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CONSTRUCTIVE COUNTERPARTS OF A QUASIORDER

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Abstract. Co-quasiorder relations, the constructive counterpart of classical quasiorder relations are examined within the framework of Bishop's constructive mathematics. Two classically equivalent, but constructively inequivalent, notions of co-quasorder are investigated. It turns out that a weak co-quasorder is a co-quasiorder if and only if it is quasi-detachable. As a consequence, the incomparability relation associated to a co-quasiorder is quasi-detachable.

1. Introduction

A quasiorder (or a preorder) relation is a binary relation which is reflexive and transitive. From a constructive point of view, the concepts of order and quasiorder are negative concepts, so it is appropriate to define them as complements of binary relations which can be defined in an affirmative way. Therefore an irreflexive and cotransitive relation (a co-quasiorder [15]) should be considered first in order to obtain a quasiorder by negation. Noticeable efforts to a constructive theory of quasiorders have been done recently by Romano and others [7, 10, 12, 13, 14, 15]. Their notion of co-quasiorder is based on a stronger version of irreflexivity which could lead to relevant results. However, one can define various notions of irreflexivity (and also of cotransitivity) which are classically equivalent but constructively inequivalent. These notions lead to different definitions of co-quasiorder.

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While cotransitivity is a key concept of constructive mathematics and it would be questionable to replace it by weaker versions, the classical (constructively weaker) irreflexivity could be used for a constructive study of quasiorders. Actually, preference relations, which are nothing else than asymmetric co-quasiorders, are examined in constructive mathematics [1, 2, 5] without assuming a stronger variant of irreflexivity or asymmetry.

The main goal of this work is to examine the relationship between co-quasiorder (a strongly irreflexive and cotransitive relation) and weak co-quasiorder (an irreflexive and cotransitive relation). A certain notion of detachability is used in order to obtain a characterization of a co-quasiorder as a weak co-quasiorder satisfying that kind of detachability. Several notions of irreflexivity, cotransitivity, order comparability and their mutual relationships are also investigated.

The framework of this paper is Bishop's constructive mathematics (*BISH*), as developed in [3, 4], a mathematics carried out with intuitionstic logic. Every theorem of *BISH* is valid in classical mathematics and also in other varieties of constructive mathematics. The strict use of intuitionistic logic requires a careful reinterpretation of the logical connectives and quantifiers and major restrictions in order to avoid non-constructive logical principles such as the *law of excluded middle* $P \vee \neg P$, the main source of non-constructivism, or the *double negation* principle $\neg\neg P \Rightarrow P$. To show that a certain proposition P is non-constructive, we can use a *Brouwerian example*, that is, we prove that P implies some non-constructive principle. The so-called omniscience principles, are frequently used to produce Brouwerian examples.

- *The limited principle of omniscience (LPO)*: for every binary sequence (a_n) either $a_n = 0$ for all n, or else there exists n such that $a_n = 1$.
- *The weak limited principle of omniscience (WLPO)*: for every binary sequence (a_n) either $a_n = 0$ for all n, or it is contradictory that $a_n = 0$ for all n.
- *The lesser limited principle of omniscience (LLPO)*: if (a_n) is a binary sequence containing at most one term equal to 1, then either $a_{2n} = 0$ for all n, or else $a_{2n+1} = 0$ for all n.

Moreover, the following two logical principles are not accepted in *BISH*.

- *Markov's principle (MP)*: if (a_n) is a binary sequence and $\neg\forall n\ (a_n = 0)$, then there exists n such that $a_n = 1$.
- *The weak Markov's principle (WMP)*: every pseudopositive number a is strictly positive.

A real number a is called *pseudopositive* if

$$\forall x \in \mathbf{R}\ (\neg\neg(0 < x) \vee \neg\neg(x < a)).$$

More details on non-constructive principles can be found in [9].

We will illustrate a main feature of constructive mathematics, the possibility of revealing distinctions between classically equivalent notions and propositions. Thus, a major task is to obtain definitions which should be classically equivalent to the classical ones. Appropriate constructive definitions could lead to one or more constructive counterparts of a classical theorem.

2. Basic definitions and notations

Each set S will be endowed with an equivalence relation, the *equality* on S, and also with an *apartness relation* $\neq$, that is, an *irreflexive*, *symmetric*, and *cotransitive relation*:

$$\forall x, y \in S\ (x \neq y \Rightarrow \neg(x = y));$$

$$\forall x, y \in S\ (x \neq y \Rightarrow y \neq x);$$

$$\forall x, y, z \in S\ (x \neq y \Rightarrow (x \neq z \vee y \neq z)).$$

The apartness $\neq$ is said to be *tight* if

$$\forall x, y \in S\ (\neg(x \neq y) \Rightarrow x = y).$$

Contrary to the classical case, the implication $\neg(x = y) \Rightarrow x \neq y$ does not hold in *BISH*. For example, one might consider the real number set $\mathbf{R}$ as constructed in [3, 4] or presented axiomatically in [6]. Then

$$(\forall x, y \in \mathbf{R}\ (\neg(x = y) \Rightarrow x \neq y)) \Rightarrow \mathbf{MP}.$$

The *Cartesian product* of the sets $(A, =_1, \neq_1)$ and $(B, =_2, \neq_2)$ is the set $(A \times B, =, \neq)$ with $A \times B = \{(a, b) : a \in A,\ b \in B\}$,

$$(a_1, b_1) = (a_2, b_2) \Leftrightarrow (a_1 =_1 b_1 \wedge a_2 =_2 b_2);$$

$$(a_1, b_1) \neq (a_2, b_2) \Leftrightarrow (a_1 \neq_1 b_1 \wedge a_2 \neq_2 b_2).$$

A *binary relation* on S is a subset of $S \times S$.

In order to construct a function, we start with $(A_1, =_1, \neq_1)$ and $(A_2, =_2, \neq_2)$. The construction of the function $f : A_1 \to A_2$ requires an algorithm which, applied to an element $x \in A_1$, produces a unique element $f(x) \in A_2$. Each function $f : A_1 \to A_2$ is *extensional*:

$$\forall x_1, x_2 \in A_1\ (x_1 =_1 x_2 \Rightarrow f(x_1) =_2 f(x_2)).$$

The function f is said to be *strongly extensional* if

$$\forall x_1, x_2 \in A_1\ (f(x_1) \neq_2 f(x_2) \Rightarrow x_1 \neq_1 x_2).$$

3. Constructive irreflexivity

A main feature of constructive mathematics is the presence of classically equivalent notions which are constructively inequivalent. We will illustrate this by examining irreflexive relations.

Definition 3.1. Let $(S, =, \neq)$ be a set with apartness and ρ a binary relation on S. The relation ρ is said to be:

- *strongly irreflexive* if
$$\forall x, y \in S\ (x\rho y \Rightarrow x \neq y);$$
- *pseudo-strongly irreflexive* if
$$\forall x, y \in S\ (x\rho y \Rightarrow \forall z\ (\neg\neg(x \neq z) \vee \neg\neg(z \neq y)));$$
- *almost strongly irreflexive* if
$$\forall x, y \in S\ (x\rho y \Rightarrow \neg\neg(x \neq y));$$
- *irreflexive* if
$$\forall x \in S\ (\neg(x\rho x)).$$

Proposition 3.2. *Let $(S, =, \neq)$ be a set with apartness and ρ a binary relation on S. Then the following implications hold.*

(i) *If the relation ρ is strongly irreflexive, then it is pseudo-strongly irreflexive.*

(ii) *If ρ is pseudo-strongly irreflexive, then ρ is almost strongly irreflexive.*

(iii) *If ρ is almost strongly irreflexive, then ρ is irreflexive.*

Proof. (i) If x and y are apart, then $\forall z\ (x \neq z \vee z \neq y)$, hence x and y are pseudo-apart, that is, $\forall z\ (\neg\neg(x \neq z) \vee \neg\neg(z \neq y))$.

(ii) If $x\rho y$ then, since x and y are pseudo-apart, it follows that $\neg\neg(x \neq x)$ or $\neg\neg(x \neq y)$. The former is contradictory to $\neg(x \neq x)$, so the latter holds.

(iii) If $x\rho x$ then, from the almost strong irreflexivity of ρ, we obtain $\neg\neg(x \neq x)$, a contradiction. $\square$

If the apartness is not tight then irreflexivity does not necessarily imply almost strong irreflexivity, as proved in the following example.

Example 3.3. *Let $S = \{a, b, c\}$ with a, b, c mutually nonequal and the apartness $\neq = \{(a, c), (c, a), (b, c), (c, b)\}$. If $\rho = \{(a, b)\}$, then ρ is irreflexive but not almost strongly irreflexive.*

When the apartness is tight, the above implication holds as we can see in the following proposition.

Proposition 3.4. *Let $(S, =, \neq)$ be a set with a tight apartness and ρ a binary relation on S. Then ρ is irreflexive if and only if it is almost strongly irreflexive.*

Proof. We need only prove that irreflexivity entails almost strong irreflexivity. To this end, assume that ρ is irreflexive and $x\rho y$. Then, $\neg(x = y)$ which is equivalent to $\neg\neg(x \neq y)$. $\square$

Classically, for a tight apartness, all these notions of irreflexivity are equivalent. However, we cannot expect to prove within *BISH* the converse implications of Proposition 3.2 (i) and (ii). More precisely, we can prove the following results.

Proposition 3.5. (i) *If each irreflexive relation on a set with tight apartness were pseudo-strongly irreflexive, then every real number a with $\neg(a \leq 0)$ would be pseudopositive.*

(ii) *If every pseudo-irreflexive relation on a set with tight apartness were strongly irreflexive, then* WMP *would hold.*

(iii) *If every irreflexive relation on a set with tight apartness were strongly irreflexive, then* MP *would hold.*

Proof. Let a be a real number with $\neg(a = 0)$ and consider the relation ρ on $\mathbf{R}$ defined by $\rho = \{(a, 0)\}$. Clearly, ρ is irreflexive and (i) follows immediately. If ρ were strongly irreflexive then $a > 0$, so *MP* would hold and, consequently, we obtain (iii).

To prove (ii), it suffices to consider a pseudopositive number a and the relation ρ on $\mathbf{R}$ defined by $\rho = \{(a, 0)\}$. $\square$

For an irreflexive relation ρ the *logical complement*

$$\neg\rho = \{(x, y) : \neg(x\rho y)\}$$

is obviously a *Reflexive* relation. Conversely, if ρ is a reflexive relation on S, then for all $x \in S$ $x\rho x$, hence $\neg\neg(x\rho x)$, that is, $\neg((x, x) \in \neg\rho)$.

Therefore the relation

$$\neg\neg\rho = \{(x, y) : \neg\neg((x, y) \in \rho)\}$$

is also reflexive and $\neg\rho$ is irreflexive. Consequently, for any irreflexive relation ρ, the relation $\neg\neg\rho$ is an irreflexive relation too. We will prove the converse implication and then obtain a similar result for almost strongly irreflexive relations.

Proposition 3.6. *Let $(S, =, \neq)$ be a set with apartness and ρ a binary relation on S.*

(i) *The relation ρ is irreflexive if and only if $\neg\neg\rho$ is irreflexive.*

(ii) *The relation ρ is almost strongly irreflexive if and only if $\neg\neg\rho$ is almost strongly irreflexive.*

Proof. (i) We have already seen that $\neg\neg\rho$ is irreflexive whenever ρ is irreflexive. To prove the converse implication we need only observe that

$$\neg\neg\neg(x\rho x) \Leftrightarrow \neg(x\rho x).$$

Alternatively, we can write the irreflexivity of a binary relation r as follows:

$$\forall x, y \in S\ (xry \Rightarrow \neg(x = y)).$$

Since $\rho \subset \neg\neg\rho$, if $\neg\neg\rho$ satisfies the above condition, then ρ does.

(ii) As above, if $\neg\neg\rho$ is almost strongly irreflexive then, taking into account that $\rho \subset \neg\neg\rho$, it follows that ρ is almost strongly irreflexive too. Conversely, assuming that ρ is almost strongly irreflexive, consider $x, y \in S$ with $(x, y) \in \neg\neg\rho$ and assume that $\neg(x \neq y)$. If $x\rho y$, then $\neg\neg(x \neq y)$, contradictory to $\neg(x \neq y)$. Therefore $\neg(x\rho y)$, in contradiction to $\neg\neg(x\rho y)$. Consequently, the condition $\neg(x \neq y)$ is impossible hence $\neg\neg(x \neq y)$. □

Corollary 3.7. *If $\neg\neg\rho$ is pseudo-strongly irreflexive, respectively strongly irreflexive, then ρ satisfes the same property.*

Proof. We can use, as above, the property $\rho \subset \neg\neg\rho$. □

Similar results can be obtained for strongly irreflexive relations by using the *apartness complement*

$$\sim\rho = \{(x, y) : \forall(a, b) \in \rho\ ((x, y) \neq (a, b))\}.$$

Lemma 3.8. *If ρ is strongly irreflexive and inhabited, then $\sim\rho$ is reflexive.*

Proof. We need only prove that for all elements x of S, $(x, x) \in\sim S$. The relation ρ is inhabited, so we can find elements of it. For an arbitrary $(a, b) \in \rho$, since ρ is strongly irreflexive, $a \neq b$ hence $a \neq x$ or $x \neq b$. As a consequence, $(x, x) \neq (a, b)$ and $(x, x) \in\sim \rho$. □

Conversely, if $\sim \rho$ is reflexive, it follows that $(x, y) \neq (x, x)$ whenever $(x, y) \in \rho$. Consequently, $x \neq y$ hence ρ is strongly irreflexive. Similarly, if ρ is reflexive, then $\sim \rho$ is strongly irreflexive.

Corollary 3.9. *Let $(S, =, \neq)$ a set with apartness and ρ an inhabited subset of $S \times S$.*

(i) *The relation ρ is strongly irreflexive if and only if $\sim\sim \rho$ is strongly irreflexive.*

(ii) *If ρ is strongly irreflexive, then for all $(x, y) \in S \times S$ and $(a, b) \in \rho$, either $(x, y) \neq (a, b)$ or else $x \neq y$.*

Proof. (i) If ρ is strongly irreflexive and inhabited then $\sim \rho$ is reflexive and therefore $\sim\sim \rho$ is strongly reflexive. Conversely, we take into account that $\rho \subset\sim\sim \rho$.

(ii) If $(x, y) \in S \times S$ and $(a, b) \in \rho$, then the strong reflexivity of ρ entails $a \neq b$ hence $a \neq x$ or $x \neq b$. By applying cotransitivity again, we obtain $a \neq x$ or $x \neq y$ or $y \neq b$. it follows that $(x, y) \neq (a, b)$ or $x \neq y$. □

Clearly, if ρ is reflexive, then $\neg\neg\rho$ is reflexive but we cannot expect to prove the converse implication within *BISH*.

Example 3.10. *Let a be a real number with $\neg a = 0$ and define the set $S = \{0, a\}$. Define the binary relation ρ on S by $\rho = \{(0, 0)\} \cup \{(a, a) \; if \; a \neq 0\}$. Then the relation $\neg\neg\rho$ is reflexive but the reflexivity of ρ entails* MP.

Proof. We have to prove that for all $x \in S$ $\neg\neg(x\rho x)$. Assume that $x \in S$ and $\neg(x\rho x)$. If $x \neq 0$ then $x = a$ and $(a, a) \in S$, contradictory to $\neg(x\rho x)$. Therefore $\neg(x \neq 0)$ hence $x = 0$ and $(x, x) \in S$, a contradiction. Consequently, the condition $\neg(x\rho x)$ is contradictory, that is, $(x, x) \in \neg\neg\rho$. If ρ is reflexive, then $(a, a) \in \rho$, condition which implies $a \neq 0$ and this, in turn, entails the Markov principle. □

4. Cotransitivity

The notion of cotransitivity plays a crucial role in constructive mathematics. For example, the cotransitivity of the real number set:

$$a < b \Rightarrow \forall x \; (a < x \vee x < b)$$

is an appropriate substitute of the trichotomy law:

$$\forall x, y \in \mathbf{R}\ (x < 0 \vee x = 0 \vee x > 0),$$

equivalent to *LPO* hence unacceptable in constructive mathematics. As in the case of irreflexivity, we may consider several definitions of cotransitivity. However, the strongest, the usual constructive cotransitivity, is the most appropriate for developing a constructive theory of order and quasiorder.

Definition 4.1. Let $(S, =, \neq)$ be a set with apartness and ρ a binary relation on S. The relation ρ is said to be:

- *cotransitive* if
$$\forall x, y, z \in S\ (x\rho y \Rightarrow (x\rho z \vee z\rho y));$$
- *pseudo-cotransitive* if
$$\forall x, y, z \in S\ (x\rho y \Rightarrow (\neg\neg(x\rho z) \vee \neg\neg(z\rho y)));$$
- *almost cotransitive* if
$$\forall x, y, z \in S\ (x\rho y \Rightarrow (\neg\neg(x\rho z \vee z\rho y)));$$
- *nearly cotransitive* if
$$\forall x, y, z \in S\ ((x\rho y \wedge \neg(z\rho y)) \vee (\neg(x\rho y) \wedge z\rho y)) \Rightarrow x\rho z;$$
- *weakly cotransitive* if
$$\forall x, y, z \in S\ ((\neg(x\rho y) \wedge \neg(y\rho z)) \Rightarrow \neg(x\rho z)).$$

Proposition 4.2. *Let $(S, =, \neq)$ a set with apartness and ρ a binary relation on S. Then the following implications hold.*

(i) *Cotransitivity of ρ implies pseudo-cotransitivity. The latter entails almost cotransitivity which, in turn, implies weak cotransitivity.*

(ii) *Cotransitivity implies near cotransitivity and the latter implies weak cotransitivity.*

Proof. (i) Since $p \Rightarrow \neg\neg p$, the first implication is straightforward. If ρ is pseudo-cotransitive, since

$$(\neg\neg p \vee \neg\neg q) \Rightarrow \neg\neg(p \vee q),$$

it follows that ρ is almost cotransitive.

For the last implication, we assume that ρ is almost cotransitive and consider the elements $x, y, z \in S$ with $\neg(x\rho y) \wedge \neg(y\rho z)$, which is equivalent to $\neg(x\rho y \vee y\rho z)$. Assuming that $x\rho z$, we obtain from the almost cotransitivity of ρ the condition $\neg\neg(x\rho y \vee y\rho z)$, a contradiction. As a consequence, we obtain $\neg(x\rho z)$.

(ii) Assume that ρ is cotransitive and let x, y, z be elements of S such that $x\rho y$ and $\neg(z\rho y)$. Then $x\rho z$ or $z\rho y$ and the latter is contradictory to the hypothesis. In a similar manner, we can easily observe that $x\rho z$ whenever $\neg(x\rho y) \wedge z\rho y$.

To prove that near cotransitivity implies weak cotransitivity, consider $x, y, z \in S$ with $(\neg(x\rho y) \wedge (\neg(y\rho z))$ and assume that $x\rho z$. If $x\rho z$ and $\neg(y\rho z)$, the near cotransitivity of ρ entails $x\rho y$, which is contradictory. Therefore $\neg(x\rho z)$ and, as a consequence, the relation ρ is weakly cotransitive. $\square$

The binary relation ρ is obviously weakly cotransitive if and only if $\neg\rho$ is transitive.

5. Constructive quasiorder relations

Classically, a binary relation $\preceq$ is said to be a *quasiorder* whenever it is reflexive and transitive. As in the case of the constructive study of partial order [11], it would be more appropriate to start with an irreflexive and cotransitive relation $\npreceq$ in order to obtain a quasiorder $\preceq = \neg \npreceq$.

We might consider a relation $\npreceq$ which satisfies one of the five conditions of cotransitivity from Definition 4.1 and also one of the conditions of irreflexivity from Definition 3.1. Then the relation $\preceq = \neg \npreceq$ is a quasiorder relation. We will define in this way two notions of co-order. Thus, $\npreceq$ is called *weak co-quasiorder* if it is irreflexive and cotransitive. Following Romano [15], we say that $\npreceq$ is a *co-quasiorder* if it is strongly irreflexive and cotransitive. Since every strongly irreflexive relation is irreflexive, any co-quasiorder is necessarily a weak co-quasiorder. If the apartness is not tight, the converse does not hold classically, let alone constructively. We illustrate this situation by the following counterexample.

Example 5.1. *Let $S = \{a, b, c\}$ with a, b, c mutually nonequal and the apartness $\neq = \{(a, c), (c, a), (b, c), (c, b)\}$. If $\npreceq = \{(a, b),\ (a, c)\}$, then $\npreceq$ is a weak co-quasiorder but not a co-quasiorder.*

Clearly, the two notions of quasiorder are classically equivalent provided the apartness is tight. However, we cannot expect to prove this equivalence in *BISH*.

Proposition 5.2. *If each weak co-quasiorder on a set with tight apartness is also a co-quasiorder, then* MP *holds.*

Proof. For a Brouwerian example, consider $a \in \mathbf{R}$ with $\neg(a = 0)$, $S = \{a, 0\}$ and $\rho = \{(a, 0)\}$, as in the proof of Proposition 3.5 (iii). □

In order to examine the relationship between these variants of co-quasiorder, we need a certain notion of detachability. Given a set with apartness $(S, =, \neq)$, a subset A of S is called *quasi-detachable* if

$$\forall x \in S \; \forall y \in A \; (x \in A \vee x \neq y).$$

One can easily prove that any co-quasiorder on the set S is a detachable subset of $S \times S$ [10]. Moreover, if $\not\preceq$ is a co-quasiorder on S, then for all $a \in S$ the subsets

$$a \not\preceq = \{x \in S : a \not\preceq x\}, \quad \not\preceq a = \{x \in S : x \not\preceq a\}$$

are quasi-detachable [7]. (Quasi-detachable subsets are called SE-subsets in [10].)

Theorem 5.3. *Let $(S, =, \neq)$ be a set with apartness and $\not\preceq$ a binary relation on S. Then the following statements are equivalent.*

(1) *The relation $\not\preceq$ is a co-quasiorder.*
(2) *The relation $\not\preceq$ is a quasi-detachable weak co-quasiorder.*
(3) *The relation $\not\preceq$ is a weak co-quasiorder which satisfies the condition $\neg \not\preceq = \sim\not\preceq$.*

Proof. Since every strongly irreflexive relation is irreflexive, every co-quasiorder is a weak co-quasiorder. Each co-quasiorder is quasi-detachable, so (1) entails (2). If $\not\preceq$ is quasi-detachable, then $\neg \not\preceq = \sim\not\preceq$ [10] and, as a consequence, (2) implies (3). To prove (3) $\Rightarrow$ (1), assume that $\not\preceq$ satisfies (3) and we need only prove that $\not\preceq$ is strongly irreflexive. To this end, consider $x, y \in S$ with $x \not\preceq y$. Since $\not\preceq$ is irreflexive, $(x, x) \in \neg \not\preceq = \sim\not\preceq$. Therefore $(x, y) \neq (x, x)$ hence $x \neq y$. It follows that $\not\preceq$ is strongly irreflexive. □

Weak co-quasiorders satisfy a weaker condition of detachability.

Proposition 5.4. *If $\not\preceq$ is a coquasiorder on S, then*

$$\forall (x, y) \in S \times S \; \forall (a, b) \in \not\preceq \; ((x, y) \in \not\preceq \vee \neg((x, y) = (a, b))).$$

Proof. Consider $(x, y) \in S \times S$ and $(a, b) \in \not\preceq$. Then $a \not\preceq x$ or $x \not\preceq y$ or $y \not\preceq b$. If $a \not\preceq x$ or $y \not\preceq b$, then $\neg(a = x)$ or $\neg(y = b)$ hence $\neg((x, y) = (a, b))$. □

In the last part of this section, we turn our attention to weak co-quasiorder relations on a metric space. Each metric space X is equipped with the standard tight apartness

$$x \neq y \Leftrightarrow d(x, y) > 0.$$

Example 5.5. *Let $f : X \to Y$ be a function between the metric spaces X and Y. Define the relation $\not\preceq$ on X by*

$$x \not\preceq y \Leftrightarrow f(x) \neq f(y).$$

Then $\not\preceq$ is a weak co-quasiorder which is co-quasiorder if and only if f is strongly extensional.

Proof. Obviously, the relation is a weak co-quasiorder. It is a co-quasiorder if and only if it is strongly irreflexive. To end the proof we need only observe that the strong irreflexivity of $\not\preceq$ is equivalent to the strong extensionality of f. □

We will use Ishihara's methods from Lemma 3 of [8] to extend that result to a more general case.

Theorem 5.6. *Let $\not\preceq$ be a weak co-quasiorder on the metric space X and let x, y be elements of X with $x \not\preceq y$. Then the subset $\{x, y\}$ is closed.*

Proof. Let (z_n) be a sequence of elements of $\{x, y\}$, convergent to a limit z in X. The cotransitivity of $\not\preceq$ entails $x \not\preceq z$ or $z \not\preceq y$. In the former case, suppose that $z \neq y$. Since (z_n) converges to z, it follows that $z_n \neq y$, hence $z_n = x$, for all sufficiently large n. Therefore $z = x$, contradictory to $x \not\preceq z$. It follows that $\neg(z \neq y)$, that is, $z = y$. In a similar way, we prove that $z = x$ in the latter case. Thus, (z_n) converges to an element of $\{x, y\}$. □

As a consequence, we can obtain Isihara's result for functions (Lemma 3 of [8]).

Corollary 5.7. *Let $f : X \to Y$ be a function between metric spaces and let x, y be elements of X such that $f(x) \neq f(y)$. Then the set $\{x, y\}$ is closed.*

Proof. We apply Theorem 5.6 for the quasiorder relation defined in Example 5.5. □

6. Comparability and incomparability

Classically, for a quasiorder $\preceq$, the elements x and y are comparable if $x \preceq y$ or $y \preceq x$, that is $(x, y) \in \preceq \cup \preceq^{-1}$. For a constructive study, it would be more convenient to define first the notion of incomparability.

Definition 6.1. Let $(S, =, \neq)$ be a set with apartness and $\npreceq$ a weak co-quasiorder on S. We define the relation $\nparallel$ of *incomparability* on S by

$$x \nparallel y \Leftrightarrow (x \npreceq y \wedge y \npreceq x).$$

If $x \nparallel y$, we say that x and y are *incomparable.*

We denote by $\preceq$ the quasiorder relation on S defined by

$$x \preceq y \Leftrightarrow \neg(x \npreceq y)$$

and by $\parallel$ the relation defined on S by

$$x \parallel y \Leftrightarrow (x \preceq y \vee y \preceq x).$$

Definition 6.2. Let $(S, =, \neq)$ be a set with apartness, $\npreceq$ a weak co-quasiorder on S, and $\nparallel$ the corresponding incomparability relation. The elements x and y of S are said to be:

- *comparable* if
$$x \parallel y;$$
- *quasicomparable* if
$$(x, y) \in \sim\nparallel;$$
- *weakly comparable* if
$$(x, y) \in \neg \nparallel .$$

Clearly, the relations of incomparability, comparability, quasicomparability, and weak comparability are symmetric and the relation $\preceq \cap \preceq^{-1}$ is an equivalence.

Lemma 6.3. *If two elements are either comparable or quasicomparable, then they are weakly comparable.*

Proof. If $x \parallel y$, then $\neg(x \npreceq y)$ or $\neg(y \npreceq x)$ hence $\neg(x \npreceq y \wedge y \npreceq x)$, that is, $\neg(x \nparallel y)$. Since $\sim\nparallel \subset \neg \nparallel$, it follows that quasicomparability implies comparability. $\square$

Example 6.4. *Consider the relation* $\npreceq$ *on* $\mathbf{R}^2$ *defined by*

$$(x_1, x_2) \npreceq (y_1, y_2) \Leftrightarrow (|x_1| > |y_1| \vee |x_2| > |y_2|).$$

Then $\npreceq$ *is a co-quasiorder, the corresponding quasiorder* $\preceq$ *is given by*

$$(x_1, x_2) \preceq (y_1, y_2) \Leftrightarrow (|x_1| \leq |y_1| \wedge |x_2| \leq |y_2|),$$

the incomparability is obtained by

$$x \not\parallel y \Leftrightarrow (|x_1| > |y_1| \wedge |x_2| < |y_2|) \vee (|x_1| < |y_1| \wedge |x_2| > |y_2|),$$

and

$$((x_1, x_2), (y_1, y_2)) \in \npreceq \cap \npreceq^{-1} \Leftrightarrow (|x_1| = |y_1| \wedge |x_2| = |y_2|).$$

Although for a tight apartness, two elements are either classically comparable or incomparabile, we cannot expect to prove this constructively. Similarly, we cannot prove constructively that any two weakly comparable elements are comparable. We prove this by using the following Brouwerian example.

Example 6.5. *Consider the co-quasiorder* $\npreceq$ *from Example 6.4.*

(i) *If for any real number* a*, we have*

$$(a, 0) \preceq (0, a) \vee (0, a) \preceq (a, 0) \vee (a, 0) \not\parallel (0, a),$$

then LPO *holds.*

(ii) *If for all* $a \in \mathbf{R}$*,*

$$((0, 0) \preceq (a, a)) \vee (a, a) \preceq (0, 0),$$

then LLPO *holds.*

Proof. (i) It follows that $a = 0$ or $a > 0$ or $a < 0$ which is equivalent to *LPO*.

(ii) If $(a, a) \not\parallel (0, 0)$, then $a > 0$ and $a < 0$, a contradiction. Therefore (a, a) and $(0, 0)$ are weakly comparable. If $(a, a) \parallel (0, 0)$, then $a \leq 0$ or $a \geq 0$ hence *LLPO* holds. □

Lemma 6.6. *If* A *and* B *are quasi-detachable subsets of the set* S*, then* $A \cap B$ *is qasi-detachable.*

Proof. Let x, y be elements of S with $y \in A \cap B$. On the one hand, $x \in A$ or $x \neq y$. On the other hand $x \in B$ or $x \neq y$. It follows that $x \neq y$ or else $x \in A \cap B$. □

Proposition 6.7. *Let $(S, =, \neq)$ be a set with apartness, $\npreceq$ a weak co-quasiorder, and $\nparallel$ its corresponding relation of incomparability. Then each of the conditions* $(1) - (3)$ *entails the next one.*

(1) *The relation $\npreceq$ is a co-quasiorder.*
(2) *The relation $\nparallel$ is quasi-detachable.*
(3) *Any two weakly comparable elements are quasicomparable.*
(4) *The relation $\nparallel$ is strongly irreflexive.*

Proof. If $\npreceq$ is a co-quasiorder, then the relation $\npreceq^{-1}$ is also a co-quasiorder and, according to Lemma 6.6, the relation $\nparallel = \npreceq \cap \npreceq^{-1}$ is quasi-detachable. Therefore $(1) \Rightarrow (2)$.

If $\nparallel$ is quasi-detachable, then $\neg \nparallel = \sim\nparallel$[10], so $(2) \Rightarrow (3)$.

We now assume (3) and let x, y be elements of S with $x \nparallel y$. Then $x \parallel x$ hence $(x, x) \in \neg \nparallel = \sim\nparallel$. It follows that $(x, y) \neq (x, x)$ hence $x \neq y$. Therefore $(3) \Rightarrow (4)$. □

We now consider weak co-quasiorders without incomparable elements. In this case, the weak co-quasiorder should be asymmetric. A binary relation ρ on the set S is called *asymmetric* if for all $x, y \in S$ $x\rho y \Rightarrow \neg(y\rho x)$. We introduce a stronger notion of asymmetry. We say that ρ is *strongly asymmetric* if

$$\forall x, y \in S (x\rho y \Rightarrow (y, x) \in \sim \rho).$$

A binary relation is said to be a *preference* if it is asymmetric and cotransitive. Therefore each preference on S is a weak co-quasiorder and for all $x, y \in S$, x and y are weakly comparable.

Definition 6.8. A binary relation is said to be a *strong preference* if it is strongly symmetric and cotransitive.

It follows that each strong preference $\succ$ is both a preference and a co-quasiorder. To prove that $\succ$ is a co-quasiorder we need only prove that $\succ$ is strongly irreflexive. If $x \succ y$ then $(y, x) \in \sim\succ$ hence $(y, x) \neq (x, y)$. Therefore $x \neq y$ and, as a consequence, $\succ$ is strongly irreflexive.

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