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fmg-CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. After the introduction of a fuzzy generalized version of closed set in [2, 3], different types of generalized versions of fuzzy closed sets have been introduced and studied. In this context, we have to mention [3, 5, 6, 7, 8, 9, 10, 11]. In this paper we study the notion of fmg-closed set, which was introduced in [9].

1. Introduction

This paper deals with the notion of fmg-closed set in fuzzy topological spaces, which was introduced in [9]. Using this concept as a basic tool, we introduce here the nption of fmg-closure operator, which is an idempotent operator. Then we establish some properties of this set operator and afterwards, the mutual relationships of this operator with the operators defined in [3, 5, 6, 7, 8, 9, 11, 12] are established. Next we introduce and characterize the notions of fmg-open function and fmg-closed function using the fmg-closure operator and we establish the mutual relationships of these two new types of functions with the functions defined in [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

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Furthermore, we introduce the notions of fmg-continuous function and fmg-irresolute function, then we characterize these two types of functions via fmg-closure operator and we establish the mutual relationships of fmg-continuous functions with the functions defined in [3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Next we introduce the notions of fmg-regular space, fmg-normal space and fmg-compact space, that are fuzzy regular space [24], fuzzy normal space [23] and fuzzy compact space [18] respectively, but the converses are not true, in general. Next we introduce the class of fmT_g-spaces in which fuzzy regularity and fuzzy normality remain invariant under fmg-continuous function and fuzzy regular space and fuzzy normal space become fmg-regular space and fmg-normal space under strongly fmg-continuous function. In the last section we introduce and study the notion of fmg-T_2 space, then some applications of the notions of fmg-open function, fmg-continuous function, fmg-irresolute function and strongly fmg-continuous function on the spaces defined above are established.

2. Preliminaries

Throughout this paper (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [18]. In [30], L.A. Zadeh introduced fuzzy set as follows: A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [30] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$ [30]. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [30] while AqB means A is quasi-coincident (q-coincident, for short) with B, if there exists $x \in X$ such that $A(x) + B(x) > 1$ [28]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy point x_t and a fuzzy set A, $x_t \in A$ means $A(x) \geq t$, i.e., $x_t \leq A$. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [18] and fuzzy interior [18] respectively. A fuzzy set A is called a fuzzy neighbourhood (fuzzy nbd, for short) of a fuzzy point x_α if there exists a fuzzy open set U in X such that $x_\alpha \in U \leq A$ [28]. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_α [28].

A fuzzy set A is called a fuzzy quasi neighbourhood (fuzzy q-nbd, for short) [28] of a fuzzy point x_α in an fts X if there is a fuzzy open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open q-nbd [28] of x_α. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [27], fuzzy α-open [17], fuzzy β-open [21], fuzzy γ-open [4]) if $A = int(clA)$ (resp., $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$, $A \leq cl(intA) \bigvee int(clA)$). A fuzzy set A is called fuzzy π-open if A is the union of finite number of fuzzy regular open sets [8]. The complement of a fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open) set is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy preclosed [27], fuzzy α-closed [17], fuzzy β-closed [21], fuzzy γ-closed [4]). The intersection of all fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed) sets containing a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy preclosure [27], fuzzy α-closure [17], fuzzy β-closure [21], fuzzy γ-closure [4]) of A, to be denoted by $sclA$ (resp., $pclA$, αclA, βclA, γclA). The collection of all fuzzy open (resp., fuzzy regular open, fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open, fuzzy π-open) sets in an fts (X, τ) is denoted by τ (resp., $FRO(X, \tau)$, $FSO(X, \tau)$, $FPO(X, \tau)$, $F\alpha O(X, \tau)$, $F\beta O(X, \tau)$, $F\gamma O(X, \tau)$, $F\pi O(X, \tau)$). The collection of all fuzzy closed (resp., fuzzy regular closed, fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed, fuzzy π-closed) sets in an fts X is denoted by τ^c (resp., $FRC(X, \tau)$, $FSC(X, \tau)$, $FPC(X, \tau)$, $F\alpha C(X, \tau)$, $F\beta C(X, \tau)$, $F\gamma C(X, \tau)$, $F\pi C(X, \tau)$).

3. fmg-Closed Set: Some Properties

The notion of fmg-closed set is introduced in [9]. Here we prove some important properties of this set and the mutual relationship of this class of sets with the classes of sets defined in [2, 3, 5, 6, 7, 9, 10, 11] are established.

First we recall the following definition from [2, 3, 9] for ready references.

Definition 3.1 [2, 3]. Let (X, τ) be an fts and $A \in I^X$. Then A is called fg-closed set if $clA \leq U$ whenever $A \leq U \in \tau$. The complement of an fg-closed set in an fts X is called fg-open set.

Definition 3.2 [9]. Let (X, τ) be an fts and $A \in I^X$. Then A is called fmg-closed set in X if $clintA \leq U$ whenever $A \leq U$ and U is fg-open set in X. The complement of fmg-closed set is called fmg-open set in X.

Remark 3.3. Union and intersection of two fmg-closed sets may not be so, as it seen from the following examples.

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Now the collection of all fg-open sets in (X, τ) is $\{0_X, 1_X, U, V\}$ where $U \geq A, V \not\geq 1_X \setminus A$. Consider two fuzzy sets B and C defined by $B(a) = 0.4, B(b) = 0.6, C(a) = 0.7, C(b) = 0.3$. As $clintB = clintC = 0_X$, clearly B and C are fmg-closed sets in (X, τ). Let $D = B \bigvee C$. Then $D(a) = 0.7, D(b) = 0.6$. Then $D \leq D$ where D is fg-open set in X. But $clintD = 1_X \not\leq D \Rightarrow D$ is not an fmg-closed set in (X, τ).

Example 3.5. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5$. Then (X, τ) is an fts. Here the collection of all fg-open sets in (X, τ) is $\{0_X, 1_X, U, V\}$ where $U(a) \geq 0.6, 0.5 \leq U(b) < 0.6, V \not\geq 1_X \setminus C$. Now consider the fuzzy sets S and T defined by $S(a) = 0.6, S(b) = 0.4, T(a) = 0.4, T(b) = 0.6$. Now $S \leq U$ where U is fg-open set in X and $clintS = 1_X \setminus C \leq U \Rightarrow S$ is fmg-closed set in (X, τ). Again as 1_X is the only fg-open set in X containing T, clearly T is fmg-closed set in (X, τ). Let $M = S \bigwedge T$. Then $M(a) = M(b) = 0.4$. Now M is fg-open set in X. So $M \leq M$, but $clintM = 1_X \setminus C \not\leq M \Rightarrow M$ is not an fmg-closed set in (X, τ).

So we can conclude that the set of all fmg-open sets in an fts (X, τ) do not form a fuzzy topology.

Remark 3.6. Fuzzy closed set, fuzzy regular closed set, fuzzy preclosed set, fuzzy α-closed set in an fts (X, τ) are fmg-closed sets in X. But the converses are not necessarily true, in general, as we can see from the following example.

Example 3.7. There exists an fmg-closed set which is none of fuzzy closed, fuzzy regular closed, fuzzy preclosed, fuzzy α-closed

Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.6$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.6$. Then 1_X is the only fg-open set in (X, τ) containing B and so B is fmg-closed set in (X, τ). But as $clintB = 1_X \not\leq B \Rightarrow B \notin FPC(X, \tau)$. Also $B \notin \tau^c$, $B \notin FRC(X)$, $B \notin F\alpha C(X)$.

Theorem 3.8. Let (X,τ) be an fts and $A, B \in I^X$. If $A \leq B \leq clintA$ and A is fmg-closed set in X, then B is also fmg-closed set in X.
Proof. Let U be an fg-open set in (X,τ) such that $B \leq U$. Then by hypothesis, $A \leq B \leq U$. As A is fmg-closed set in X, $clintA \leq U$. As $B \leq clintA$, so $clintB \leq clint(clintA) \leq clintA \leq U \Rightarrow B$ is fmg-closed set in X.
Theorem 3.9. Let (X,τ) be an fts and $A, B \in I^X$. If $intclA \leq B \leq A$ and A is fmg-open set in X, then B is also fmg-open set in X.
Proof. $intclA \leq B \leq A \Rightarrow 1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus intclA = clint(1_X \setminus A)$ where $1_X \setminus A$ is fmg-closed set in X. By Theorem 3.8, $1_X \setminus B$ is fmg-closed set in $X \Rightarrow B$ is fmg-open set in X.
Theorem 3.10. Let (X,τ) be an fts and $A \in I^X$. Then A is fmg-open set in X if and only if $K \leq intclA$ whenever $K \leq A$ and K is fg-closed set in (X,τ).
Proof. Let $A(\in I^X)$ be fmg-open set in X and $K \leq A$ where K is fg-closed set in (X,τ). Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus A$ is fmg-closed set in X and $1_X \setminus K$ is fg-open set in (X,τ). By hypothesis, $clint(1_X \setminus A) \leq 1_X \setminus K \Rightarrow 1_X \setminus intclA \leq 1_X \setminus K \Rightarrow K \leq intclA$.

Conversely, let $K \leq intclA$ whenever $K \leq A$, K be fg-closed set in (X,τ). Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus K$ is fg-open set in (X,τ). Now $1_X \setminus intclA \leq 1_X \setminus K \Rightarrow clint(1_X \setminus A) \leq 1_X \setminus K$ (by hypothesis) $\Rightarrow 1_X \setminus A$ is fmg-closed set in $X \Rightarrow A$ is fmg-open set in X.
Theorem 3.11. Let (X,τ) be an fts and $A, B \in I^X$. If A is fmg-closed set in X and B is fg-closed set in (X,τ) with $A \not{q} B$. Then $clintA \not{q} B$.
Proof. By hypothesis, $A \not{q} B \Rightarrow A \leq 1_X \setminus B$ which is fg-open set in $(X,\tau) \Rightarrow clintA \leq 1_X \setminus B \Rightarrow clintA \not{q} B$.
Remark 3.12. The converse of Theorem 3.11 may not be true, in general, as it seen from the following example.
Example 3.13. Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.4, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5, C(a) = 0.8, C(b) = 1$. Then (X,τ) is an fts. Consider the fuzzy set D defined by $D(a) = 0.4, D(b) = 0.5$. Then $D < A$ which is fg-open set in X. But $clintD = 1_X \setminus B \not\leq A \Rightarrow D$ is not fmg-closed set in X. Again $D \not{q}(1_X \setminus C)$ which is fg-closed set in X and $clintD \not{q}(1_X \setminus C)$ also.

Let us now recall the following definitions from [3, 5, 6, 7, 9, 11, 12] for ready references.

Definition 3.14. Let (X, τ) be an fts and $A \in I^X$. Then A is called (i) fgp-closed set [3] if $pclA \leq U$ whenever $A \leq U \in \tau$,

(ii) fpg-closed set [3] if $pclA \leq U$ whenever $A \leq U \in FPO(X, \tau)$, (iii) $fg\alpha$-closed set [3] if $\alpha clA \leq U$ whenever $A \leq U \in \tau$, (iv) $f\alpha g$-closed set [3] if $\alpha clA \leq U$ whenever $A \leq U \in F\alpha O(X, \tau)$, (v) $fg\beta$-closed set [7] if $\beta clA \leq U$ whenever $A \leq U \in \tau$, (vi) $f\beta g$-closed set [7] if $\beta clA \leq U$ whenever $A \leq U \in F\beta O(X, \tau)$, (vii) fgs-closed set [3] if $sclA \leq U$ whenever $A \leq U \in \tau$,

(viii) fsg-closed set [3] if $sclA \leq U$ whenever $A \leq U \in FSO(X, \tau)$, (ix) fgs^*-closed set [5] if $clA \leq U$ whenever $A \leq U \in FSO(X, \tau)$, (x) fs^*g-closed set [6] if $clA \leq U$ whenever $A \leq U$ and U is fg-open set in X, (xi) $fswg$-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in FSO(X, \tau)$, (xii) $frwg$-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in FRO(X, \tau)$, (xiii) $f\pi g$-closed set [9] if $clA \leq U$ whenever $A \leq U$ where $U \in F\pi O(X)$, (xiv) fwg-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in \tau$, (xv) $fg\gamma$-closed set [10] if $\gamma clA \leq U$ whenever $A \leq U \in \tau$, (xvi) $fg\gamma^*$-closed set [11] if $\gamma clA \leq U$ whenever $A \leq U \in FSO(X, \tau)$, (xvii) $fgpr$-closed set [9] if $pclA \leq U$ whenever $A \leq U \in FRO(X)$.

Remark 3.15. From above discussion we can conclude that

(i) every fmg-closed set is fgp-closed, $fgpr$-closed, $fg\alpha$-closed, $fg\beta$-closed, $fg\gamma$-closed, $fg\gamma^*$-closed, fwg-closed, $frwg$-closed, (ii) every fs^*g-closed set is fmg-closed, (iii) the concept of fg-closed set, $f\pi g$-closed set, fpg-closed set, $f\alpha g$-closed set, $f\beta g$-closed set, fgs-closed set, fsg-closed set, fgs^*-closed set, $fswg$-closed set are independent of the concept of fmg-closed set.

Example 3.16. There exists an fmg-closed set which is not fg-closed, fgs-closed, fsg-closed, fgs^*-closed, fs^*g-closed

Consider Example 3.4. Here B is fmg-closed set in X. Now $B \leq A \in \tau$ (also A is fg-open set in $X, A \in FSO(X)$). But $clB = sclB = 1_X \not\leq A \Rightarrow B$ is not fg-closed set, fgs^*-closed set, fs^*g-closed set, fgs-closed set, fsg-closed set.

Example 3.17. There exists an fmg-closed set which is not $f\pi g$-closed set Consider Example 3.5 and the fuzzy set S. Here S is fmg-closed set in X. Now $S < C \in F\pi O(X)$, but $clS = 1_X \setminus B \not\leq C \Rightarrow S$ is not $f\pi g$-closed set in X.

Example 3.18. There exists an fmg-closed set which is none of $f\beta g$-closed, $fswg$-closed, fpg-closed, $f\alpha g$-closed

Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.7$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = 0.6, B(b) = 0.7$. Then $B \in F\beta O(X), B \in FSO(X)$, $B \leq B$. But $B \notin F\beta C(X)$ and so $\beta clB \nleq B \Rightarrow B$ is not $f\beta g$-closed set in X. Again $clintB = 1_X \nleq B \Rightarrow B$ is not $fswg$-closed set in X. Now fg-open sets in X is $\{0_X, 1_X, U\}$ where $U \ngeq 1_X \setminus A$. Here $B \geq 1_X \setminus A$ and so 1_X is the only fg-open set in X containing B which implies that B is fmg-closed set in X. Again as $intclB = 1_X \geq B \Rightarrow B \in FPO(X)$ and so $B \leq B \in FPO(X)$. But as $B \notin FPC(X)$, $pclB \nleq B \Rightarrow B$ is not fpg-closed set in X. Also $B \in F\alpha O(X)$ and so $B \leq B \in F\alpha O(X)$. But as $B \notin F\alpha C(X)$, $\alpha clB \nleq B \Rightarrow B$ is not an $f\alpha g$-closed set in X.
Example 3.19. There exists a set which is fg-closed, $f\pi g$-closed, $fgpr$-closed, fwg-closed, $frwg$-closed, $fg\gamma$-closed but it is not an fmg-closed set Consider Example 3.4 and the fuzzy set D. Here D is not fmg-closed set in X. As $1_X \in \tau$ (resp., $1_X \in FRO(X), 1_X \in F\pi O(X)$) only containing D, D is fg-closed, $fgpr$-closed, $f\pi g$-closed, fwg-closed, $frwg$-closed and $fg\gamma$-closed.
Example 3.20. There exists a set which is $fg\beta$-closed, $f\beta g$-closed, fgp-closed, $fg\alpha$-closed, $f\alpha g$-closed, fgs-closed, fsg-closed, $fg\gamma^*$-closed but it is not an fmg-closed set Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here fg-open sets in X is $\{0_X, 1_X, U\}$ where $U \ngeq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $B \leq B$ which is fg-open set in X. But $clintB = 1_X \setminus A \nleq B \Rightarrow B$ is not fmg-closed set in X. Now $B \in F\gamma C(X), B \in F\beta C(X)$, $B \in FSC(X)$ and so B is $fg\gamma^*$-closed set, $fg\beta$-closed set, $f\beta g$-closed set, fsg-closed set, fgs-closed set in X. Again $1_X \in \tau$ (resp., $1_X \in F\alpha O(X)$, $1_X \in FPO(X)$) only containing B and so B is $fg\alpha$-closed set, $f\alpha g$-closed set, fgp-closed set.
Example 3.21. There exists a set which is $fswg$-closed as well as fgs^*-closed but it is not an fmg-closed set Consider Example 3.5 and the fuzzy set U defined by $U(a) = 0.35, U(b) = 0.6$. Then $U \leq U$ which is fg-open set in X. But $clintU = 1_X \setminus C \nleq U \Rightarrow U$ is not fmg-closed set in X. But $FSO(X) = \{0_X, 1_X, V, W\}$ where $A \leq V \leq 1_X \setminus C$, $C \leq W \leq 1_X \setminus B$. Then $U < 1_X \setminus B \in FSO(X)$ and $clU = 1_X \setminus B \leq 1_X \setminus B \Rightarrow U$ is fgs^*-closed set in X. Again $clintU = 1_X \setminus C < 1_X \setminus B \Rightarrow U$ is $fswg$-closed set in X.
Example 3.22. There exists a set which is fpg-closed but it is not an fmg-closed set Let $X = \{a\}$, $\tau = \{0_X, 1_X, B, C\}$ where $B(a) =$

$0.4, C(a) = 0.45$. Then (X, τ) is an fts. Consider the fuzzy set A defined by $A(a) = 0.5$. Then $A \leq A$ which is fg-open set in X. But $clintA = 1_X \setminus C \not\leq A \Rightarrow A$ is not an fmg-closed set in X. But $A < U \in FPO(X)$ where $U(a) > 0.6$. Then $pclA = 1_X \setminus C < U \Rightarrow A$ is fpg-closed set in X.

Definition 3.23. An fts (X, τ) is called fmT_g-space if every fmg-closed set in X is fuzzy closed set in X.

Now we recall the definitions of some spaces from [3, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16] in which the reverse implications in Remark 3.16 hold.

Definition 3.24. An fts (X, τ) is said to be (i) $f\beta T_b$-space [8] if every $f\beta g$-closed set in X is fuzzy closed set in X, (ii) fT_β-space [8] if every $fg\beta$-closed set in X is fuzzy closed set in X, (iii) fT_α-space [3] if every $fg\alpha$-closed set in X is fuzzy closed set in X, (iv) $f\alpha T_b$-space [3] if every $f\alpha g$-closed set in X is fuzzy closed set in X, (v) fT_b-space [3] if every fgs-closed set in X is fuzzy closed set in X, (vi) fT_{sg}-space [3] if every fsg-closed set in X is fuzzy closed set in X, (vii) fT_γ-space [11] if every $fg\gamma$-closed set in X is fuzzy closed set in X, (viii) fT_{γ^*}-space [12] if every $fg\gamma^*$-closed set in X is fuzzy closed set in X, (ix) frT_g-space [16] if every $frwg$-closed set in X is fuzzy closed set in X, (x) fsT_g-space [15] if every $fswg$-closed set in X is fuzzy closed set in X, (xi) fT_p-space [3] if every fgp-closed set in X is fuzzy closed set in X, (xii) fpT_b-space [3] if every fpg-closed set in X is fuzzy closed set in X, (xiii) fT_{pr}-space [10] if every $fgpr$-closed set in X is fuzzy closed set in X, (xiv) fT_w-space [14] if every fwg-closed set in X is fuzzy closed set in X, (xv) fT_π-space [13] if every $f\pi g$-closed set in X is fuzzy closed set in X, (xvi) fgT_{s^*}-space [5] if every fgs^*-closed set in X is fuzzy closed set in X, (xvii) fT_g-space [3] if every fg-closed set in X is fuzzy closed set in X.

Note 3.25. (i) In fmT_g-space, fmg-closed set is fg-closed set, $f\pi g$-closed set, fpg-closed set, $f\alpha g$-closed set, $f\beta g$-closed set, fsg-closed set, fgs^*-closed set, $fswg$-closed set, fs^*g-closed set, (ii) In fT_g-space (resp., fT_β-space, $f\beta T_b$-space, fT_α-space, $f\alpha T_b$-space, fT_b-space, fT_{sg}-space, fgT_{s^*}-space, fT_p-space, fpT_b-space, fT_γ, fT_{γ^*}-space, frT_g-space, fsT_g-space, fT_w-space, fT_π-space, fT_{pr}-space), fg-closed set (resp., $fg\beta$-closed set, $f\beta g$-closed set, $fg\alpha$-closed set, $f\alpha g$-closed set, fgs-closed set, fsg-closed set, fgs^*-closed set, fgp-closed set, fpg-closed set, $fg\gamma$-closed set, $fg\gamma^*$-closed set, $frwg$-closed set,

$fswg$-closed set, fwg-closed set, $f\pi g$-closed set, $fgpr$-closed set) is fmg-closed set.

Now we introduce a new type of generalized version of neighbourhood system in an fts.

Definition 3.26. Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called fmg-neighbourhood (fmg-nbd, for short) of x_α, if there exists an fmg-open set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is fmg-open set in X, then A is called an fmg-open nbd of x_α.

Definition 3.27. Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called fmg-quasi neighbourhood (fmg-q-nbd, for short) of x_α if there is an fmg-open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fmg-open set in X, then A is called an fmg-open q-nbd of x_α.

Note 3.28. (i) It is clear from definitions that every fmg-open set is an fmg-open nbd of each of its points. But it is possible to have an fmg-nbd of x_α that is not an fmg-open set containing x_α , as follows from the next example. (ii) Also every fuzzy open nbd (resp., fuzzy open q-nbd) of a fuzzy point x_α is an fmg-open nbd (resp., fmg-open q-nbd) of x_α. But the converses are not necessarily true, in general, as it seen from the following example.

Example 3.29. Consider Example 3.5 and the fuzzy set $1_X \setminus M$ and the fuzzy point $a_{0.4}$. Then $a_{0.4} \in 1_X \setminus S \leq 1_X \setminus M$ where $1_X \setminus S$ is an fmg-open set in X. So $1_X \setminus M$ is an fmg-nbd of $a_{0.4}$, but as $1_X \setminus M$ is not an fmg-open set in X, $1_X \setminus M$ is not an fmg-open nbd of $a_{0.4}$. Again consider the fuzzy point $b_{0.5}$. Then $b_{0.5} q (1_X \setminus S) \leq 1_X \setminus M \Rightarrow 1_X \setminus M$ is an fmg-q-nbd of $b_{0.5}$, though $1_X \setminus M$ is not an fmg-open q-nbd of $b_{0.5}$. Again $b_{0.5} \not{q} U \leq 1_X \setminus M$ where U is any fuzzy open set in X, therefore $1_X \setminus M$ is not a fuzzy q-nbd and hence fuzzy open q-nbd of $b_{0.5}$. Again consider the fuzzy point $b_{0.6}$. Then $b_{0.6} \in 1_X \setminus S \leq 1_X \setminus M \Rightarrow 1_X \setminus M$ is an fmg-nbd of $b_{0.6}$. But there does not exist any fuzzy open set U in X such that $b_{0.6} \in U \leq 1_X \setminus M$. Hence $1_X \setminus M$ is not a fuzzy nbd of $b_{0.6}$.

Theorem 3.30. Let (X, τ) be an fts and x_α, a fuzzy point in X. If $F (\in I^X)$ is an fmg-closed set in X with $x_\alpha \in 1_X \setminus F$, then there exists an fmg-open nbd G of x_α in X such that $G \not{q} F$.

Proof. By hypothesis, $1_X \setminus F$ being an fmg-open set in X is an fmg-open nbd of x_α. So there exists an fmg-open set G in X such that $x_\alpha \in G \leq 1_X \setminus F \Rightarrow G \not{q} F$.

4. fmg-Closure Operator, fmg-open Function and fmg-Closed Function

Using fmg-closed set as a basic tool, here we introduce and study fmg-closure operator which is an idempotent operator. Afterwards, fmg-open function and fmg-closed function are introduced and characterized by fmg-closure operator.

Definition 4.1. Let (X, τ) be an fts and $A \in I^X$. Then fmg-closure and fmg-interior of A, denoted by $fmgcl(A)$ and $fmgint(A)$, are defined as follows: $fmgcl(A) = \bigwedge\{F : A \leq F, F$ is fmg-closed set in $X\}$,

$fmgint(A) = \bigvee\{G : G \leq A, G$ is fmg-open set in $X\}$.

Remark 4.2. It is clear from definition that for any $A \in I^X$, $A \leq fmgcl(A) \leq clA$. If A is fmg-closed set in an fts X, then $A = fmgcl(A)$. Similarly, $intA \leq fmgint(A) \leq A$. If A is fmg-open set in an fts X, then $A = fmgint(A)$. It follows from Remark 3.3 that $fmgcl(A)$ (resp., $fmgint(A)$) may not be fmg-closed (resp., fmg-open) set in an fts X.

Theorem 4.3. Let (X, τ) be an fts and $A \in I^X$. Then for a fuzzy point x_t in X, $x_t \in fmgcl(A)$ for every fmg-open q-nbd U of x_t, we have UqA.

Proof. Let $x_t \in fmgcl(A)$ for any fuzzy set A in an fts X and F be any fmg-open q-nbd of x_t. Now $x_t qF$ implies $x_t \not\in 1_X \setminus F$, and $1_X \setminus F$ is fmg-closed set in X. Then by Definition 4.1, $A \not\leq 1_X \setminus F$. Hence there exists $y \in X$ such that $A(y) > 1 - F(y) \Rightarrow AqF$.

Conversely, let for every fmg-open q-nbd F of x_t, FqA. If possible, let $x_t \not\in fmgcl(A)$. Then by Definition 4.1, there exists an fmg-closed set U in X with $A \leq U$, $x_t \not\in U$. Then $x_t q(1_X \setminus U)$ which being fmg-open set in X is fmg-open q-nbd of x_t. By assumption, $(1_X \setminus U)qA \Rightarrow (1_X \setminus A)qA$, a contradiction.

Theorem 4.4. Let (X, τ) be an fts and $A, B \in I^X$. Then the following statements are true: (i) $fmgcl(0_X) = 0_X$, (ii) $fmgcl(1_X) = 1_X$, (iii) $A \leq B \Rightarrow fmgcl(A) \leq fmgcl(B)$, (iv) $fmgcl(A \bigvee B) = fmgcl(A) \bigvee fmgcl(B)$, (v) $fmgcl(A \wedge B) \leq fmgcl(A) \wedge fmgcl(B)$

(equality does not hold, in general as follows from Example 3.5), (vi) $fmgcl(fmgcl(A)) = fmgcl(A)$.

Proof. (i), (ii) and (iii) are obvious. (iv) From (iii), $fmgcl(A) \bigvee fmgcl(B) \leq fmgcl(A \bigvee B)$. To prove the converse, let $x_\alpha \in fmgcl(A \bigvee B)$. Then by Theorem 4.3, for any fmg-open set U in X with $x_\alpha q U$, $Uq(A \bigvee B) \Rightarrow$ there exists $y \in X$ such that $U(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $U(y) + A(y) > 1$ or $U(y) + B(y) > 1 \Rightarrow$ either UqA or $UqB \Rightarrow$ either $x_\alpha \in fmgcl(A)$ or $x_\alpha \in fmgcl(B) \Rightarrow x_\alpha \in fmgcl(A) \bigvee fmgcl(B)$.

(v) Follows from (iii). (vi) As $A \leq fmgcl(A)$, for any $A \in I^X$, $fmgcl(A) \leq fmgcl(fmgcl(A))$ (by (iii)). Conversely, let $x_\alpha \in fmgcl(fmgcl(A)) = fmgcl(B)$ where $B = fmgcl(A)$. Let U be any fmg-open set in X with $x_\alpha q U$. Then UqB implies that there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = t$. Then $y_t q U$ and $y_t \in B = fmgcl(A) \Rightarrow UqA \Rightarrow x_\alpha \in fmgcl(A) \Rightarrow fmgcl(fmgcl(A)) \leq fmgcl(A)$. Consequently, $fmgcl(fmgcl(A)) = fmgcl(A)$.

Theorem 4.5. Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold: (i) $fmgcl(1_X \setminus A) = 1_X \setminus fmgint(A)$ (ii) $fmgint(1_X \setminus A) = 1_X \setminus fmgcl(A)$.

Proof (i). Let $x_t \in fmgcl(1_X \setminus A)$ for a fuzzy set A in an fts (X, τ). If possible, let $x_t \notin 1_X \setminus fmgint(A)$. Then $1 - (fmgint(A))(x) < t \Rightarrow [fmgint(A)](x) + t > 1 \Rightarrow fmgint(A) q x_t \Rightarrow$ there exists at least one fmg-open set $F \leq A$ with $x_t q F \Rightarrow x_t q A$. As $x_t \in fmgcl(1_X \setminus A), Fq(1_X \setminus A) \Rightarrow Aq(1_X \setminus A)$, a contradiction. Hence

$$fmgcl(1_X \setminus A) \leq 1_X \setminus fmgint(A)...(1)$$

Conversely, let $x_t \in 1_X \setminus fmgint(A)$. Then $1 - [(fmgint(A)](x) \geq t \Rightarrow x_t \not{q}\, (fmgint(A))$, hence

$$x_t \not{q}\, F \text{ for every } fmg \text{ -open set } F \text{ contained in } A...(2).$$

Let U be any fmg-closed set in X such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is fmg-open set in X contained in A. By (2), $x_t \not{q}\, (1_X \setminus U) \Rightarrow x_t \in U \Rightarrow x_t \in fmgcl(1_X \setminus A)$ and so

$$1_X \setminus fmgint(A) \leq fmgcl(1_X \setminus A)...(3).$$

Combining (1) and (3), (i) follows. (ii) Putting $1_X \setminus A$ for A in (i), we get $fmgcl(A) = 1_X \setminus fmgint(1_X \setminus A) \Rightarrow fmgint(1_X \setminus A) = 1_X \setminus fmgcl(A)$.

Let us now recall the following definition from [29] for ready references.

Definition 4.6 [29]. A function $f : X \to Y$ is called fuzzy open (resp., fuzzy closed) if $f(U)$ is fuzzy open (resp., fuzzy closed) set in Y for every fuzzy open (resp., fuzzy closed) set U in X.

Let us now introduce the following concept.

Definition 4.7. A function $h : X \to Y$ is called fmg-open function if $h(U)$ is fmg-open set in Y for every fuzzy open set U in X.

Remark 4.8. Since fuzzy open set is fmg-open set, we say that fuzzy open function is fmg-open function. But the converse need not be true, as it seen from the following example.

Example 4.9. There exists an fmg-open function which is not a fuzzy open function Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_2) is fmg-open set in (X, τ_2), clearly i is fmg-open function. But $A \in \tau_1$, $i(A) = A \notin \tau_2 \Rightarrow i$ is not a fuzzy open function.

Theorem 4.10. For a bijective function $h : X \to Y$, the following statements are equivalent: (i) h is fmg-open, (ii) $h(intA) \leq fmgint(h(A))$, for all $A \in I^X$, (iii) for each fuzzy point x_α in X and each fuzzy open set U in X containing x_α, there exists an fmg-open set V in Y containing $h(x_\alpha)$ such that $V \leq h(U)$.

Proof (i) $\Rightarrow$ (ii). Let $A \in I^X$. Then $intA$ is a fuzzy open set in X. By (i), $h(intA)$ is fmg-open set in Y. Since $h(intA) \leq h(A)$ and $fmgint(h(A))$ is the union of all fmg-open sets contained in $h(A)$, we have $h(intA) \leq fmgint(h(A))$. (ii) $\Rightarrow$ (i). Let U be any fuzzy open set in X. Then $h(U) = h(intU) \leq fmgint(h(U))$ (by (ii)) $\Rightarrow h(U)$ is fmg-open set in $Y \Rightarrow h$ is fmg-open function. (ii) $\Rightarrow$ (iii). Let x_α be a fuzzy point in X, and U, a fuzzy open set in X such that $x_\alpha \in U$. Then $h(x_\alpha) \in h(U) = h(intU) \leq fmgint(h(U))$ (by (ii)). Then $h(U)$ is fmg-open set in Y. Let $V = h(U)$. Then $h(x_\alpha) \in V$ and $V \leq h(U)$. (iii) $\Rightarrow$ (i). Let U be any fuzzy open set in X and y_α, any fuzzy point in $h(U)$, i.e., $y_\alpha \in h(U)$. Then there exists unique $x \in X$ such that $h(x) = y$ (as h is bijective). Then $[h(U)](y) \geq \alpha \Rightarrow U(h^{-1}(y)) \geq \alpha \Rightarrow U(x) \geq \alpha \Rightarrow x_\alpha \in U$. By (iii), there exists fmg-open set V in Y such that $h(x_\alpha) \in V$ and $V \leq h(U)$. Then $h(x_\alpha) \in V = fmgint(V) \leq fmgint(h(U))$. Since y_α is taken arbitrarily and $h(U)$ is the union of all fuzzy points in

$h(U)$, $h(U) \leq fmgint(f(U)) \Rightarrow h(U)$ is fmg-open set in $Y \Rightarrow h$ is an fmg-open function.

Theorem 4.11. If $h : X \to Y$ is fmg-open, bijective function, then the following statements are true: (i) for each fuzzy point x_α in X and each fuzzy open q-nbd U of x_α in X, there exists an fmg-open q-nbd V of $h(x_\alpha)$ in Y such that $V \leq h(U)$, (ii) $h^{-1}(fmgcl(B)) \leq cl(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_α be a fuzzy point in X and U be any fuzzy open q-nbd of x_α in X. Then $x_\alpha q U = intU \Rightarrow h(x_\alpha) q h(intU) \leq fmgint(h(U))$ (by Theorem 4.10 (i)$\Rightarrow$(ii)) implies that there exists at least one fmg-open q-nbd V of $h(x_\alpha)$ in Y with $V \leq h(U)$. (ii) Let x_α be any fuzzy point in X such that $x_\alpha \notin cl(h^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open q-nbd U of x_α in X such that $U \not{q} h^{-1}(B)$. Now

$$h(x_\alpha) q h(U) ...(1)$$

where $h(U)$ is fmg-open set in Y. Now $h^{-1}(B) \leq 1_X \setminus U$ which is a fuzzy closed set in $X \Rightarrow B \leq h(1_X \setminus U)$ (as h is injective) $\leq 1_Y \setminus h(U) \Rightarrow B \not{q} h(U)$. Let $V = 1_Y \setminus h(U)$. Then $B \leq V$ which is fmg-closed set in Y. We claim that $h(x_\alpha) \notin V$. If possible, let $h(x_\alpha) \in V = 1_Y \setminus h(U)$. Then $1 - [h(U)](h(x)) \geq \alpha \Rightarrow h(U) \not{q} h(x_\alpha)$, contradicting (1). So $h(x_\alpha) \notin V \Rightarrow h(x_\alpha) \notin fmgcl(B) \Rightarrow x_\alpha \notin h^{-1}(fmgcl(B)) \Rightarrow h^{-1}(fmgcl(B)) \leq cl(h^{-1}(B))$.

Theorem 4.12. An injective function $h : X \to Y$ is fmg-open if and only if for each $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$, there exists an fmg-closed set V in Y such that $B \leq V$ and $h^{-1}(V) \leq F$.

Proof. Let $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$. Then $1_X \setminus h^{-1}(B) \geq 1_X \setminus F$ where $1_X \setminus F$ is a fuzzy open set in $X \Rightarrow h(1_X \setminus F) \leq h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus B$ (as h is injective) where $h(1_X \setminus F)$ is an fmg-open set in Y. Let $V = 1_Y \setminus h(1_X \setminus F)$. Then V is fmg-closed set in Y such that $B \leq V$. Now $h^{-1}(V) = h^{-1}(1_Y \setminus h(1_X \setminus F)) = 1_X \setminus h^{-1}(h(1_X \setminus F)) \leq F$.

Conversely, let F be a fuzzy open set in X. Then $1_X \setminus F$ is a fuzzy closed set in X. We have to show that $h(F)$ is an fmg-open set in Y. Now $h^{-1}(1_Y \setminus h(F)) \leq 1_X \setminus F$ (as h is injective). By assumption, there exists an fmg-closed set V in Y such that

$$1_Y \setminus h(F) \leq V ...(1)$$

and $h^{-1}(V) \leq 1_X \setminus F$. Therefore, $F \leq 1_X \setminus h^{-1}(V)$ implies that

$$h(F) \leq h(1_X \setminus h^{-1}(V)) \leq 1_Y \setminus V...(2)$$

(as h is injective). Combining (1) and (2), $h(F) = 1_Y \setminus V$ which is an fmg-open set in Y. Hence h is fmg-open function.

Definition 4.13. A function $h : X \to Y$ is called fmg-closed function if $h(A)$ is fmg-closed set in Y for each fuzzy closed set A in X.

Remark 4.14. Since fuzzy closed set is fmg-closed set in an fts, we can conclude that every fuzzy closed function is fmg-closed function, but the converse may not be true as it follows from Example 4.9. Here $1_X \setminus A \in \tau_1^c$, but $i(1_X \setminus A) = 1_X \setminus A \notin \tau_2^c \Rightarrow i$ is not a fuzzy closed function. But since every fuzzy set in (X, τ_2) is fmg-closed set in (X, τ_2), clearly i is fmg-closed function.

Theorem 4.15. A bijective function $h : X \to Y$ is fmg-closed function if and only if $fmgcl(h(A)) \leq h(clA)$, for all $A \in I^X$.

Proof. Let us suppose that $h : X \to Y$ be an fmg-closed function and $A \in I^X$. Then $h(cl(A))$ is fmg-closed set in Y. Since $h(A) \leq h(clA)$ and $fmgcl(h(A))$ is the intersection of all fmg-closed sets in Y containing $h(A)$, we have $fmgcl(h(A)) \leq h(clA)$.

Conversely, let for any $A \in I^X$, $fmgcl(h(A)) \leq h(clA)$. Let U be any fuzzy closed set in X. Then $h(U) = h(clU) \geq fmgcl(h(U)) \Rightarrow h(U)$ is an fmg-closed set in $Y \Rightarrow h$ is an fmg-closed function.

Theorem 4.16. If $h : X \to Y$ is an fmg-closed bijective function, then the following statements hold: (i) for each fuzzy point x_α in X and each fuzzy closed set U in X with $x_\alpha \not q U$, there exists an fmg-closed set V in Y with $h(x_\alpha) \not q V$ such that $V \geq h(U)$, (ii) $h^{-1}(fmgint(B)) \geq int(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_α be a fuzzy point in X and U be any fuzzy closed set in X with $x_\alpha \not q U = clU \Rightarrow h(x_\alpha) \not q h(clU) \geq fmgcl(h(U))$ (by Theorem 4.15) $\Rightarrow h(x_\alpha) \not q V$ for some fmg-closed set V in Y with $V \geq h(U)$. (ii). Let $B \in I^Y$ and x_α be any fuzzy point in X such that $x_\alpha \in int(h^{-1}(B))$. Then there exists a fuzzy open set U in X with $U \leq h^{-1}(B)$ such that $x_\alpha \in U$. Then $1_X \setminus U \geq 1_X \setminus h^{-1}(B) \Rightarrow h(1_X \setminus U) \geq h(1_X \setminus h^{-1}(B))$ where $h(1_X \setminus U)$ is an fmg-closed set in Y. Let $V = 1_Y \setminus h(1_X \setminus U)$. Then V is an fmg-open set in Y and $V = 1_Y \setminus h(1_X \setminus U) \leq 1_Y \setminus h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus (1_Y \setminus B) = B$ (as h is injective). Now $U(x) \geq \alpha \Rightarrow x_\alpha \not q (1_X \setminus U) \Rightarrow h(x_\alpha) \not q h(1_X \setminus U)$ $\Rightarrow h(x_\alpha) \leq 1_Y \setminus h(1_X \setminus U) = V \Rightarrow h(x_\alpha) \in V = fmgint(V) \leq$

$fmgint(B) \Rightarrow x_\alpha \in h^{-1}(fmgint(B))$. Since x_α is taken arbitrarily, $int(h^{-1}(B)) \leq h^{-1}(fmgint(B))$, for all $B \in I^Y$.

Remark 4.17. Composition of two fmg-closed (resp., fmg-open) functions need not be so, as it seen from the following example.

Example 4.18. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.5$, $B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fmg-closed functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. We claim that i_3 is not fmg-closed function. Now $1_X \setminus A \in \tau_1^c$. $(i_2 \circ i_1)(1_X \setminus A) = 1_X \setminus A \leq A$ which is fg-open set in (X, τ_3). But $cl_{\tau_3} int_{\tau_3}(1_X \setminus A) = 1_X \setminus B \not\leq A \Rightarrow 1_X \setminus A$ is not fmg-closed set in $(X, \tau_3) \Rightarrow i_2 \circ i_1$ is not fmg-closed function.

Similarly we can show that $i_2 \circ i_1$ is not fmg-open function though i_1 and i_2 are so.

Theorem 4.19. If $h_1 : X \to Y$ is fuzzy closed (resp., fuzzy open) function and $h_2 : Y \to Z$ is fmg-closed (resp., fmg-open) function, then $h_2 \circ h_1 : X \to Z$ is fmg-closed (resp., fmg-open) function.

Proof. Obvious.

Now to establish the mutual relationships of fmg-closed function with the functions defined in [3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16], we have to recall he following definitions first.

Definition 4.20. Let $(X, \tau_1) \to (Y, \tau_2)$ be a function. Then h is called an (i) fg-closed function [3] if $h(A)$ is fg-closed set in Y for every $A \in \tau_1^c$, (ii) $fg\beta$-closed function [7] if $h(A)$ is $fg\beta$-closed set in Y for every $A \in \tau_1^c$, (iii) $f\beta g$-closed function [7] if $h(A)$ is $f\beta g$-closed set in Y for every $A \in \tau_1^c$, (iv) $fg\alpha$-closed function [3] if $h(A)$ is $fg\alpha$-closed set in Y for every $A \in \tau_1^c$, (v) $f\alpha g$-closed function [3] if $h(A)$ is $f\alpha g$-closed set in Y for every $A \in \tau_1^c$, (vi) fgp-closed function [3] if $h(A)$ is fgp-closed set in Y for every $A \in \tau_1^c$, (vii) fpg-closed function [3] if $h(A)$ is fpg-closed set in Y for every $A \in \tau_1^c$, (viii) fgs-closed function [3] if $h(A)$ is fgs-closed set in Y for every $A \in \tau_1^c$, (ix) fsg-closed function [3] if $h(A)$ is fsg-closed set in Y for every $A \in \tau_1^c$, (x) fgs^*-closed function [5] if $h(A)$ is fgs^*-closed set in Y for every $A \in \tau_1^c$, (xi) fs^*g-closed function [6] if $h(A)$ is fs^*g-closed set in Y for every $A \in \tau_1^c$, (xii) $fg\gamma$-closed function [11] if $h(A)$ is $fg\gamma$-closed set in Y for every $A \in \tau_1^c$, (xiii) $fg\gamma^*$-closed function [12] if $h(A)$ is $fg\gamma^*$-closed set in Y for every $A \in \tau_1^c$, (xiv) $fswg$-closed function [15] if $h(A)$ is $fswg$-closed set in Y for every $A \in \tau_1^c$, (xv) $frwg$-closed function [16] if $h(A)$ is $frwg$-closed set in Y for every $A \in \tau_1^c$, (xvi) $f\pi g$-closed function [13] if $h(A)$ is $f\pi g$-closed set in Y for every $A \in \tau_1^c$, (xvii) fwg-closed function [14] if $h(A)$ is fwg-closed set in Y for every $A \in \tau_1^c$, (xviii) $fgpr$-closed function [10] if $h(A)$ is $fgpr$-closed set in Y for every $A \in \tau_1^c$.

Remark 4.21. (i) fs^*g-closed function is fmg-closed function. (ii) fmg-closed function is fgp-closed function, $fgpr$-closed function, $fg\alpha$-closed function, $fg\beta$-closed function, $fg\gamma$-closed function, $fg\gamma^*$-closed function, fwg-closed function, $frwg$-closed function. (iii) fmg-closed function is independent concept of fg-closed function, $f\pi g$-closed function,, fpg-closed function,, $f\alpha g$-closed function,, $f\beta g$-closed function,, fgs-closed function,, fsg-closed function,, fgs^*-closed function, $fswg$-closed function.

Example 4.22. There exists a function which is fmg-closed but it is not an fg-closed, fgs-closed, fsg-closed, fgs^*-closed, fs^*g-closed Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.6, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_1^c, i(1_X \setminus B) = 1_X \setminus B$. Then $cl_{\tau_2} int_{\tau_2}(1_X \setminus B) = 0_X \Rightarrow 1_X \setminus B$ is fmg-closed set in (X, τ_2) and hence i is an fmg-closed function. Now $A \in \tau_2$, $A \in FSO(X, \tau_2)$ and also A is fg-open set in (X, τ_2). Now $1_X \setminus B < A$. But $cl_{\tau_2}(1_X \setminus B) = scl_{\tau_2}(1_X \setminus B) = 1_X \not\leq A$ and so i is not fg-closed function, fgs-closed function, fsg-closed function, fgs^*-closed function, fs^*g-closed function.

Example 4.23. There exists a function which is fmg-closed but it is not an $f\pi g$-closed function Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, S\}$, $\tau_2 = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, S(a) = 0.4, S(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus S \in \tau_1^c$, $i(1_X \setminus S) = 1_X \setminus S$. Now fg-open sets in (X, τ_2) is $\{0_X, 1_X, U, V\}$ where $U(a) \geq 0.6, 0.5 \leq U(b) < 0.6, V \not\geq 1_X \setminus C$. Then $1_X \setminus S < U_1$ where $U_1(a) \geq 0.6, 0.5 \leq U_1(b) < 0.6$, is fg-open set in (X, τ_2). $cl_{\tau_2} int_{\tau_2}(1_X \setminus S) = 1_X \setminus C < U_1 \Rightarrow 1_X \setminus S$ is fmg-closed set in $(X, \tau_2) \Rightarrow i$ is an fmg-closed function. Now $1_X \setminus S < C \in F\pi O(X)$. But $cl_{\tau_2}(1_X \setminus S) = 1_X \setminus B \not\leq C \Rightarrow 1_X \setminus S$ is not an $f\pi g$-closed set in $(X, \tau_2) \Rightarrow i$ is not an $f\pi g$-closed function.

Example 4.24. There exists a function which is fmg-closed but it is not an $f\beta g$-closed, $f\alpha g$-closed, $fswg$-closed, fpg-closed Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.7$, $B(a) = 0.4, B(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_1^c$, $i(1_X \setminus B) = 1_X \setminus B$. Now fg-open sets in (X, τ_2) is $\{0_X, 1_X, U\}$ where $U \ngeq 1_X \setminus A$. Since $1_X \setminus B \geq 1_X \setminus A$, 1_X is the only fg-open set in (X, τ_2) containing $1_X \setminus B$ and so $1_X \setminus B$ is fmg-closed set in $(X, \tau_2) \Rightarrow i$ is an fmg-closed function. Now $(1_X \setminus B) \in F\beta O(X, \tau_2)$ as well as $(1_X \setminus B) \in FSO(X, \tau_2)$ and so $1_X \setminus B \leq 1_X \setminus B$, but $\beta cl(1_X \setminus B) \neq 1_X \setminus B \Rightarrow \beta cl(1_X \setminus B) \nleq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not $f\beta g$-closed set in $(X, \tau_2) \Rightarrow i$ is not an $f\beta g$-closed function. Again $cl_{\tau_2} int_{\tau_2}(1_X \setminus B) = 1_X \nleq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not $fswg$-closed set in $(X, \tau_2) \Rightarrow i$ is not an $fswg$-closed function. Again $1_X \setminus B \in FPO(X, \tau_2)$, but $1_X \setminus B \notin FPC(X, \tau_2)$ and so $1_X \setminus B \leq 1_X \setminus B \in FPO(X, \tau_2)$, but $pcl_{\tau_2}(1_X \setminus B) \nleq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not an fpg-closed set in $(X, \tau_2) \Rightarrow i$ is not an fpg-closed function. Again $(1_X \setminus B) \in F\alpha O(X, \tau_2)$, but $(1_X \setminus B) \notin F\alpha C(X, \tau_2)$ and so $1_X \setminus B \leq 1_X \setminus B \in F\alpha O(X, \tau_2)$, but $\alpha cl(1_X \setminus B) \nleq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not $f\alpha g$-closed set in $(X, \tau_2) \Rightarrow i$ is not an $f\alpha g$-closed function.

Example 4.25. The notion fg-closed function, $f\pi g$-closed function, $fgpr$-closed function, fwg-closed function, $frwg$-closed function, $fg\gamma$-closed function do not imply fmg-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, D\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$, $D(a) = 0.3, D(b) = 0.4$. Then $X, \tau_1)$ and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now$1_X \setminus D \in \tau_1^c$, $i(1_X \setminus D) = 1_X \setminus D$. Now $1_X \setminus D \leq 1_X \setminus D$ which is fg-open set in (X, τ_2). But $cl_{\tau_2} int_{\tau_2}(1_X \setminus D) = 1_X \nleq 1_X \setminus D \Rightarrow 1_X \setminus D$ is not fmg-closed set in $(X, \tau_2) \Rightarrow i$ is not an fmg-closed function. Now $1_X \in \tau_2$ (resp., $1_X \in F\pi O(X, \tau_2)$, $1_X \in FRO(X, \tau_2)$) only containing $1_X \setminus D$, so $1_X \setminus D$ is fg-closed set, $f\pi g$-closed set, $fgpr$-closed set, fwg-closed set, $frwg$-closed set, $fg\gamma$-closed set in $(X, \tau_2) \Rightarrow i$ is fg-closed function, $f\pi g$-closed function, $fgpr$-closed function, fwg-closed function, $frwg$-closed function, $fg\gamma$-closed function.

Example 4.26. There exists a function which is fpg-closed but it is not an fmg-closed function Let $X = \{a\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B, C\}$ where $A(a) = 0.5, B(a) = 0.4, C(a) = 0.45$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function

$i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A) =\leq 1_X \setminus A$ which is fg-open set in (X, τ_2). But $cl_{\tau_2} int_{\tau_2}(1_X \setminus A) = 1_X \setminus C \not\leq 1_X \setminus A \Rightarrow 1_X \setminus A$ is not an fmg-closed set in $(X, \tau_2) \Rightarrow i$ is not an fmg-closed function. Now $1_X \setminus A < U \in FPO(X, \tau_2)$ where $U > 1_X \setminus B$. So $pcl_{\tau_2}(1_X \setminus A) = 1_X \setminus C < U \Rightarrow 1_X \setminus A$ is fpg-closed set in $(X, \tau_2) \Rightarrow i$ is an fpg-closed function.

Example 4.27. There exists a function which is not an fmg-closed function but it is an $fg\beta$-closed, $f\beta g$-closed, fgp-closed, $fg\alpha$-closed, $f\alpha g$-closed, fgs-closed, fsg-closed, $fg\gamma^*$-closed Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$, $B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_1^c$, $i(1_X \setminus B) = 1_X \setminus B \leq 1_X \setminus B$ which is fg-open set in (X, τ_2). But $cl_{\tau_2} int_{\tau_2}(1_X \setminus B) = 1_X \setminus A \not\leq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not an fmg-closed set in $(X, \tau_2) \Rightarrow i$ is not an fmg-closed function. Now $1_X \setminus B \in FSC(X, \tau_2)$, $1_X \setminus B \in F\beta C(X, \tau_2)$, $1_X \setminus B \in F\gamma C(X, \tau_2)$ and $1_X \setminus B \leq 1_X \setminus B$. Consequently, $1_X \setminus B$ is fgs-closed set, fsg-closed set, $fg\beta$-closed set, $f\beta g$-closed set, $fg\gamma^*$-closed set in $(X, \tau_2) \Rightarrow i$ is an fgs-closed function, fsg-closed function, $fg\beta$-closed function, $f\beta g$-closed function, $fg\gamma^*$-closed function. Again 1_X is the only fuzzy open set as well as fuzzy α-open set in (X, τ_2) containing $1_X \setminus B \Rightarrow 1_X \setminus B$ is fgp-closed set, $fg\alpha$-closed set, $f\alpha g$-closed set in $(X, \tau_2) \Rightarrow i$ is an fgp-closed function, $fg\alpha$-closed function, $f\alpha g$-closed function.

Example 4.28. There exists a function which is not an fmg-closed function but it is an $fswg$-closed, fgs^*-closed. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, D\}$, $\tau_2 = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.3, A(b) = 0.4$, $B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.65, D(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus D \in \tau_1^c$, $i(1_X \setminus D) = 1_X \setminus D \leq 1_X \setminus D$ which is an fg-open set in (X, τ_2). But $cl_{\tau_2} int_{\tau_2}(1_X \setminus D) = 1_X \setminus C \not\leq 1_X \setminus D \Rightarrow 1_X \setminus D$ is not an fmg-closed set in $(X, \tau_2) \Rightarrow i$ is not an fmg-closed function. Now $FSO(X, \tau_2) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus C, C \leq V \leq 1_X \setminus B$. Then $1_X \setminus D < 1_X \setminus B \in FSO(X \tau_2)$. Then $cl_{\tau_2}(1_X \setminus D) = 1_X \setminus B \leq 1_X \setminus B \Rightarrow 1_X \setminus D$ is fgs^*-closed set in $(X, \tau_2) \Rightarrow i$ is an fgs^*-closed function. Again $cl_{\tau_2} int_{\tau_2}(1_X \setminus D) = 1_X \setminus C < 1_X \setminus B \Rightarrow 1_X \setminus D$ is an $fswg$-closed set in $(X, \tau_2) \Rightarrow i$ is an $fswg$-closed function.

Remark 4.29. (i) Let $h : X \to Y$ be a function where Y is an fT_g-space (resp., fT_β-space, $f\beta T_b$-space, fT_α-space, $f\alpha T_b$-space, fT_b-space, fT_{sg}-space, fgT_{s^*}-space, fT_p-space, fpT_b-space, fT_γ-space, fT_{γ^*}-space, frT_g-space, fsT_g-space, fT_w-space, fT_π-space, fT_{pr}-space). If h is an fg-closed function (resp., $fg\beta$-closed function, $f\beta g$-closed function, $fg\alpha$-closed function, $f\alpha g$-closed function, fgs-closed function, fsg-closed function, fgs^*-closed function, fgp-closed function, fpg-closed function, $fg\gamma$-closed function, $fg\gamma^*$-closed function, $frwg$-closed function, $fswg$-closed function, fwg-closed function, $f\pi g$-closed function, $fgpr$-closed function), then h is an fmg-closed function. (ii) Let $h : X \to Y$ be a function where Y is an fmT_g-space. If h is an fmg-closed function, then h is an fg-closed function, fgs^*-closed function, fs^*g-closed function, $fswg$-closed function, fsg-closed function, $f\beta g$-closed function, $f\alpha g$-closed function, $f\pi g$-closed function, fpg-closed function.

5. fmg-Regular, fmg-Normal and fmg-Compact Spaces

In this section new types of generalized versions of fuzzy regularity, fuzzy normality and fuzzy compactness are introduced and studied. It is also shown that these three concepts are weak concepts of fuzzy regularity [24], fuzzy normality [23] and fuzzy compactness [18] respectively.

Definition 5.1. An fts (X, τ) is said to be fmg-regular space if for any fuzzy point x_t in X and each fmg-closed set F in X with $x_t \notin F$, there exist $U, V \in \tau$ such that $x_t \in U, F \leq V$ and $U \not{q} V$.

Theorem 5.2. In an fts (X, τ), the following statements are equivalent: (i) X is fmg-regular, (ii) for each fuzzy point x_t in X and any fmg-open q-nbd U of x_t, there exists $V \in \tau$ such that $x_t \in V$ and $clV \leq U$,

(iii) for each fuzzy point x_t in X and each fmg-closed set A of X with $x_t \notin A$, there exists $U \in \tau$ with $x_t \in U$ such that $clU \not{q} A$.

Proof (i) $\Rightarrow$ (ii). Let x_t be a fuzzy point in X and U, any fmg-open q-nbd of x_t. Then $x_t q U \Rightarrow U(x) + t > 1 \Rightarrow x_t \notin 1_X \setminus U$ which is an fmg-closed set in X. By (i), there exist $V, W \in \tau$ such that $x_t \in V, 1_X \setminus U \leq W$ and $V \not{q} W$. Then $V \leq 1_X \setminus W \Rightarrow clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$. (ii) $\Rightarrow$ (iii). Let x_t be a fuzzy point in X and A, an fmg-closed set in X with $x_t \notin A$. Then $A(x) < t \Rightarrow x_t q(1_X \setminus A)$ which being fmg-open set in X is fmg-open q-nbd of x_t. So by (ii),

there exists $V \in \tau$ such that $x_t \in V$ and $clV \leq 1_X \setminus A$. Then $clV \not{q} A$.
(iii) $\Rightarrow$ (i). Let x_t be a fuzzy point in X and F be any fmg-closed set in X with $x_t \notin F$. Then by (iii), there exists $U \in \tau$ such that $x_t \in U$ and $clU \not{q} F$. Then $F \leq 1_X \setminus clU$ (=V, say). So $V \in \tau$ and $V \not{q} U$ as $U \not{q} (1_X \setminus clU)$. Consequently, X is fmg-regular space.

Definition 5.3. An fts (X, τ) is called fmg-normal space if for each pair of fmg-closed sets A, B in X with $A \not{q} B$, there exist $U, V \in \tau$ such that $A \leq U, B \leq V$ and $U \not{q} V$.

Theorem 5.4. An fts (X, τ) is fmg-normal space if and only if for every fmg-closed set F and fmg-open set G in X with $F \leq G$, there exists $H \in \tau$ such that $F \leq H \leq clH \leq G$.

Proof. Let X be fmg-normal space and let F be fmg-closed set and G be fmg-open set in X with $F \leq G$. Then $F \not{q} (1_X \setminus G)$ where $1_X \setminus G$ is fmg-closed set in X. By hypothesis, there exist $H, T \in \tau$ such that $F \leq H, 1_X \setminus G \leq T$ and $H \not{q} T$. Then $H \leq 1_X \setminus T \leq G$. Therefore, $F \leq H \leq clH \leq cl(1_X \setminus T) = 1_X \setminus T \leq G$. Conversely, let A, B be two fmg-closed sets in X with $A \not{q} B$. Then $A \leq 1_X \setminus B$. By hypothesis, there exists $H \in \tau$ such that $A \leq H \leq clH \leq 1_X \setminus B \Rightarrow A \leq H, B \leq 1_X \setminus clH$ (=V, say). Then $V \in \tau$ and so $B \leq V$. Also as $H \not{q} (1_X \setminus clH)$, $H \not{q} V$. Consequently, X is fmg-normal space.

Let us now recall the following definitions from [18, 22] for ready references.

Definition 5.5. Let (X, τ) be an fts and $A \in I^X$. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A if $\bigcup \mathcal{U} \geq A$ [22]. If each member of $\mathcal{U}$ is fuzzy open (resp., fuzzy regular open, fmg-open) in X, then $\mathcal{U}$ is called a fuzzy open [22] (resp., fuzzy regular open [1], fmg-open) cover of A. If, in particular, $A = 1_X$, we get the definition of fuzzy cover of X as $\bigcup \mathcal{U} = 1_X$ [18].

Definition 5.6. Let (X, τ) be an fts and $A \in I^X$. Then a fuzzy cover $\mathcal{U}$ of A (resp., of X) is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$ [22]. If, in particular $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$ [18].

Definition 5.7. Let (X, τ) be an fts and $A \in I^X$. Then A is called fuzzy compact [18] (resp., fuzzy almost compact [19], fuzzy nearly compact [25]) set if every fuzzy open (resp., fuzzy open, fuzzy regular open) cover $\mathcal{U}$ of A has a finite subcollection $\mathcal{U}_0$ such that $\bigcup \mathcal{U}_0 \geq A$ (resp., $\bigcup\limits_{U \in \mathcal{U}_0} clU \geq A$, $\bigcup \mathcal{U}_0 \geq A$). If, in particular, $A = 1_X$, we get the definition of fuzzy compact [18] (resp., fuzzy almost compact [19], fuzzy nearly compact [20]) space as $\bigcup \mathcal{U}_0 = 1_X$ (resp., $\bigcup_{U \in \mathcal{U}_0} clU = 1_X$, $\bigcup \mathcal{U}_0 = 1_X$).

Let us now introduce the following concept.

Definition 5.8. Let (X, τ) be an fts and $A \in I^X$. Then A is called fmg-compact if every fuzzy cover $\mathcal{U}$ of A by fmg-open sets of X has a finite subcover. If, in particular, $A = 1_X$, we get the definition of fmg-compact space X.

Theorem 5.9. Every fmg-closed set in an fmg-compact space X is fmg-compact.

Proof. Let $A(\in I^X)$ be an fmg-closed set in an fmg-compact space X. Let $\mathcal{U}$ be a fuzzy cover of A by fmg-open sets of X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy cover of X by fmg-open sets of X. As X is fmg-compact space, $\mathcal{V}$ has a finite subcollection $\mathcal{V}_0$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcover of A. Hence A is fmg-compact set.

Next we recall the following two definitions from [24, 23] for ready references.

Definition 5.10 [24]. An fts (X, τ) is called fuzzy regular space if for each fuzzy point x_t in X and each fuzzy closed set F in X with $x_t \notin F$, there exist $U, V \in \tau$ such that $x_t \in U$, $F \leq V$ and $U \not{q} V$.

Definition 5.11 [23]. An fts (X, τ) is called fuzzy normal space if for each pair of fuzzy closed sets A, B of X with $A \not{q} B$, there exist $U, V \in \tau$ such that $A \leq U, B \leq V$ and $U \not{q} V$.

Remark 5.10. It is clear from above discussion that (i) fmg-regular (resp., fmg-normal) space is fuzzy regular (resp., fuzzy normal) space, but the converses are not true, in general, follow from the following example. (ii) fmg-compact space is fuzzy compact, fuzzy almost compact, fuzzy nearly compact space, but the converses are not true, in general, follow from the following example. (iii) In fmT_g-space, fuzzy regularity, fuzzy normality and fuzzy compactness imply fmg-regularity, fmg-normality and fmg-compactness.

Example 5.13. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Clearly (X, τ) is fuzzy regular space, fuzzy normal space and fuzzy compact space. Here every fuzzy set is fmg-open as well as fmg-closed set in X. Consider the fuzzy set F defined by $F(a) = 0.4$ and the fuzzy point $a_{0.6}$. Then $a_{0.6} \notin F$. But there does not exist fuzzy open sets U, V in X such that $a_{0.6} \leq U, F \leq V$ and $U \not{q} V$. So (X, τ) is not an fmg-regular space. Again consider two fmg-closed sets A, B defined by $A(a) = 0.4, B(a) = 0.5$. Then A, B are fmg-closed sets in X with $A \not{q} B$. But there does not exist $U, V \in \tau$ such that $A \leq U, B \leq V$ and $U \not{q} V$. Hence (X, τ) is not an fmg-normal space. Now considering fmg-open covering $\mathcal{U} = \{U_n : n \in N\}$ where $U_n(a) = \frac{n}{n+1}$, for all $n \in N$ of X. But $\mathcal{U}$ has no finite subcovering of X. Hence (X, τ) is not an fmg-compact space.

6. fmg-Continuous and fmg-Irresolute Functions

With the help of fmg-closed set as a basic tool, here we introduce and characterize fmg-continuous function the class of which is strictly larger than the class of fuzzy continuous function [18] and then introduce and characterize fmg-irresolute function. It is shown that that the fmg-continuous image of an fmg-regular (resp., fmg-normal, fmg-compact) space is fuzzy regular (resp., fuzzy normal, fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space. Also under fmg-irresolute function, fmg-regularity (resp., fmg-normality and fmg-compactness) remains invariant. Afterwards, the mutual relationship of fmg-continuous function with the functions defined in [3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16] are established.

Now we first introduce the following concept.

Definition 6.1. A function $h : X \to Y$ is said to be fmg-continuous function if $h^{-1}(V)$ is fmg-closed set in X for every fuzzy closed set V in Y.

Theorem 6.2. Let $h : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent: (i) h is fmg-continuous function, (ii) for each fuzzy point x_α in X and each fuzzy open nbd V of $h(x_\alpha)$ in Y , there exists an fmg-open nbd U of x_α in X such that $h(U) \leq V$, (iii) $h(fmgcl(A)) \leq cl(h(A))$, for all $A \in I^X$, (iv) $fmgcl(h^{-1}(B)) \leq h^{-1}(clB)$, for all $B \in I^Y$.

Proof (i) $\Rightarrow$ (ii). Let x_α be a fuzzy point in X and V, any fuzzy open nbd of $h(x_\alpha)$ in Y. Then $x_\alpha \in h^{-1}(V)$ which is fmg-open set in X (by (i)). Let $U = h^{-1}(V)$. Then $h(U) = h(h^{-1}(V)) \leq V$. (ii) $\Rightarrow$ (i). Let A be any fuzzy open set in Y and x_α, a fuzzy point in X such that $x_\alpha \in h^{-1}(A)$. Then $h(x_\alpha) \in A$ where A is a fuzzy open nbd of $h(x_\alpha)$ in Y. By (ii), there exists an fmg-open nbd U of x_α in X such that $h(U) \leq A$. Then $x_\alpha \in U \leq$

$h^{-1}(A) \Rightarrow x_\alpha \in U = fmgint(U) \leq fmgint(h^{-1}(A))$. Since x_α is taken arbitrarily and $h^{-1}(A)$ is the union of all fuzzy points in $h^{-1}(A)$, $h^{-1}(A) \leq fmgint(h^{-1}(A)) \Rightarrow h^{-1}(A)$ is an fmg-open set in $X \Rightarrow h$ is an fmg-continuous function. (i) $\Rightarrow$ (iii). Let $A \in I^X$. Then $cl(h(A))$ is a fuzzy closed set in Y. By (i), $h^{-1}(cl(h(A)))$ is fmg-closed set in X. Now $A \leq h^{-1}(h(A)) \leq h^{-1}(cl(h(A)))$ and so $fmgcl(A) \leq fmgcl(h^{-1}(cl(h(A)))) = h^{-1}(cl(h(A))) \Rightarrow h(fmgcl(A)) \leq cl(h(A))$. (iii) $\Rightarrow$ (i). Let V be a fuzzy closed set in Y. Put $U = h^{-1}(V)$. Then $U \in I^X$. By (iii), $h(fmgcl(U)) \leq cl(h(U)) = cl(h(h^{-1}(V))) \leq clV = V \Rightarrow fmgcl(U) \leq h^{-1}(V) = U \Rightarrow U$ is fmg-closed set in $X \Rightarrow h$ is fmg-continuous function. (iii) $\Rightarrow$ (iv). Let $B \in I^Y$ and $A = h^{-1}(B)$. Then $A \in I^X$. By (iii), $h(fmgcl(A)) \leq cl(h(A)) \Rightarrow h(fmgcl(h^{-1}(B))) \leq cl(h(h^{-1}(B))) \leq clB \Rightarrow fmgcl(h^{-1}(B)) \leq h^{-1}(clB)$. (iv) $\Rightarrow$ (iii). Let $A \in I^X$. Then $h(A) \in I^Y$. By (iv), $fmgcl(h^{-1}(h(A))) \leq h^{-1}(cl(h(A))) \Rightarrow fmgcl(A) \leq fmgcl(h^{-1}(h(A))) \leq h^{-1}(cl(h(A))) \Rightarrow h(fmgcl(A)) \leq cl(h(A))$.

Remark 6.3. Composition of two fmg-continuous functions need not be so, as it seen from the following example.

Example 6.4. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Then clearly i_1 and i_2 are fmg-continuous functions. Now $1_X \setminus B \in \tau_3^c$. So $(i_2 \circ i_1)^{-1}(1_X \setminus B) = 1_X \setminus B \leq 1_X \setminus B$ which is an fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1}(1_X \setminus B) = 1_X \not\leq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not an fmg-closed set in $(X, \tau_1) \Rightarrow i_2 \circ i_1$ is not an fmg-continuous function.

Let us now recall the following definition from [18] for ready references.

Definition 6.5 [18]. A function $h : X \to Y$ is called fuzzy continuous function if $h^{-1}(V)$ is fuzzy closed set in X for every fuzzy closed set V in Y.

Remark 6.6. Since every fuzzy closed set is fmg-closed set, it is clear that fuzzy continuous function is fmg-continuous function. But the converse is not necessarily true , as follows from the next example.

Example 6.7. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1)

is fmg-closed set in (X, τ), clearly i is fmg-continuous function. But $A \in \tau_2^c$, $i^{-1}(A) = A \notin \tau_1^c \Rightarrow i$ is not fuzzy continuous function.

Theorem 6.8. If $h_1 : X \to Y$ and $h_2 : Y \to Z$ are fuzzy continuous functions, then $h_2 \circ h_1 : X \to Z$ is fmg-continuous function.

Proof. Obvious.

Theorem 6.9. If a bijective function $h : X \to Y$ is an fmg-continuous, fuzzy open function from an fmg-regular space X onto an fts Y, then Y is fuzzy regular space.

Proof. Let y_α be a fuzzy point in Y and F, a fuzzy closed set in Y with $y_\alpha \notin F$. As h is bijective, there exists unique $x \in X$ such that $h(x) = y$. So $h(x_\alpha) \notin F \Rightarrow x_\alpha \notin h^{-1}(F)$ where $h^{-1}(F)$ is fmg-closed set in X (as h is an fmg-continuous function). As X is fmg-regular space, there exist $U, V \in \tau$ such that $x_\alpha \in U, h^{-1}(F) \leq V$ and $U \not{q} V$. Then $h(x_\alpha) \in h(U)$, $F = h(h^{-1}(F))$ (as h is bijective) $\leq h(V)$ and $h(U) \not{q} h(V)$ where $h(U)$ and $h(V)$ are fuzzy open sets in Y. (Indeed, $h(U)qh(V) \Rightarrow$ there exists $z \in Y$ such that $[h(U)](z) + [h(V)](z) > 1 \Rightarrow U(h^{-1}(z)) + V(h^{-1}(z)) > 1$ as h is bijective $\Rightarrow UqV$, a contradiction). Hence Y is a fuzzy regular space.

In a similar manner we can state the following theorems which have similar proofs to that of Theorem 6.9.

Theorem 6.10. If a bijective function $h : X \to Y$ is fmg-continuous, fuzzy open function from an fmg-normal space X onto an fts Y, then Y is fuzzy normal space.

Theorem 6.11. If a bijective function $h : X \to Y$ is fmg-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal), fmT_g-space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Definition 6.12. A function $h : X \to Y$ is called fmg-irresolute function if $h^{-1}(U)$ is an fmg-open set in X for every fmg-open set U in Y.

Theorem 6.13. A function $h : X \to Y$ is fmg-irresolute function if and only if for each fuzzy point x_α in X and each fmg-open nbd V in Y of $h(x_\alpha)$, there exists an fmg-open nbd U in X of x_α such that $h(U) \leq V$.

Proof. Let $h : X \to Y$ be an fmg-irresolute function. Let x_α be a fuzzy point in X and V be any fmg-open nbd of $h(x_\alpha)$ in Y. Then $x_\alpha \in h^{-1}(V)$ being an fmg-open set in X is an fmg-open nbd of x_α in X. Put $U = h^{-1}(V)$. Then U is an fmg-open nbd of x_α in X and

$h(U) = h(h^{-1}(V)) \leq V$. Conversely, let A be an fmg-open set in Y and x_α be any fuzzy point in X such that $x_\alpha \in h^{-1}(A)$. Then $h(x_\alpha) \in A$. By hypothesis, there exists an fmg-open nbd U of x_α in X such that $h(U) \leq A \Rightarrow x_\alpha \in U = fmgint(U) \leq fmgint(h^{-1}(A))$. Since x_α is taken arbitrarily and $h^{-1}(A)$ is the union of all fuzzy points in $h^{-1}(A)$, $h^{-1}(A) \leq fmgint(h^{-1}(A)) \Rightarrow h^{-1}(A) = fmgint(h^{-1}(A)) \Rightarrow h^{-1}(A)$ is fmg-open set in $X \Rightarrow h$ is an fmg-irresolute function.

Now we state the following two theorems which have similar proofs to that of Theorem 6.9.

Theorem 6.14. If a bijective function $h : X \to Y$ is fmg-irresolute, fuzzy open function from an fmg-regular (resp., fmg-normal) space X onto an fts Y, then Y is fmg-regular (resp., fmg-normal) space.

Theorem 6.15. If a bijective function $h : X \to Y$ is fmg-irresolute, fuzzy open function from an fmg-regular (resp., fmg-normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.16. If a bijective function $h : X \to Y$ is fmg-irresolute, fuzzy open function from a fuzzy regular (resp., fuzzy normal), fmT_g-space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.17. Let $h : X \to Y$ be an fmg-continuous function from X onto an fts Y and $A(\in I^X)$ be an fmg-compact set in X. Then $h(A)$ is a fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) set in Y.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of $h(A)$ by fuzzy open (resp., fuzzy open, fuzzy regular open) sets of Y. Then $h(A) \leq \bigcup_{\alpha\in\Lambda} U_\alpha \Rightarrow A \leq h^{-1}(\bigcup_{\alpha\in\Lambda} U_\alpha) = \bigcup_{\alpha\in\Lambda} h^{-1}(U_\alpha)$. Then $\mathcal{V} = \{h^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of A by fmg-open sets of X as h is an fmg-continuous function. As A is fmg-compact set in X, there exists a finite subcollection Λ_0 of Λ such that $A \leq \bigcup_{\alpha\in\Lambda_0} h^{-1}(U_\alpha) \Rightarrow h(A) \leq h(\bigcup_{\alpha\in\Lambda_0} h^{-1}(U_\alpha) \leq \bigcup_{\alpha\in\Lambda_0} U_\alpha \Rightarrow h(A)$ is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) set in Y.

Since fuzzy open set is fmg-open, we can state the following theorems easily the proofs of which are same as that of Theorem 6.17.

Theorem 6.18. Let $h : X \to Y$ be an fmg-irresolute function from X onto an fts Y and $A(\in I^X)$ be an fmg-compact set in X. Then

$h(A)$ is fmg-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) set in Y.

Theorem 6.19. Let $h : X \to Y$ be an fmg-continuous function from an fmg-compact space X onto an fts Y . Then Y is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.20. Let $h : X \to Y$ be an fmg-irresolute function from an fmg-compact space X onto an fts Y . Then Y is fmg-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.21. Let $h : X \to Y$ be an fmg-continuous function from a fuzzy compact, fmT_g-space X onto an fts Y . Then Y is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.22. Let $h : X \to Y$ be an fmg-irresolute function from a fuzzy compact, fmT_g-space X onto an fts Y . Then Y is fmg-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space.

Remark 6.23. It is clear from definitions that (i) fmg-irresolute function is fmg-continuous, but the converse may not be true, as it seen from the following example. Also (ii) fuzzy continuity and fmg-irresoluteness are independent concepts follow from the following examples.

Example 6.24. There exists a function which is Fuzzy continuous, fmg-continuous but it is not an fmg-irresolute Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is fmg-continuous as well as fuzzy continuous function. Now every fuzzy set in (X, τ_2) is fmg-closed set in $(X.\tau_2)$. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.7$. Then B is fmg-closed set in (X, τ_2). Then $B \leq B$ which is an fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1} B = 1_X \not\leq B \Rightarrow B$ is not an fmg-closed set in $(X, \tau_1) \Rightarrow i$ is not an fmg-irresolute function.

Example 6.25. fmg-irresoluteness does not imply fuzzy continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is fmg-closed set in (X, τ_1), clearly i is fmg-irresolute function. Also i is not fuzzy continuous function as $A \in \tau_2, i^{-1}(A) = A \notin \tau_1$.

Theorem 6.26. Let $h : X \to Y$ be an fmg-continuous function where Y is an fmT_g-space. Then h is fmg-irresolute function.
Proof. Obvious.
Theorem 6.27. It is clear from definition that composition of two fmg-irresolute functions is fmg-irresolute function. Again if $h_1 : X \to Y$ is fmg-irresolute function and $h_2 : Y \to Z$ is fmg-continuous function, then $h_2 \circ h_1 : X \to Z$ is an fmg-continuous function.

To establish the mutual relationship of fmg-continuous function with the classes of functions defined in [3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16], we first recall the definitions of the functions introduced in [3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16].
Definition 6.28. Let $h : (X, \tau_1) \to (Y, \tau_2)$ be a function. Then h is called (i) fg-continuous function [3] if $h^{-1}(V)$ is fg-closed set in X for every $V \in \tau_2^c$, (ii) $fg\beta$-continuous function [7] if $h^{-1}(V)$ is $fg\beta$-closed set in X for every $V \in \tau_2^c$, (iii) $f\beta g$-continuous function [7] if $h^{-1}(V)$ is $f\beta g$-closed set in X for every $V \in \tau_2^c$, (iv) fgp-continuous function [3] if $h^{-1}(V)$ is fgp-closed set in X for every $V \in \tau_2^c$, (v) fpg-continuous function [3] if $h^{-1}(V)$ is fpg-closed set in X for every $V \in \tau_2^c$, (vi) $fg\alpha$-continuous function [3] if $h^{-1}(V)$ is $fg\alpha$-closed set in X for every $V \in \tau_2^c$, (vii) $f\alpha g$-continuous function [3] if $h^{-1}(V)$ is $f\alpha g$-closed set in X for every $V \in \tau_2^c$, (viii) fgs-continuous function [3] if if $h^{-1}(V)$ is fgs-closed set in X for every $V \in \tau_2^c$, (ix) fsg-continuous function [3] if $h^{-1}(V)$ is fsg-closed set in X for every $V \in \tau_2^c$, (x) fgs^*-continuous function [5] if $h^{-1}(V)$ is fgs^*-closed set in X for every $V \in \tau_2^c$, (xi) fs^*g-continuous function [6] if $h^{-1}(V)$ is fs^*g-closed set in X for every $V \in \tau_2^c$, (xii) $fg\gamma$-continuous function [11] if $h^{-1}(V)$ is $fg\gamma$-closed set in X for every $V \in \tau_2^c$, (xiii) $fg\gamma^*$-continuous function [12] if $h^{-1}(V)$ is $fg\gamma^*$-closed set in X for every $V \in \tau_2^c$, (xiv) $frwg$-continuous function [16] if $h^{-1}(V)$ is $frwg$-closed set in X for every $V \in \tau_2^c$, (xv) $fswg$-continuous function [15] if $h^{-1}(V)$ is $fswg$-closed set in X for every $V \in \tau_2^c$, (xvi) $fgpr$-continuous function [10] if $h^{-1}(V)$ is $fgpr$-closed set in X for every $V \in \tau_2^c$, (xvii) fwg-continuous function [14] if $h^{-1}(V)$ is fwg-closed set in X for every $V \in \tau_2^c$, (xviii) $f\pi g$-continuous function [13] if $h^{-1}(V)$ is $f\pi g$-closed set in X for every $V \in \tau_2^c$.
Remark 6.29. It is clear from definitions that (i) fs^*g-continuity implies fmg-continuity, (ii) fmg-continuity implies fgp-continuity, $fgpr$-continuity, $fg\alpha$-continuity, $fg\beta$-continuity, $fg\gamma$-continuity, $fg\gamma^*$-continuity, fwg-continuity, $frwg$-continuity, (iii)

fmg-continuity is independent concept of fg-continuity, $f\pi g$-continuity, fpg-continuity, $f\alpha g$-continuity, $f\beta g$-continuity, fgs-continuity, fsg-continuity, fgs^*-continuity, $fswg$-continuity. But the reverse implications are not necessarily true, as it seen from the following examples.

Example 6.30. fmg-continuity does not imply fg-continuity, fgs-continuity, fsg-continuity, fgs^*-continuity and fs^*g-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.6, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1 \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_2^c$, $i^{-1}(1_X \setminus B) = 1_X \setminus B$ and $cl_{\tau_1} int_{\tau_1}(1_X \setminus B) = 0_X \Rightarrow 1_X \setminus B$ is fmg-closed set in $(X, \tau_1) \Rightarrow i$ is fmg-continuous function. But $1_X \setminus B < A$ where $A \in \tau_1$ (resp., $A \in FSO(X, \tau_1)$ and also A is fg-open set in (X, τ_1). But $cl_{\tau_1}(1_X \setminus B) = scl_{\tau_1}(1_X \setminus B) = 1_X \not\leq A \Rightarrow 1_X \setminus B$ is not an fg-closed set, fgs-closed set, fsg-closed set, fgs^*-closed set and fs^*g-closed set in $(X, \tau_1) \Rightarrow i$ is not an fg-continuous function, fgs-continuous function, fsg-continuous function, fgs^*-continuous function, fs^*g-continuous function.

Example 6.31. fmg-continuity does not imply $f\pi g$-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B, C\}$, $\tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.4, D(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus D \in \tau_2^c$, $i^{-1}(1_X \setminus D) = 1_X \setminus D$. Now fg-open sets in (X, τ_1) is $\{0_X, 1_X, U, V\}$ where $U(a) \geq 0.6, 0.5 \leq U(b) < 0.6, V \not\geq 1_X \setminus C$. Then $1_X \setminus D < U_1$ where U_1 is fg-open set in (X, τ_1) defined by $U_1(a) \geq 0.6, 0.5 \leq U_1(b) < 0.6$. So $cl_{\tau_1} int_{\tau_1}(1_X \setminus D) = 1_X \setminus C < U_1 \Rightarrow 1_X \setminus D$ is fmg-closed set in $(X, \tau_1) \Rightarrow i$ is fmg-continuous function. Now $1_X \setminus D < C \in F\pi O(X, \tau_1)$. But $cl_{\tau_1}(1_X \setminus D) = 1_X \setminus D \not\leq C \Rightarrow 1_X \setminus D$ is not $f\pi g$-closed set in $(X, \tau_1) \Rightarrow i$ is not $f\pi g$-continuous function.

Example 6.32. fmg-continuity does not imply any of $fswg$-continuity, fpg-continuity, $f\beta g$-continuity and $f\alpha g$-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.4, A(b) = 0.7, B(a) = 0.4, B(b) = 0.3$. Then (X, τ_1) and $(X.\tau_2 0$ are fts's. Consider the identity function $i : (X.\tau_1) \to (X, \tau_2)$. $1_X \setminus B \in \tau_2^c$, $i^{-1}(1_X \setminus B) = 1_X \setminus B$. Now fg-open sets in (X, τ_1) is $\{0_X, 1_X, U\}$ where $U \not\geq 1_X \setminus A$. Since $1_X \setminus B \geq 1_X \setminus A$, so 1_X is the only fg-open set in (X, τ_1) containing $1_X \setminus B$ and so $1_X \setminus B$ is fmg-closed set in $(X, \tau_1) \Rightarrow i$

is an fmg-continuous function. Again $1_X \setminus B \in F\beta O(X, \tau_1)$ as well as $1_X \setminus B \in FSO(X, \tau_1)$, $1_X \setminus B \in FPO(X, \tau_1)$, $1_X \setminus B \in F\alpha O(X, \tau_1)$. But as $1_X \setminus B \notin F\beta C(X, \tau_1)$, $1_X \setminus B \notin FPC(X, \tau_1)$, $1_X \setminus B \notin F\alpha C(X, \tau_1)$, we conclude that $1_X \setminus B$ is not $f\beta g$-closed set, fpg-closed set and $f\alpha g$-closed set in $(X, \tau_1) \Rightarrow i$ is not an $f\beta g$-continuous function, fpg-continuous function and $f\alpha g$-continuous function. Again $cl_{\tau_1} int_{\tau_1}(1_X \setminus B) = 1_X \not\leq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not $fswg$-closed set in $(X, \tau_1) \Rightarrow i$ is not an $fswg$-continuous function.

Example 6.33. None of fg-continuity, $f\pi g$-continuity, $fgpr$-continuity, fwg-continuity, $frwg$-continuity, $fg\gamma$-continuity implies fmg-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_2^c$, $i^{-1}(1_X \setminus B) = 1_X \setminus B$. Here $1_X \setminus B \leq 1_X \setminus B$ which is fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1}(1_X \setminus B) = 1_X \not\leq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not an fmg-closed set in $(X, \tau_1) \Rightarrow i$ is not an fmg-continuous function. Now $1_X \in \tau_1$ (resp., $1_X \in F\pi O(X, \tau_1)$, $1_X \in FRO(X, \tau_1)$) only containing $1_X \setminus B$ and so $1_X \setminus B$ is fg-closed set (resp., $fg\gamma$-closed set, $f\pi g$-closed set, $fgpr$-closed set, fwg-closed set, $frwg$-closed set) in $(X, \tau_1) \Rightarrow i$ is an fg-continuous function (resp., $fg\gamma$-continuous function, $f\pi g$-continuous function, $fgpr$-continuous function, fwg-continuous function, $frwg$-continuous function).

Example 6.34. fpg-continuity does not imply fmg-continuity

Let $X = \{a\}$, $\tau = \{0_X, 1_X, B, C\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, B(a) = 0.4, C(a) = 0.45$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A \leq 1_X \setminus A$ which is fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1}(1_X \setminus A) = 1_X \setminus C \not\leq 1_X \setminus A \Rightarrow 1_X \setminus A$ is not fmg-closed set in $(X, \tau_1) \Rightarrow i$ is not an fmg-continuous function. Now $1_X \setminus A < U \in FPO(X, \tau_1)$ where $U > 1_X \setminus B$. So $pcl_{\tau_1}(1_X \setminus A) = 1_X \setminus C < U \Rightarrow 1_X \setminus A$ is fpg-closed set in $(X, \tau_1) \Rightarrow i$ is an fpg-continuous function.

Example 6.35. None of $fg\beta$-continuity, $f\beta g$-continuity, fgp-continuity, $fg\alpha$-continuity, $f\alpha g$-continuity, fgs-continuity, fsg-continuity and $fg\gamma^*$-continuity implies fmg-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = B(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus B \in \tau_2^c$,

$i^{-1}(1_X \setminus B) = 1_X \setminus B \leq 1_X \setminus B$ which is an fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1}(1_X \setminus B) = 1_X \setminus A \not\leq 1_X \setminus B \Rightarrow 1_X \setminus B$ is not an fmg-closed set in $(X, \tau_1) \Rightarrow i$ is not an fmg-continuous function. Again since $1_X \setminus B \in F\beta C(X, \tau_1)$, $1_X \setminus B$ is $fg\beta$-closed set as well as $f\beta g$-closed set in $(X, \tau_1) \Rightarrow i$ is an $fg\beta$-continuous function as well as $f\beta g$-continuous function. Also $1_X \setminus B \in FSC(X, \tau_1)$ as well as $1_X \setminus B \in F\gamma C(X, \tau_1)$ and so $1_X \setminus B$ is fgs-closed set, fsg-closed set and $fg\gamma^*$-closed set in $(X, \tau_1) \Rightarrow i$ is an fgs-continuous function, fsg-continuous function and $fg\gamma^*$-continuous function. Now $1_X \in \tau_1$ also $1_X \in F\alpha O(X, \tau_1)$ only containing $1_X \setminus B \Rightarrow 1_X \setminus B$ is fgp-closed set, $fg\alpha$-closed set and $f\alpha g$-closed set in $(X, \tau_1) \Rightarrow i$ is an fgp-continuous function, $fg\alpha$-continuous function and $f\alpha g$-continuous function.

Example 6.36. None of $fswg$-continuity, fgs^*-continuity implies fmg-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B, C\}$, $\tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.65, D(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus D \in \tau_2^c$, $i^{-1}(1_X \setminus D) = 1_X \setminus D \leq 1_X \setminus D$ which is an fg-open set in (X, τ_1). But $cl_{\tau_1} int_{\tau_1}(1_X \setminus D) = 1_X \setminus C \not\leq 1_X \setminus D \Rightarrow 1_X \setminus D$ is not an fmg-closed set in $(X, \tau_1) \Rightarrow i$ is not an fmg-continuous function. Now $FSO(X, \tau_1) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus C, C \leq V \leq 1_X \setminus B$. Then $1_X \setminus D < 1_X \setminus B \in FSO(X, \tau_1)$. Then $cl_{\tau_1}(1_X \setminus D) = 1_X \setminus B \leq 1_X \setminus B \Rightarrow 1_X \setminus D$ is fgs^*-closed set in $(X, \tau_1) \Rightarrow i$ is an fgs^*-continuous function. Again $cl_{\tau_1} int_{\tau_1}(1_X \setminus D) = 1_X \setminus C < 1_X \setminus B \Rightarrow 1_X \setminus D$ is an $fswg$-closed set in $(X, \tau_1) \Rightarrow i$ is an $fswg$-continuous function.

Remark 6.37. (i) Let $h : X \to Y$ be a function where X is an fmT_g-space. Then if h is an fmg-continuous function, then h is an fg-continuous function, $f\pi g$-continuous function, fpg-continuous function, $f\alpha g$-continuous function, $f\beta g$-continuous function, fsg-continuous function, fgs^*-continuous function, fs^*g-continuous function, $fswg$-continuous function. (ii) Let $h : X \to Y$ be a function where X is an fT_g-space (resp., fT_β-space, $f\beta T_b$-space, fT_α-space, $f\alpha T_b$-space, fT_b-space, fT_{sg}-space, fgT_{s^*}-space, fT_p-space, fpT_b-space, fT_γ-space, fT_{γ^*}-space, frT_g-space, fsT_g-space, fT_w-space, fT_π-space, fT_{pr}-space). If h is fg-continuous function (resp., $fg\beta$-continuous function, $f\beta g$-continuous function, $fg\alpha$-continuous function, $f\alpha g$-continuous function, fgs-continuous function, fsg-continuous function, fgs^*-continuous function, fgp-continuous function, fpg-continuous function, $fg\gamma$-continuous function, $fg\gamma^*$-continuous function, $frwg$-continuous function, $fswg$-continuous function, fwg-continuous function, $f\pi g$-continuous function, $fgpr$-continuous function), then h is fmg-continuous function.

7. fmg-T_2 Space

In this section a strong form of fuzzy T_2-space is introduced and established. Afterwards, a strong form of fmg-continuity is introduced and the applications of this newly defined function is established.

We first recall the definition and theorem from [24, 25] for ready references.

Definition 7.1 [24]. An fts (X, τ) is called fuzzy T_2-space if for any two distinct fuzzy points x_α and y_β; when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1, U_1 \not{q} V_1$ and $x_\alpha q U_2, y_\beta \in V_2, U_2 \not{q} V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy open sets U and V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Theorem 7.2 [25]. An fts (X, τ) is fuzzy T_2-space if and only if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fuzzy open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy open nbd U and y_β has a fuzzy open q-nbd V such that $U \not{q} V$.

Now we introduce the following concept.

Definition 7.3. An fts (X, τ) is called fmg-T_2-Space if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fmg-open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), x_α has an fmg-open nbd U and y_β has an fmg-open q-nbd V such that $U \not{q} V$.

Theorem 7.4. If an injective function $h : X \to Y$ is fmg-continuous function from an fts X onto a fuzzy T_2-space Y, then X is fmg-T_2-space.

Proof. Let x_α and y_β be two distinct fuzzy points in X. Then $h(x_\alpha)$ $(= z_\alpha$, say) and $h(y_\beta)$ $(= w_\beta$, say) are two distinct fuzzy points in Y. Case I. Suppose $x \neq y$. Then $z \neq w$. Since Y is fuzzy T_2-space, there exist fuzzy open sets U, V in Y such that $z_\alpha q U, w_\beta q V$ and $U \not{q} V$. As h is fmg-continuous function, $h^{-1}(U)$ and $h^{-1}(V)$ are fmg-open sets in X with $x_\alpha q h^{-1}(U), y_\beta q h^{-1}(V)$ and $h^{-1}(U) \not{q} h^{-1}(V)$ [Indeed, $z_\alpha q U \Rightarrow U(z) + \alpha > 1 \Rightarrow U(h(x)) + \alpha > 1 \Rightarrow [h^{-1}(U)](x) + \alpha >$

$1 \Rightarrow x_\alpha q h^{-1}(U)$. Again, $h^{-1}(U) q h^{-1}(V) \Rightarrow$ there exists $t \in X$ such that $[h^{-1}(U)](t) + [h^{-1}(V)](t) > 1 \Rightarrow U(h(t)) + V(h(t)) > 1 \Rightarrow UqV$, a contradiction]. Case II. Suppose $x = y$ and $\alpha < \beta$ (say). Then $z = w$ and $\alpha < \beta$. Since Y is fuzzy T_2-space, there exist a fuzzy open nbd U of x_α and a fuzzy open q-nbd V of w_β such that $U \not{q} V$. Then $U(z) \geq \alpha \Rightarrow [h^{-1}(U)](x) \geq \alpha \Rightarrow x_\alpha \in h^{-1}(U), y_\beta q h^{-1}(V)$ and $h^{-1}(U) \not{q} h^{-1}(V)$ where $h^{-1}(U)$ and $h^{-1}(V)$ are fmg-open sets in X as h is fmg-continuous function. Consequently, X is fmg-T_2-space.

Similarly we can state the following theorems easily the proofs of which are similar to that of Theorem 7.4.

Theorem 7.5. If a bijective function $h : X \to Y$ is fmg-irresolute function from an fts X onto an fmg-T_2-space Y, then X is fmg-T_2-space.

Theorem 7.6. If a bijective function $h : X \to Y$ is fmg-continuous function from an fmT_g-space X onto a fuzzy T_2-space Y, then X is fuzzy T_2-space.

Theorem 7.7. If a bijective function $h : X \to Y$ is fmg-irresolute function from an fmT_g-space X onto an fmg-T_2-space Y, then X is fuzzy T_2-space.

Theorem 7.8. If a bijective function $h : X \to Y$ is fmg-open function from a fuzzy T_2-space X onto an fts Y, then Y is fmg-T_2-space.

Theorem 7.9. If a bijective function $h : X \to Y$ is fmg-open function from a fuzzy T_2-space X onto an fmT_g-space Y, then Y is fuzzy T_2-space.

It is clear from definitions that every fuzzy T_2-space is fmg-T_2-space, but the converse is not necessarily true, follows from the following example.

Example 7.10. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Clearly (X, τ) is not a fuzzy T_2-space. Here every fuzzy set in (X, τ) is fmg-open set in (X, τ). Consider two fuzzy points $a_{0.1}$ and $a_{0.4}$. Then there exist two fmg-open sets U, V in X where $U(a) = 0.2, V(a) = 0.61$ such that $a_{0.1} \in U$, $a_{0.4} qV$ and $U \not{q} V$ and this is true for every pair of distinct fuzzy points in X. So (X, τ) is an fmg-T_2-space.

Now we introduce the strong form of fmg-continuous function.

Definition 7.11. A function $h : X \to Y$ is called strongly fmg-continuous function if $h^{-1}(V)$ is fuzzy closed set in X for every fmg-closed set V in Y.

Theorem 7.12. A function $h : X \to Y$ is strongly fmg-continuous function iff for each fuzzy point x_α in X and each fmg-open nbd V in Y of $h(x_\alpha)$, there exists a fuzzy open nbd U in X of x_α such that $h(U) \leq V$.

Proof. Let $h : X \to Y$ be a strongly fmg-continuous function. Let x_α be a fuzzy point in X and V be any fmg-open nbd of $h(x_\alpha)$ in Y. Then $h(x_\alpha) \in V \Rightarrow x_\alpha \in h^{-1}(V)$ which being a fuzzy open set in X is a fuzzy open nbd of x_α in X. Put $U = h^{-1}(V)$. Then $h(U) = h(h^{-1}(V)) \leq V$. Conversely, let A be an fmg-open set in Y and x_α be any fuzzy point in X such that $x_\alpha \in h^{-1}(A)$. Then $h(x_\alpha) \in A$. By hypothesis, there exists a fuzzy open nbd U of x_α in X such that $h(U) \leq A \Rightarrow x_\alpha \in U = int(U) \leq int(h^{-1}(A))$. Since x_α is taken arbitrarily and $h^{-1}(A)$ is the union of all fuzzy points in $h^{-1}(A)$, $h^{-1}(A) \leq int(h^{-1}(A)) \Rightarrow h^{-1}(A) = int(h^{-1}(A)) \Rightarrow h^{-1}(A)$ is fuzzy open set in $X \Rightarrow h$ is a strongly fmg-continuous function.

Remark 7.13. It is clear from above discussion that strongly fmg-continuous function implies fuzzy continuous, fmg-continuous and fmg-irresolute functions. But the converses are not true, in general, follow from the following examples.

Example 7.14. None of fuzzy continuity, fmg-continuity implies strongly fmg-continuity Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since 0_X and 1_X are the only fuzzy closed sets in (X, τ_2), clearly i is fuzzy continuous as well as fmg-continuous function. As every fuzzy set in (X, τ_2) is fmg-closed set in (X, τ_2), considering the fuzzy set B, defined by $B(a) = B(b) = 0.5$. Then B is fmg-closed set in (X, τ_2). Now $i^{-1}(B) = B \notin \tau_1^c \Rightarrow i$ is not strongly fmg-continuous function.

Example 7.15. fmg-irresoluteness does not imply strongly fmg-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is fmg-closed set in (X, τ_1), clearly i is fmg-irresolute function. Now $A \in \tau_2$ is fmg-closed set in (X, τ_2). $i^{-1}(A) = A \notin \tau_1^c \Rightarrow i$ is not strongly fmg-continuous function.

Note 7.16. Clearly composition of two fmg-irresolute functions is also so.

Theorem 7.17. If $h_1 : X \to Y$ is strongly fmg-continuous function and $h_2 : Y \to Z$ is fmg-continuous function, then $h_2 \circ h_1 : X \to Z$ is fuzzy continuous function.

Proof. Obvious.

Since fuzzy open set is fmg-open set, we have the following theorems.

Theorem 7.18. If a bijective function $h : X \to Y$ is strongly fmg-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal) space X onto an fts Y, then Y is fmg-regular (resp., fmg-normal) space.

Theorem 7.19. If a bijective function $h : X \to Y$ is strongly fmg-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 7.20. If a bijective function $h : X \to Y$ is strongly fmg-continuous function from an fts X onto an fmg-T_2-space Y, then X is fuzzy T_2-space.

Theorem 7.21. If a bijective function $h : X \to Y$ is strongly fmg-continuous function from a fuzzy compact space X onto an fts Y, then Y is fmg-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space.

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