.11. $H(s_1, ..., s_5) = \alpha \max \{s_1, s_2, s_3\} + (1 - \alpha)(as_4 + bt_5)$, $\alpha \in (0, 1)$, $a, b \geq 0$ *and* $a + b \leq 1$.

Example 3.12. $H(s_1, ..., s_5) = s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2.$

Example 3.13. $H(s_1, ..., s_5) = \frac{s_1}{1+s_2} + \frac{s_2}{1+s_3} + \frac{s_3}{1+s_4} + \frac{s_4}{1+s_5} + \frac{s_5}{1+s_1}.$

Example 3.14. $H(s_1, ..., s_5) = \frac{s_1+s_2+s_3+s_4+s_5}{1+s_1}.$

Example 3.15. $H(s_1, ..., s_5) = \frac{1}{s_1+s_2+s_3+s_4+s_5}.$

Definition 3.16. A function $\phi(t_1, ..., t_6, s_1, ..., s_5) = F(t_1, ..., t_6) + H(s_1, ..., s_5)$ is called a mixed implicit function.

The following theorem is proved in [32].

Theorem 3.17 (Theorem 3.2 [32]). *Let X be a metric space and A, B, S and T be self mappings of X satisfying the inequality*

$$\begin{array}{r} F\left(\begin{array}{c} \psi\left(d\left(Ax,By\right)\right),\psi\left(d\left(Sx,Ty\right)\right),\psi\left(d\left(Sx,Ax\right)\right), \\ \psi\left(d\left(Ty,By\right)\right),\psi\left(d\left(Sx,By\right)\right),\psi\left(d\left(Ax,Ty\right)\right) \end{array}\right)+ \\ H\left(\begin{array}{c} \psi\left(d\left(Sx,Ty\right)\right),\psi\left(d\left(Sx,Ax\right)\right),\psi\left(d\left(Ty,By\right)\right), \\ \psi\left(d\left(Sx,By\right)\right),\psi\left(d\left(Ax,Ty\right)\right) \end{array}\right)\leq 0 \end{array}$$

for all $x, y \in X$, some $F \in \mathcal{F}^*$ and some $H \in \mathcal{H}^*$ and $\psi(t)$ is an almost altering distance.

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then $\mathcal{C}(A, S) \neq \emptyset$ and $\mathcal{C}(B, T) \neq \emptyset$. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

The purpose of this paper is to extend Theorem 3.17 to G - metric spaces. As applications we obtain new results for mappings satisfying contractive conditions of integral type and for mappings satisfying φ - contractive conditions.

4. Main results

Lemma 4.1 ([2]). *Let f and g be weakly compatible mappings of a nonempty set X. If f and g have a unique point of coincidence $w = fx = gx$ for some $x \in X$, then w is the unique common fixed point of f and g.*

Theorem 4.2. *Let (X, G) be a G - metric space and A, B, S and T be self mappings of X satisfying the inequality*

$$\tag{4.1} \begin{array}{r} F\left(\begin{array}{c} \psi\left(G(Ax,By,By)\right),\psi\left(G(Sx,Ty,Ty)\right), \\ \psi\left(G(Ax,Sx,Sx)\right),\psi\left(G(Ty,By,By)\right), \\ \psi\left(G(Sx,By,By)\right),\psi\left(G(Ax,Ty,Ty)\right) \end{array}\right)+ \\ H\left(\begin{array}{c} \psi\left(G(Sx,Ty,Ty)\right),\psi\left(G(Ax,Sx,Sx)\right), \\ \psi\left(G(Ty,By,By)\right),\psi\left(G(Sx,By,By)\right), \\ \psi\left(G(Ax,Ty,Ty)\right) \end{array}\right)\leq 0, \end{array}$$

for all $x, y \in X$, some $F \in \mathcal{F}^*$, some $H \in \mathcal{H}^*$ and ψ an almost altering distance. If there exist $u, v \in X$ such that $Av = Sv$ and $Tu = Bu$, then there exists $t \in X$ such that t is the unique point of coincidence of A and S, as well t is the unique point of coincidence of B and T.

Proof. First we prove that $Sv = Tu$. Suppose that $Sv \neq Tu$. By (4.1) for $x = v$ and $y = u$ we get

$$F\left(\begin{array}{c}\psi\left(G(Av,Bu,Bu)\right),\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Av,Sv,Sv)\right),\\ \psi\left(G(Tu,Bu,Bu)\right),\psi\left(G(Sv,Bu,Bu)\right),\psi\left(G(Av,Tu,Tu)\right)\end{array}\right)+$$
$$H\left(\begin{array}{c}\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Av,Sv,Sv)\right),\psi\left(G(Tu,Bu,Bu)\right),\\ \psi\left(G(Sv,Bu,Bu)\right),\psi\left(G(Av,Tu,Tu)\right)\end{array}\right)\leq 0,$$

$$F\left(\begin{array}{c}\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right),0,\\ 0,\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right)\end{array}\right)+$$
$$H\left(\begin{array}{c}\psi\left(G(Sv,Tu,Tu)\right),0,0,\\ \psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right)\end{array}\right)\leq 0.$$

Since

$$H\left(\ \psi\left(G(Sv,Tu,Tu)\right),0,0,\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right)\ \right)>0$$

because $H \in \mathcal{H}^*$, then

$$F\left(\begin{array}{c}\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right),0,\\ 0,\psi\left(G(Sv,Tu,Tu)\right),\psi\left(G(Sv,Tu,Tu)\right)\end{array}\right)<0,$$

a contradiction of (F_3^*). Hence, $\psi\left(G(Sv,Tu,Tu)\right) = 0$ which implies $Sv = Tu$. Hence, $Sv = Av = Tu = Bu = t$ for some $t \in X$.

Suppose that there exists w with $Sw = Aw \neq Av$. Then by (4.1) for $x = w$ and $y = u$ we obtain

$$F\left(\begin{array}{c}\psi\left(G(Aw,Bu,Bu)\right),\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Aw,Sw,Sw)\right),\\ \psi\left(G(Tu,Bu,Bu)\right),\psi\left(G(Sw,Bu,Bu)\right),\psi\left(G(Aw,Tu,Tu)\right)\end{array}\right)+$$
$$H\left(\begin{array}{c}\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Aw,Sw,Sw)\right),\psi\left(G(Tu,Bu,Bu)\right),\\ \psi\left(G(Sw,Bu,Bu)\right),\psi\left(G(Aw,Tu,Tu)\right)\end{array}\right)\leq 0,$$

$$F\left(\begin{array}{c}\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right),0,\\ 0,\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right)\end{array}\right)+$$
$$H\left(\ \psi\left(G(Sw,Tu,Tu)\right),0,0,\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right)\ \right)\leq 0.$$

Since

$$H\left(\ \psi\left(G(Sw,Tu,Tu)\right),0,0,\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right)\ \right)>0$$

because $H \in \mathcal{H}^*$, then

$$F\left(\begin{array}{c}\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right),0,\\ 0,\psi\left(G(Sw,Tu,Tu)\right),\psi\left(G(Sw,Tu,Tu)\right)\end{array}\right)<0,$$

a contradiction of (F_3^*). Hence, $Sw = Aw = Sv = Av = Tu = Bu = t$ and t is the unique point of coincidence of A and S. Similarly, t is the unique point of coincidence of B and T. $\square$

Theorem 4.3. *Let (X,G) be a G - metric space and A,B,S and T be self mappings of X satisfying the inequality (4.1) for all $x,y\in X$, some $F\in\mathcal{F}^*$, some $H\in\mathcal{H}^*$ and ψ is an almost altering distance. If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then $\mathcal{C}(A,S)\neq\emptyset$ and $\mathcal{C}(B,T)\neq\emptyset$. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.*

Proof. Since (A,S) and T satisfy $CLR_{(A,S)T}$ - property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}Sx_n=z$, with $z\in S(X)\cap T(X)$. Since $z\in T(X)$, there exists $u\in X$ such that $z=Tu$. By (4.1) for $x=x_n$ and $y=u$ we obtain

$$\begin{array}{l} F\left(\begin{array}{c}\psi(G(Ax_n,Bu,Bu)),\psi(G(Sx_n,Tu,Tu)),\psi(G(Ax_n,Sx_n,Sx_n)),\\ \psi(G(Tu,Bu,Bu)),\psi(G(Sx_n,Bu,Bu)),\psi(G(Ax_n,Tu,Tu))\end{array}\right)+\\ H\left(\begin{array}{c}\psi(G(Sx_n,Tu,Tu)),\psi(G(Ax_n,Sx_n,Sx_n)),\psi(G(Tu,Bu,Bu)),\\ \psi(G(Sx_n,Bu,Bu)),\psi(G(Ax_n,Tu,Tu))\end{array}\right)\leq 0.\end{array}$$

Letting n tend to infinity we obtain

$$\begin{array}{c} F\left(\ \psi(G(z,Bu,Bu)),0,0,\psi(G(z,Bu,Bu)),\psi(G(z,Bu,Bu)),0\ \right)+\\ H\left(\ 0,0,\psi(G(z,Bu,Bu)),\psi(G(z,Bu,Bu)),0\ \right)\leq 0.\end{array}$$

If $G(z,Bu,Bu)>0$, then

$$H(0,0,\psi(G(z,Bu,Bu)),\psi(G(z,Bu,Bu)),0)>0$$

which implies

$$F\left(\begin{array}{c}\psi(G(z,Bu,Bu)),0,0,\psi(G(z,Bu,Bu)),\\ \psi(G(z,Bu,Bu)),0\end{array}\right)<0,$$

a contradiction of (F_2^*). Hence, $\psi(G(z,Bu,Bu))=0$ which implies $z=Bu=Tu$ and $\mathcal{C}(B,T)\neq\emptyset$.

On the other hand, $z\in S(X)$. Hence, there exists $v\in X$ such that $z=Sv$. By (4.1) for $x=v$ and $y=u$ we obtain

$$\begin{array}{l} F\left(\begin{array}{c}\psi(G(Av,Bu,Bu)),\psi(G(Sv,Tu,Tu)),\psi(G(Av,Sv,Sv)),\\ \psi(G(Tu,Bu,Bu)),\psi(G(Sv,Bu,Bu)),\psi(G(Av,Tu,Tu))\end{array}\right)+\\ H\left(\begin{array}{c}\psi(G(Sv,Tu,Tu)),\psi(G(Av,Sv,Sv)),\psi(G(Tu,Bu,Bu)),\\ \psi(G(Sv,Bu,Bu)),\psi(G(Av,Tu,Tu))\end{array}\right)\leq 0,\end{array}$$

$$\begin{array}{c} F(\psi(G(Av,z,z)),0,\psi(G(Av,z,z)),0,0,\psi(G(Av,z,z)))+\\ H(0,\psi(G(Av,z,z)),0,0,\psi(G(Av,z,z)))\leq 0.\end{array}$$

If $\psi(G(Av,z,z))>0$, then

$$H(0,\psi(G(Av,z,z)),0,0,\psi(G(Av,z,z)))>0$$

which implies that

$$F(\psi\left(G(Av,z,z)\right),0,\psi\left(G(Av,z,z)\right),0,0,\psi\left(G(Av,z,z)\right))<0,$$

a contradiction of (F_1^*). Hence, $\psi\left(G(Av,z,z)\right)=0$ which implies $z=Av=Sv$ and $\mathcal{C}\left(A,S\right)\neq\emptyset$. Therefore $z=Sv=Av=Bu=Tu$. By Theorem 4.2, z is the unique point of coincidence of A and S, and of B and T.

Moreover, if (A,S) and (B,T) are weakly compatible, by Lemma 4.1, z is the unique common fixed point for A and S and for B and T. $\square$

If $\psi\left(t\right)=t$ we obtain

Theorem 4.4. *Let (X,G) be a G - metric space and A,B,S and T be self mappings of X such that for all $x,y\in X$*

$$\begin{array}{c} F\left(\begin{array}{c} G(Ax,By,By),G(Sx,Ty,Ty),G(Ax,Sx,Sx), \\ G(Ty,By,By),G(Sx,By,By),G(Ax,Ty,Ty) \end{array}\right)+ \\ H\left(\begin{array}{c} G(Sx,Ty,Ty),G(Ax,Sx,Sx),G(Ty,By,By), \\ G(Sx,By,By),G(Ax,Ty,Ty) \end{array}\right)\leq 0, \end{array} \tag{4.2}$$

for some $F\in\mathcal{F}^*$ and some $H\in\mathcal{H}^*$. If (A,S) and T satisfy $CLR_{(A,S)T}$ - property, then $\mathcal{C}\left(A,S\right)\neq\emptyset$ and $\mathcal{C}\left(B,T\right)\neq\emptyset$. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point.

5. Applications

5.1. **Fixed points for mappings satisfying contractive conditions of integral type in G - metric spaces.** In [6], Branciari established the following theorem which opened the way to the study of fixed points for mappings satisfying contractive conditions of integral type.

Theorem 5.1 ([6])**.** *Let (X,d) be a complete metric space, $c\in(0,1)$ and $f:X\to X$ such that for all $x,y\in X$*

$$\int_0^{d(fx,fy)}h(t)dt\leq c\int_0^{d(x,y)}h(t)dt$$

where $h:[0,\infty)\to[0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0,\infty)$, such that $\int_0^{\varepsilon}h(t)dt>0$, for all $\varepsilon>0$. Then, f has a unique fixed point z such that for all $x\in X$, $z=\lim_{n\to\infty}f^nx$.

Some fixed point results for mappings satisfying contractive conditions of integral type are obtained in [26], [27], [33] and in other papers.

Lemma 5.2. *Let* $h : [0,\infty) \to [0,\infty)$ *be as in Theorem 5.1. Then* $\psi(t) = \int_0^t h(x)dx$ *is an almost altering distance.*

Proof. The proof it follows by Lemma 2.5 [27]. $\square$

Theorem 5.3. *Let* A, B, S *and* T *be self mappings of a* G *- metric space* (X, G) *such that for all* $x, y \in X$

$$F\left(\begin{array}{ccc} \int\limits_0^{G(Ax,By,By)} h(t)dt, & \int\limits_0^{G(Sx,Ty,Ty)} h(t)dt, & \int\limits_0^{G(Ax,Sx,Sx)} h(t)dt, \\ \int\limits_0^{G(Ty,By,By)} h(t)dt, & \int\limits_0^{G(Sx,By,By)} h(t)dt, & \int\limits_0^{G(Ax,Ty,Ty)} h(t)dt \end{array}\right)+$$

$$(5.1)\qquad H\left(\begin{array}{cc} \int\limits_0^{G(Sx,Ty,Ty)} h(t)dt, & \int\limits_0^{G(Ax,Sx,Sx)} h(t)dt, \\ \int\limits_0^{G(Ty,By,By)} h(t)dt, & \int\limits_0^{G(Sx,By,By)} h(t)dt, \\ \multicolumn{2}{c}{\int\limits_0^{G(Ax,Ty,Ty)} h(t)dt} \end{array}\right) \leq 0,$$

for some $F \in \mathcal{F}^*$*, some* $H \in \mathcal{H}^*$ *and* $h(t)$ *is as in Theorem 5.1.*

If (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then* $\mathcal{C}(A, S) \neq \emptyset$ *and* $\mathcal{C}(B, T) \neq \emptyset$*. Moreover, if* (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. By Lemma 5.2, $\psi(t) = \int_0^t h(x)dx$ is an almost altering distance. Hence,

$$\int\limits_0^{G(Ax,By,By)} h(t)dt = \psi(G(Ax, By, By)), \quad \int\limits_0^{G(Sx,Ty,Ty)} h(t)dt = \psi(G(Sx, Ty, Ty)),$$
$$\int\limits_0^{G(Ax,Sx,Sx)} h(t)dt = \psi(G(Ax, Sx, Sx)), \quad \int\limits_0^{G(Ty,By,By)} h(t)dt = \psi(G(Ty, By, By)),$$
$$\int\limits_0^{G(Sx,By,By)} h(t)dt = \psi(G(Sx, By, By)), \quad \int\limits_0^{G(Ax,Ty,Ty)} h(t)dt = \psi(G(Ax, Ty, Ty)).$$

By (5.1) we obtain

$$F\left(\begin{array}{c} \psi(G(Ax, By, By)), \psi(G(Sx, Ty, Ty)), \psi(G(Ax, Sx, Sx)), \\ \psi(G(Ty, By, By)), \psi(G(Sx, By, By)), \psi(G(Ax, Ty, Ty)) \end{array}\right)+$$
$$H\left(\begin{array}{c} \psi(G(Sx, Ty, Ty)), \psi(G(Ax, Sx, Sx)), \psi(G(Ty, By, By)), \\ \psi(G(Sx, By, By)), \psi(G(Ax, Ty, Ty)) \end{array}\right) \leq 0,$$

which is inequality (4.2). Hence, the conditions of Theorem 4.3 are satisfied and Theorem 5.3 follows by Theorem 4.3. □

For example, by Examples 3.1, 3.9 and Theorem 5.3 we obtain

Theorem 5.4. *Let A, B, S and T be self mappings of a G - metric space (X, G) such that for all $x, y \in X$*

$$\int_0^{G(Ax,By,By)} h(t)dt \leq kM_1 - M_2 \tag{5.2}$$

where $h(t)$ is as in Theorem 5.1, $k \in [0,1]$ and

$$M_1 = \max \left\{ \begin{array}{c} \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Ax,Sx,Sx)} h(t)dt, \int_0^{G(Ty,By,By)} h(t)dt, \\ \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt) \end{array} \right\},$$

$$M_2 = \max \left\{ \begin{array}{c} \int_0^{G(Sx,Ty,Ty)} h(t)dt, \int_0^{G(Ax,Sx,Sx)} h(t)dt, \int_0^{G(Ty,By,By)} h(t)dt, \\ \int_0^{G(Sx,By,By)} h(t)dt, \int_0^{G(Ax,Ty,Ty)} h(t)dt \end{array} \right\}.$$

If (A, S) and T satisfy $CLR_{(A,S)T}$ - property, then $\mathcal{C}(A, S) \neq \emptyset$ and $\mathcal{C}(B, T) \neq \emptyset$. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Remark 5.5. *1) Theorem 5.3 is a generalization for G - metric space of Theorem 4.3* [32].
2) Theorem 5.4 is a generalization for G - metric space of Theorem 4.4 [32].

5.2. **Fixed points for mappings satisfying φ - contractive conditions in G - metric spaces.** As in [17], let Φ be the set of all real nondecreasing continuous functions $\varphi : [0, \infty) \to [0, \infty)$ with
i) $\varphi(t) < t$ for all $t > 0$,
ii) $\varphi(0) = 0$.

The following functions are from $\mathcal{F}^*$.

Example 5.6. $F(t_1, ..., t_6) = t_1 - \varphi(\max\{t_2, t_3, t_4, t_5, t_6\})$.

Example 5.7. $F(t_1,...,t_6) = t_1 - \varphi\left(\max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}\right)$.

Example 5.8. $F(t_1,...,t_6) = t_1 - \varphi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$.

Example 5.9. $F(t_1,...,t_6) = t_1 - \varphi\left(\max\left\{t_2, \sqrt{t_3t_6}, \sqrt{t_4t_5}, \sqrt{t_5t_6}\right\}\right)$.

Example 5.10. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + bt_3 + ct_4 + dt_5 + et_6)$, $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e \leq 1$.

Example 5.11. $F(t_1,...,t_6) = t_1 - \varphi\left(at_2 + b\frac{\sqrt{t_5t_6}}{1+t_3+t_4}\right)$, $a, b \geq 0$ *and* $a + b \leq 1$.

Example 5.12. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + b\max\{t_3, t_4\} + c\max\{\frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\})$, $a, b, c \geq 0$ *and* $a + b + c \leq 1$.

Example 5.13. $F(t_1,...,t_6) = t_1 - \varphi(at_2 + b\max\{\frac{2t_4+t_5}{3}, \frac{2t_4+t_6}{3}, \frac{t_3+t_5+t_6}{3}\})$, $a, b \geq 0$ *and* $a + b \leq 1$.

For example, from Theorem 4.4 and Example 5.6 we obtain

Theorem 5.14. *Let* (X, G) *be a* G *- metric space and* A, B, S *and* T *be self mappings of* X *such that for all* $x, y \in X$

$$G(Ax, By, By) \leq \varphi\left(\max\left\{\begin{array}{c} G(Sx, Ty, Ty), G(Ax, Sx, Sx), \\ G(Ty, By, By), G(Sx, By, By), \\ G(Ax, Ty, Ty) \end{array}\right\}\right) - H\left(\begin{array}{c} G(Sx, Ty, Ty), G(Ax, Sx, Sx), G(Ty, By, By), \\ G(Sx, By, By), G(Ax, Ty, Ty) \end{array}\right),$$

for $\varphi \in \Phi$ *and* $H \in \mathcal{H}^*$.

If (A, S) *and* T *satisfy* $CLR_{(A,S)T}$ *- property, then* $\mathcal{C}(A, S) \neq \emptyset$ *and* $\mathcal{C}(B, T) \neq \emptyset$. *Moreover, if* (A, S) *and* (B, T) *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

References

[1] A. Aamri, D. El - Moutawakil, **Some new common fixed point theorems under strict contractive conditions**, Math. Anal. Appl. 270 (2002), 181-188.

[2] M. Abbas, B. E. Rhoades, **Common fixed point results for noncommuting mappings without continuity in generalized metric spaces**, Appl. Math. Comput. 215 (2009), 262-269.

[3] Ya. I. Alber, S. Guerre - Delabrierre, **Principle of weakly contractive maps in Hilbert spaces. New results in operator theory and its applications**, Adv. Appl. Math., Birkhäuser Verlag Basel 98 (1997), 7-22.

[4] J. Ali, M. Imdad, **An implicit function implies several contractive conditions**, Sarajevo J. Math. 17 (4) (2008), 269-285.

[5] I. Beg, M. Abbas, **Coincidence points and invariant approximations for mappings satisfying generalized weak contractive condition**, Fixed Point Theory Appl. (2006), Article ID 74503, 7 pages.

[6] A. Branciari, **A fixed point theorem for mappings satisfying a general contractive condition of integral type**, Int. J. Math. Math. Sci. 29 (2) (2002), 531-536.

[7] B. C. Dhage, **Generalized metric spaces and mappings with fixed point**, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[8] B. C. Dhage, **Generalized metric spaces and topological structures I**, Anal. St. Univ. Al. I. Cuza, Iasi Ser. Mat. 46 (1) (2000), 3-24.

[9] D. Doric, **Common fixed point for generalized (ψ, φ) - weak contractions**, Appl. Math. Lett. 22 (2009), 1896-1900.

[10] M. Imdad, M. Pant, S. Chauhan, **Fixed point theorems in Menger spaces using $CLR_{(\mathbf{ST})}$ - property and applications**, J. Nonlinear Anal. Optim. 3 (2) (2012), 225-237.

[11] M. Imdad, S. Chauhan, **Employing common limit range property to prove unified metrical common fixed point theorems**, Int. J. Anal. (2013), Article ID 763261, 10 pages.

[12] M. Imdad, S. Chauhan, Z. Kadelburg, **Fixed point theorems for mappings with common limit range property satisfying generalized (ψ, φ) - weak contractive conditions**, Math. Sci. 7 (16) (2013), doi:10.1186/2251-7456-7-16.

[13] G. Jungck, **Compatible mappings and common fixed points**, Int. J. Math. Math. Sci. 9 (1986), 771-779.

[14] G. Jungck, **Common fixed points for noncommuting nonself mappings on nonnumeric spaces**, Far East J. Math. Sci. 4 (2) (1996), 195-215.

[15] M. S. Khan, M. Swaleh, S. Sessa, **Fixed point theorems by altering distance between two points**, Bull. Austral. Math. Sci. 30 (1984), 1-9.

[16] Y. Liu, J. Wu, Z. Li, **Common fixed points of single valued and multi – valued maps**, Int. J. Math. Math. Sci. 19 (2005) , 3045-3055.

[17] J. Matkowski, **Fixed point theorems for mappings with a contractive iterate at a point**, Proc. Amer. Math. Soc. 62 (2) (1977), 344-348.

[18] Z. Mustafa, B. Sims, **Some remarks concerning D - metric spaces**, Proc. Intern. Conf. Fixed Point Theory and Applications, Valencia (Spain) (2003), 189-198.

[19] Z. Mustafa, B. Sims, **A new approach to generalized metric spaces**, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[20] Z. Mustafa, H. Obiedat, F. Awawdeh, **Some fixed point theorems for mappings on G - complete metric spaces**, Fixed Point Theory Appl. (2008), Article ID 189870, 12 pages.

[21] R. P. Pant, $R-$ **weak commutativity and common fixed points of noncompatible maps**, Ganita 49 (1998), 19-27.

[22] R. P. Pant, $R-$ **weak commutativity and common fixed points**, Soochow J. Math. 25 (1999), 37-42.

[23] V. Popa, **Fixed point theorems for implicit contractive mappings**, Stud. Cerc. Ştiinţ., Ser. Mat., Univ. Bacău 7 (1997), 127-134.

[24] V. Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, Demonstr. Math. 32 (1) (1999), 157-163.

[25] V. Popa, **Fixed point theorems for two pairs of mappings satisfying a new type of common limit range property**, Filomat 31 (11) (2017), 3181-3192.

[26] V. Popa, M. Mocanu, **A new viewpoint in the study of fixed points for mappings satisfying a contractive condition of integral type**, Bul. Inst. Politeh. Iaşi, Sec. Mat. Mec. Teor. Fiz. 53 (57) (2007), 269-286.

[27] V. Popa, M. Mocanu, **Altering distances and common fixed points under implicit relations**, Hacet. J. Math. Stat. 38 (3) (2009), 329-337.

[28] V. Popa, A.-M. Patriciu, **A general fixed point theorem for weakly compatible mappings in G - metric spaces**, J. Nonlinear Sci. Appl. 5 (2) (2012), 151-160.

[29] V. Popa, A.-M. Patriciu, **Fixed point theorems for mappings satisfying an implicit relation in G - metric spaces**, Bul. Inst. Politeh. Iaşi, Sec. Mat. Mec. Teor. Fiz. 59 (63) (2013), 97-126.

[30] V. Popa, A.-M. Patriciu, **A general fixed point theorem for a pair of self mappings with common limit range property in G - metric spaces**, Facta Univ. Ser. Math. Inform. 29 (4) (2014), 351-370.

[31] V. Popa, A.-M. Patriciu, **Fixed point theorems for two pairs of mappings satisfying common limit range property in G - metric spaces**, Bul. Inst. Politeh. Iaşi, Sec. Mat. Mec. Teor. Fiz. 62 (66) (2016), 19-43.

[32] V. Popa, A.-M. Patriciu, **A general fixed point theorem for two pairs of mappings satisfying a mixed implicit relation**, Math. Slovaca 68 (3) (2018), 655-666.

[33] B. E. Rhoades, **Two fixed point theorems for mappings satisfying a general contractive condition of integral type**, Intern. J. Math. Math. Sci. 63 (2003), 4007-4013.

[34] B. E. Rhoades, **Some theorems on weakly contractive maps**, Nonlinear Anal. 47 (2001), 2683-2693.

[35] K. P. Sastri, G. V. R. Babu, **Fixed point theorems in metric spaces by altering distances**, Bull. Calcutta Math. Soc. 90 (1998), 175-182.

[36] W. Sintunavarat, P. Kumam, **Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces**, J. Appl. Math. (2011), Article ID 637958, 14 pages.

VALERIU POPA
"Vasile Alecsandri" University of Bacău
157 Calea Mărăşeşti, Bacău, 600115, Romania
e-mail address: vpopa@ub.ro

ALINA-MIHAELA PATRICIU
"Dunărea de Jos" University of Galaţi,
Faculty of Sciences and Environment,
Department of Mathematics and Computer Sciences,
111 Domnească Street, Galaţi, 800201, Romania
e-mail address: Alina.Patriciu@ugal.ro