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ω-SEMI-SYMMETRIC SASAKIAN MANIFOLDS ADMITTING GENERAL CONNECTION

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Abstract. The object of the present paper is to study the properties of Sasakian manifold in the light of general connection, which is induced with quarter symmetric metric connection, Tanaka Webster connection, Schouten-Van Kampen connection and Zamkovoy connection. We consider ω^*-semi-symmetric Sasakian manifold. Furthermore, we discuss the Sasakian manifold satisfying $R^*(X,Y) \cdot Z^* = 0$. Where ω^* is quasi conformal curvature tensor and Z^* is Z-tensor with respect to general connection respectively.

1. Introduction

Throughout our paper, we denote the symboles ∇, ∇^* and ω for Levi-Civita connection, general connection and quasi-conformal like curvature tensor respectively.

Recently, Biswas and Baishya ([2],[3]) introduced and studied a new connection, named general connection in the context of Sasakian geometry. The general connection ∇^* is defined as

$$\nabla^*_U V = \nabla_U V + \lambda \left[(\nabla_U \eta)(V)\xi - \eta(V)\nabla_U \xi \right] + \mu \eta(U)\phi V, \tag{1}$$

for all $U, V \in \chi(M)$ and the pair (λ, μ) being real constants. The beauty of such connection ∇^* lies in the fact that it has the flavour of

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(i) quarter symmetric metric connection([7], [4]) for $(\lambda, \mu) \equiv (0, -1)$;

(ii) Tanaka Webster connection[13] for $(\lambda, \mu) \equiv (1, -1)$;

(iii) Schouten-Van Kampen connection[12] for $(\lambda, \mu) \equiv (1, 0)$ and

(iv) Zamkovoy connection[18] for $(\lambda, \mu) \equiv (1, 1)$.

Generalized quasi-conformal curvature tensor (hereafter we abrevriate it as ω-tensor)[1] is a linear combination of four tensors namely conformal curvature tensor C, Concircular curvature tensor E, Projective curvature tensor P and Conharmonic curvature tensor K respectively. *i.e* The components of such *curvature tensor* ω in a Riemannian manifold $(M^n, g)(n > 1)$, are given by

$$\begin{aligned} \omega(U,V)W &= \frac{n-2}{n}\left[(1+(n-1)a-b)-\{1+(n-1)(a+b)\}c\right]C(U,V)W \\ &\quad +[1-b+(n-1)a]E(U,V)W+(n-1)(b-a)P(U,V)W \\ &\quad +\frac{n-2}{n}(c-1)\{1+(n-1)(a+b)\}K(U,V)W. \end{aligned} \tag{2}$$

After straightforward calculation (2) reduces to

$$\begin{aligned} &\omega(U,V)W \\ = {} & R(U,V)W+a[S(V,W)U-S(U,W)V] \\ &+b\left[g(V,W)QU-g(U,W)QV\right] \\ &-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\left[g(V,W)U-g(U,W)V\right], \end{aligned} \tag{3}$$

for all U, V & $W \in \chi(M)$, set of all vector fields of the manifold M, where $(a,\ b,\ c)$ is a scalar triple of real constants. It is to be noted that such *curvature tensor* is that it is a Riemann curvature tensor R if the scalar triple $(a,\ b,\ c) \equiv (0,0,0)$, Conharmonic curvature tensor K[8] if $(a,b,c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 0)$, Conformal curvature tensor C [5] if $(a,b,c) \equiv \left(-\frac{1}{n-2}, -\frac{1}{n-2}, 1\right)$,concircular curvature tensor E[16] if $(a,\ b,\ c) \equiv (0,0,1)$, m-projective curvature tensor H[10], if $(a,\ b,\ c) \equiv (-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0)$ and projective curvature tensor P[16], if $(a,b,c) \equiv (-\frac{1}{n-1}, 0, 0)$.

A n-dimensional Sasakian manifold M is said to be ω semi- symmetric if the ω-tensor satisfies the following condition([4])

$$R(U,V)\cdot\omega = 0,$$

for all U, $V \in \chi(M)$, set of all vector fields of the manifold M.

This paper is structured as follows: After introduction, a short description of Sasakian manifold is given in section 2. In section 3, we have studied Sasakian manifold admitting general connection and we obtained curvature tensor R^*, Ricci tensor S^*, scalar curvature r^* of ∇^* in Sasakian manifold. Section 4 deals with the ω^*-semi-symmetric Sasakian manifold and tabled the nature of the Ricci tensor for different curvature restrictions in the light of quarter symmetric metric connection, Tanaka Webster connection, Schouten-Van Kampen connection and Zamkovoy connection. Finally in section 5, we have discussed Sasakian manifold satisfying $R^*(X,Y)\cdot Z^*=0$ with respect to general connection and we found an n-dimensional Sasakian manifold M admitting the general connction satisfying $R^*(X,Y)\cdot Z^*=0$, is an η- Einstein manifold.

2. Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure $(\phi,\ \xi,\ \eta,\ g)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\begin{aligned}
(4)\qquad \phi^2 V &= -V+\eta(V)\xi, \eta(\xi)=1, \eta(\phi U)=0,\ \phi\xi=0,\\
(5)\quad g(U,V) &= g(\phi U,\phi V)+\eta(U)\eta(V),\\
(6)\, g(U,\phi V) &= -g(\phi U,V), \eta(V)=g(V,\xi), \text{for all } U,V\in\chi(M),
\end{aligned}$$

where $\chi(M)$ is set of all vector fields of the manifold M. An almost contact metric manifold M is said to be (a) a contact metric manifold if

$$g(U,\phi V)=d\eta(U,V), \text{for all } U,V\in\chi(M); \tag{7}$$

(b) a K-contact manifold if the vector field ξ is Killing equivalently

$$\nabla_V\xi=-\phi V, \tag{8}$$

where ∇ is Riemannian connection and (c) a Sasakian manifold if

$$(\nabla_U\phi)V=g(U,V)\xi-\eta(V)U, \text{for all } U,\ V\in\chi(M). \tag{9}$$

Further, for Sasakian manifold with structure (ϕ,ξ,η,g), the following relations hold([11],[17]):

$$R(U,V)\xi=\eta(V)U-\eta(U)V,\ \text{ for all } U,V\in\chi(M), \tag{10}$$

$$(\nabla_U \eta) V = g(U, \phi V), \tag{11}$$

$$R(\xi, U)V = g(U,V)\xi - \eta(V)U, \tag{12}$$

$$S(U, \xi) = (n-1)\eta(U), \tag{13}$$

$$R(U, \xi) V = \eta(V) U - g(U, V)\xi, \tag{14}$$

$$Q\xi = (n-1)\xi, \tag{15}$$

where S and Q are Ricci tensor and Ricci operator.

Definition 1. *A n-dimensional Sasakian manifold M is said to be η-Einstein if the Ricci tensor S of type $(0,2)$ is of the form*

$$S(U,V) = k_1 g(U,V) + k_2 \eta(U)\eta(V),$$

for all U, $V \in \chi(M)$, set of all vector fields of the manifold M and k_1 and k_2 are scalars.

3. Some properties of Sasakian manifold admitting general connection

Let us consider a Sasakian manifold admitting general connection. Then by virtue of (11), the relation (1) takes the following form

$$\nabla^*_U V = \nabla_U V + \lambda\left[g(U, \phi V)\xi + \eta(V)\phi U\right] + \mu\eta(U)\phi V. \tag{16}$$

Putting $V = \xi$ in (1)

$$\nabla^*_U \xi = -\phi U + \lambda\phi U. \tag{17}$$

Now, in view of (8), (9) and (16) we get the following

$$\begin{aligned} &\nabla^*_U \eta(V) \\ &= \eta(\nabla_U V) + \lambda g(U, \phi V) - g(V, \phi U) + \lambda g(V, \phi U), \end{aligned} \tag{18}$$

$$\begin{aligned} &\nabla^*_U g(V, \phi W) \\ &= g(\nabla_U V, \phi W) + \mu\eta(U) g(\phi V, \phi W) + g(V, \nabla_U(\phi W)) \\ &\quad -\mu\eta(U) g(V, W) + \mu\eta(U)\eta(V)\eta(W), \end{aligned} \tag{19}$$

$$\begin{aligned} &\nabla^*_U(\phi V) \\ &= \nabla_U(\phi V) - \lambda g(\phi U, \phi V)\xi - \mu\eta(U) V + \mu\eta(U)\eta(V)\xi. \end{aligned} \tag{20}$$

By virtue of (16), (17), (18), (19) and (20), we obtain the following

$$
\begin{aligned}
& \nabla^*_V \nabla^*_U W \\
= \; & \nabla_V \nabla_U W + \lambda g(V, \phi \nabla_U W)\xi + \lambda \eta(\nabla_U W)\phi V + \mu \eta(V)\phi \nabla_U W \\
& + \lambda g(\nabla_V U, \phi W)\xi + AB\eta(V) g(\phi U, \phi W)\xi + \lambda g(U, \nabla_V(\phi W))\xi \\
& - \lambda\mu\eta(V) g(U, W)\xi + \lambda\mu\eta(V)\eta(U)\eta(W)\xi - \lambda g(U, \phi W)\phi V \\
& + \lambda^2 g(U, \phi W)\phi V + \lambda\eta(\nabla_V W)\phi U + \lambda^2 g(V, \phi W)\phi U \\
& - \lambda g(W, \phi V)\phi U + \lambda^2 g(W, \phi V)\phi U + \lambda\eta(W)\nabla_V(\phi U) \\
& - \lambda^2 \eta(W) g(\phi V, \phi U)\xi - \lambda\mu\eta(W)\eta(V) U + \lambda\mu\eta(W)\eta(V)\eta(U)\xi \\
& + \mu\eta(\nabla_V U)\phi W + \lambda\mu g(V, \phi U)\phi W - \mu g(U, \phi V)\phi W \\
& + \lambda\mu g(U, \phi V)\phi W + \mu\eta(U)\nabla_V(\phi W) - \lambda\mu\eta(U) g(\phi V, \phi W)\xi \\
& - \mu^2\eta(U)\eta(V) W + \mu^2 \eta(U)\eta(V)\eta(W)\xi
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
& \nabla^*_{[U,V]} W \\
= \; & \nabla_{[U,V]} W + \lambda g(\nabla_U V, \phi W)\xi - \lambda g(\nabla_V U, \phi W)\xi + \lambda\eta(W)\phi\nabla_U V \\
& - \lambda\eta(W)\phi\nabla_V U + \mu\eta(\nabla_U V)\phi W - \mu\eta(\nabla_V U)\phi W.
\end{aligned} \tag{22}
$$

Interchanging U and V in (21), we obtain

$$
\begin{aligned}
& \nabla^*_U \nabla^*_V W \\
= \; & \nabla_U \nabla_V W + \lambda g(U, \phi \nabla_V W)\xi + \lambda\eta(\nabla_V W)\phi U + \mu\eta(U)\phi\nabla_V W \\
& + \lambda g(\nabla_U V, \phi W)\xi + \lambda\mu\eta(U) g(\phi V, \phi W)\xi + \lambda g(V, \nabla_U(\phi W))\xi \\
& - \lambda\mu\eta(U) g(V, W)\xi + \lambda\mu\eta(U)\eta(V)\eta(W)\xi - \lambda g(V, \phi W)\phi U \\
& + \lambda^2 g(V, \phi W)\phi U + \lambda\eta(\nabla_U W)\phi V + \lambda^2 g(U, \phi W)\phi V \\
& + \lambda^2 g(W, \phi U)\phi V + \lambda\eta(W)\nabla_U(\phi V) - \lambda^2\eta(W) g(\phi U, \phi V)\xi \\
& - \lambda\mu\eta(W)\eta(U) V + \lambda\mu\eta(W)\eta(U)\eta(V)\xi + \mu\eta(\nabla_U V)\phi W \\
& + \lambda\mu g(U, \phi V)\phi W - \mu g(V, \phi U)\phi W + \lambda\mu g(V, \phi U)\phi W \\
& + \mu\eta(V)\nabla_U(\phi W) - \lambda\mu\eta(V) g(\phi U, \phi W)\xi - \lambda g(W, \phi U)\phi V \\
& - \mu^2\eta(V)\eta(U) W + \mu^2\eta(V)\eta(U)\eta(W)\xi,
\end{aligned} \tag{23}
$$

Finally, in view of (21), (22) and (23) we get

$$\begin{aligned} & R^*(U,V)W \\ = \ & \nabla^*_U\nabla^*_V W - \nabla^*_V\nabla^*_U W - \nabla^*_{[U,V]}W \\ = \ & R(U,V)W + \left(\lambda^2 - 2\lambda\right)\left[g(W,\phi U)\phi V + g(V,\phi W)\phi U\right] \\ & -2\mu g(V,\phi U)\phi W \\ & +(\lambda - \lambda\mu + \mu)\left[g(U,W)\eta(V)\xi - \eta(U)g(V,W)\xi\right] \\ & +(\lambda - \lambda\mu + \mu)\left[\eta(U)\eta(W)V - \eta(V)\eta(W)U\right]. \end{aligned} \tag{24}$$

In consequence of (4), (5), (6), (9), (11), (12), (13), (14) and (28), one can easily bring out the following:

$$S^*(V,W) = S(V,W) - \overline{A}g(V,W) + \overline{B}\eta(V)\eta(W), \tag{25}$$

$$S^*(V,\xi) = -(n-1)\overline{C}\eta(V), \tag{26}$$

$$S^*(\xi,W) = -(n-1)\overline{C}\eta(W), \tag{27}$$

$$Q^*V = QY - \overline{A}V + \overline{B}\eta(V)\xi, \tag{28}$$

$$Q^*\xi = -(n-1)\overline{C}\xi, \tag{29}$$

$$r^* = r - \overline{A}n + \overline{B}, \tag{30}$$

$$R^*(U,V)\xi = \overline{C}\left[\eta(U)V - \eta(V)U\right], \tag{31}$$

$$R^*(\xi,V)W = \overline{C}\left[\eta(W)V - g(V,W)\xi\right], \tag{32}$$

$$R^*(U,\xi)W = \overline{C}\left[g(U,W)\xi - \eta(W)U\right], \tag{33}$$

where S^*, Q^*, r^* are the Ricci tensor, Ricci operator and scalar curvature with respect to the general connection and

$$\overline{A} = \left(\lambda^2 - \lambda - \mu - \lambda\mu\right), \tag{34}$$

$$\overline{B} = \left[\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu)\right], \tag{35}$$

$$\overline{C} = (\lambda - \lambda\mu + \mu - 1). \tag{36}$$

Therefore for quarter-symmetric metric connection

$$\overline{A} = 1; \overline{B} = n; \overline{C} = -2, \tag{37}$$

for Tanaka Webster connection

$$\overline{A} = 2; \overline{B} = 3 - n; \overline{C} = 0, \tag{38}$$

for Schouten-Van Kampen connection

$$\overline{A} = 0; \overline{B} = 1 - n; \overline{C} = 0, \tag{39}$$

and for Zamkovoy connection

$$\overline{A} = -2; \overline{B} = -1 - n; \overline{C} = 0 \tag{40}$$

respectively. Thus, we can state the following:

Proposition 2. *Let M be an n-dimensional Sasakian manifold admitting general connection ∇^*, Then*
(i) The curvature tensor R^* of ∇^* is given by (32),
(ii) The Ricci tensor S^* of ∇^* is given by (33),
(iii) The scalar curvature r^* of ∇^* is given by (37),
(iv) The Ricci tensor S^* of ∇^* is symmetric.

Now, if we suppose that the Sasakian manifold is Ricci flat with respect to the general connection. Then we get

$$S(U,V) = \overline{A}g(U,V) - \overline{B}\eta(U)\eta(V).$$

This leads to the following

Theorem 3. *If the Sasakian manifold M^n is Ricci flat with respect to the general connection if and only if it is an η-Einstein.*

Let us denote quasi conformal like curvature tensor with respect to general connection by the symbole ω^*, defined by

$$\begin{aligned} & \omega^*(U,V)W \\ = \; & R^*(U,V)W + a[S^*(V,W)U - S^*(U,W)V] \\ & +b[g(V,W)Q^*U - g(U,W)Q^*V] \\ & -\frac{cr^*}{n}\left(\frac{1}{n-1} + a + b\right)[g(V,W)U - g(U,W)V]. \end{aligned} \tag{41}$$

4. ω^*-semi-symmetric Sasakian manifold

In this section, we consider an ω^*-semi-symmetric Sasakian manifold. Then, we have

$$R^*(V,X) \circ \omega^*(Y,Z)U = 0, \tag{42}$$

for all U, $V \in \chi(M)$, set of all vector fields of the manifold M. The above equation can also be written as

$$\begin{aligned} 0 = \; & R^*(V,X)\omega^*(Y,Z)U - \omega^*(R^*(V,X)Y,Z)U \\ & -\omega^*(Y,R^*(V,X)Z)U - \omega^*(Y,Z)R^*(V,X)U. \end{aligned} \tag{43}$$

Putting in $V = \xi$ in (43), we get

$$\begin{aligned} 0 &= R^*(\xi, X)\omega * (Y,Z)U - \omega^*(R^*(\xi, X)Y, Z)U \\ &\quad -\omega^*(Y, R^*(\xi, X)Z)U - \omega^*(Y,Z)R^*(\xi, X)U. \end{aligned} \tag{44}$$

Using (31), (32), (33) and (41) we obtain the following

$$\begin{aligned} &\omega^*(R^*(\xi, X)Y, Z)U \\ &= \overline{C}\left[\eta(Y)\,\omega^*(X,Z)\,U - g(X,Y)\,\omega^*(\xi, Z)\,U\right], \end{aligned} \tag{45}$$

$$\begin{aligned} &\omega^*(Y, R^*(\xi, X)Z)U \\ &= \overline{C}\left[\eta(Z)\,\omega^*(Y,X)\,U - g(X,Z)\,\omega^*(Y,\xi)\,U\right], \end{aligned} \tag{46}$$

$$\begin{aligned} &\omega^*(Y,Z)\,R^*(\xi,X)\,U \\ &= \overline{C}\left[\eta(U)\,\omega^*(Y,Z)\,X - g(X,U)\,\omega^*(Y,Z)\,\xi\right], \end{aligned} \tag{47}$$

$$\begin{aligned} &R^*(\xi,X)\,\omega^*(Y,Z)U \\ &= \overline{C}\left[\omega^*(Y,Z,U,\xi)X - g(X,\omega^*(Y,Z)U)\,\xi\right]. \end{aligned} \tag{48}$$

Using (45), (46), (47), (48) in (44), we obtain

$$\begin{aligned} &\omega^*(Y,Z,U,\xi)X - g(X,\omega^*(Y,Z)U)\,\xi \\ =\ &\eta(Y)\,\omega^*(X,Z)\,U - g(X,Y)\,\omega^*(\xi,Z)\,U + \eta(Z)\,\omega^*(Y,X)\,U \\ &-g(X,Z)\,\omega^*(Y,\xi)\,U + \eta(U)\,\omega^*(Y,Z)\,X - g(X,U)\,\omega^*(Y,Z)\,\xi. \end{aligned} \tag{49}$$

Taking covariant derivative with ξ in (49) and then contracting over X and Y we get

$$\begin{aligned} &\eta(U)\,\omega^*(e_i,Z,e_i,\xi) + g(e_i,\omega^*(e_i,Z)U) + \eta(Z)\,\omega^*(e_i,e_i,U,\xi) \\ &n\omega^*(\xi,Z,U,\xi) + g(e_i,Z)\,\omega^*(e_i,\xi,U,\xi) + g(e_i,U)\,\omega^*(e_i,Z,\xi,\xi). \end{aligned} \tag{$\cancel{50}$}$$

By the help of (25), (31), (32), (33) and (41) we get the following

$$\begin{aligned} &\eta(U)\,\omega^*(e_i,Z,e_i,\xi) \\ =\ &(n-1)\,\overline{C}\eta(U)\,\eta(Z) - a(n-1)\,\overline{C}\eta(U)\,\eta(Z) \\ &-ar\eta(U)\,\eta(Z) + an\overline{A}\eta(U)\,\eta(Z) - a\overline{B}\eta(U)\,\eta(Z) \\ &-b(n-1)\,\overline{C}\eta(U)\,\eta(Z) + bn(n-1)\,\overline{C}\eta(U)\,\eta(Z) \\ &-\frac{cr^*}{n}\left(\frac{1}{n-1} + a + b\right)\left[\eta(U)\,\eta(Z) - n\eta(U)\,\eta(Z)\right], \end{aligned} \tag{51}$$

$$\begin{aligned}
& g\left(e_i, \omega^*\left(e_i, Z\right) U\right) \\
= \; & S^*\left(Z, U\right) + anS^*\left(Z, U\right) - aS^*\left(Z, U\right) + bg\left(Z, U\right) r^* \\
& -bS^*\left(Z, U\right) - \frac{cr^*}{n}\left(\frac{1}{n-1} + a + b\right)\left[ng\left(Z, U\right) - g\left(Z, U\right)\right],
\end{aligned} \tag{52}$$

$$\eta\left(Z\right)\omega^*\left(e_i, e_i, U, \xi\right) = 0, \tag{53}$$

$$\begin{aligned}
& n\omega^*\left(\xi, Z, U, \xi\right) \\
= \; & n\overline{C}\eta\left(U\right)\eta\left(Z\right) - n\overline{C}g\left(Z, U\right) \\
& + anS^*\left(Z, U\right) + an\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) \\
& - bn\left(n-1\right)\overline{C}g\left(Z, U\right) + bn\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) \\
& - \frac{cr^*}{n}\left(\frac{1}{n-1} + a + b\right)\left[ng\left(Z, U\right) - n\eta\left(U\right)\eta\left(Z\right)\right],
\end{aligned} \tag{54}$$

$$\begin{aligned}
& g\left(e_i, Z\right)\omega^*\left(e_i, \xi, U, \xi\right) \\
= \; & \overline{C}\left[g\left(Z, U\right) - \eta\left(U\right)\eta\left(Z\right)\right] - a\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) - aS^*\left(Z, U\right) \\
& - b\left(n-1\right)\overline{C}\eta\left(U\right)\eta\left(Z\right) + b\left(n-1\right)\overline{C}g\left(Z, U\right) \\
& - \frac{cr^*}{n}\left(\frac{1}{n-1} + a + b\right)\left[\eta\left(U\right)\eta\left(Z\right) - g\left(Z, U\right)\right],
\end{aligned} \tag{55}$$

$$g\left(e_i, U\right)\omega^*\left(e_i, Z, \xi, \xi\right) = 0. \tag{56}$$

Using (51), (52), (53), (54) (55) and (56) in (50), we obtain

$$\left(1-b\right)S\left(Z, U\right) = M_1 g\left(Z, U\right) + M_2\eta\left(U\right)\eta\left(Z\right). \tag{57}$$

where

$$M_1 = \frac{n + \left(-n+1\right)\left(\lambda - \lambda\mu + \mu\right) + bn^2 - 2bn + bn\overline{B} - br - 2b\overline{B} + b + \overline{A} - 1}{1-b}$$

$$M_1 = \frac{n\left(\lambda - \lambda\mu + \mu\right) - \overline{A} - an\overline{B} + a\overline{B} + b\overline{B} - an^2 + an - \left(\lambda - \lambda\mu + \mu\right) + ar}{1-b}.$$

This leads to the following

Theorem 4. *Every ω^*-semi-symmetric Sasakian manifold M is an η- Einstein manifold (provided $b \neq 1$).*

Also, from (57), we infer the following

Theorem 5. *Let M be a n-dimensional Sasakian manifold admitting quarter symmetric metric connection, then the following table holds:*

Curvature restrictions	*Expressions for Ricci Tensor*
$R^*(\xi, X) \cdot R^* = 0$ *(Obtained if $a = b = c = 0$)*	$S = (2n-1)\, g - n\eta \otimes \eta$
$R^*(\xi, X) \cdot C^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}$, $c = 1$)*	$S = \dfrac{1-n+r}{n-1} g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot K^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}, c = 0$)*	$S = \dfrac{1-n+r}{n-1} g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot E^* = 0$ *(Obtained if $a = b = 0, c = 1$)*	$S = (2n-1)\, g - n\eta \otimes \eta$
$R^*(\xi, X) \cdot P^* = 0$ *(Obtained if $a = -\dfrac{1}{n-1}, b = c = 0$)*	$S = (2n-1)\, g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot H^* = 0$ *(Obtained if $a = b = -\dfrac{1}{2n-2}$, $c = 0$)*	$S = \left(\dfrac{2n^2+r}{2n-1} - 1\right) g - \left(\dfrac{r+n}{2n-1}\right) \eta \otimes \eta.$

Theorem 6. *Let M be a n-dimensional Sasakian manifold admitting Tanaka Webster connection, then the following table holds:*

Curvature restrictions	*Expressions for Ricci Tensor*
$R^*(\xi, X) \cdot R^* = 0$ *(Obtained if $a = b = c = 0$)*	$S = 2g + (n-3)\, \eta \otimes \eta$
$R^*(\xi, X) \cdot C^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}$, $c = 1$)*	$S = \left(-1 + \frac{r}{n-1}\right) g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot K^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}, c = 0$)*	$S = \left(-1 + \dfrac{r}{n-1}\right) g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot E^* = 0$ *(Obtained if $a = b = 0, c = 1$)*	$S = 2g + (n-3)\, \eta \otimes \eta$
$R^*(\xi, X) \cdot P^* = 0$ *(Obtained if $a = -\dfrac{1}{n-1}, b = c = 0$)*	$S = 2g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot H^* = 0$ *(Obtained if $a = b = -\dfrac{1}{2n-2}$, $c = 0$)*	$S = \left(\dfrac{n+1+r}{2n-1}\right) g + \left(\dfrac{2n^2-4n-r}{2n-1}\right) \eta \otimes \eta.$

Theorem 7. *Let M be a n-dimensional Sasakian manifold admitting Schouten-Van Kampen connection, then the following table holds:*

Curvature restrictions	*Expressions for Ricci Tensor*
$R^*(\xi, X) \cdot R^* = 0$ *(Obtained if $a = b = c = 0$)*	$S = (n-1)\,\eta \otimes \eta$
$R^*(\xi, X) \cdot C^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2},\ c = 1$)*	$S = \left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot K^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}, c = 0$)*	$S = \left(\frac{r}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot E^* = 0$ *(Obtained if $a = b = 0, c = 1$)*	$S = (n-1)\,\eta \otimes \eta$
$R^*(\xi, X) \cdot P^* = 0$ *(Obtained if $a = -\dfrac{1}{n-1}, b = c = 0$)*	$S = \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot H^* = 0$ *(Obtained if $a = b = -\dfrac{1}{2n-2},\ c = 0$)*	$S = \left(\frac{1+r-n}{2n-1} - 1\right) g + \left(\frac{2n^2-2n-r}{2n-1}\right) \eta \otimes \eta.$

Theorem 8. *Let M be a n-dimensional Sasakian manifold admitting Zamkovoy connection, then the following table holds:*

Curvature restrictions	*Expressions for Ricci Tensor*
$R^*(\xi, X) \cdot R^* = 0$ *(Obtained if $a = b = c = 0$)*	$S = -2g + (n+1)\,\eta \otimes \eta$
$R^*(\xi, X) \cdot C^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2},\ c = 1$)*	$S = \left(\frac{r-2n^2}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot K^* = 0$ *(Obtained if $a = b = -\dfrac{1}{n-2}, c = 0$)*	$S = \left(\frac{r-2n^2}{n-1} - 1\right) g + \left(n - \frac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot E^* = 0$ *(Obtained if $a = b = 0, c = 1$)*	$S = -2g + (n+1)\,\eta \otimes \eta$
$R^*(\xi, X) \cdot P^* = 0$ *(Obtained if $a = -\dfrac{1}{n-1}, b = c = 0$)*	$S = -2g + \left(n - \dfrac{r}{n-1}\right) \eta \otimes \eta$
$R^*(\xi, X) \cdot H^* = 0$ *(Obtained if $a = b = -\dfrac{1}{2n-2},\ c = 0$)*	$S = \left(\frac{-2n^2+r-3n+1}{2n-1}\right) g + \left(\frac{2n^2-r}{2n-1}\right) \eta \otimes \eta.$

5. Sasakian manifold admitting $R^{*}(X,Y)\cdot Z^{*}=0$

In this section we considered Sasakian manifold satisfying the following condition

$$R^{*}(X,Y)\cdot Z^{*}(U,V)=0, \tag{58}$$

for any vector fields X , Y , U and V , where the $Z^{*}(U,V)$ is a Z-tensor with respect to general connection which is defined [9] by

$$Z^{*}(U,V)=S^{*}(U,V)-\frac{r^{*}}{2}g(U,V). \tag{59}$$

Taking account of (25) and (30) we get from (59)

$$\begin{aligned} & Z^{*}(U,V) \\ = \;& S(U,V)-\overline{A}g(U,V)+\overline{B}\eta(U)\eta(V) \\ & -\frac{r}{2}g(U,V)+\frac{1}{2}\overline{A}ng(U,V)-\frac{1}{2}\overline{B}g(U,V) \end{aligned} \tag{60}$$

Equation (58) can be written as

$$Z^{*}(R^{*}(X,Y)U,V)+Z^{*}(U,R^{*}(X,Y)V)=0. \tag{61}$$

Putting $X=U=\xi$ in (61), we get

$$Z^{*}(R^{*}(\xi,Y)\xi,V)+Z^{*}(\xi,R^{*}(\xi,Y)V)=0. \tag{62}$$

By the help of (31), (32), (33) we get the following

$$Z^{*}(R^{*}(\xi,Y)\xi,V)=\overline{C}\left[Z^{*}(Y,V)-\eta(Y)Z^{*}(\xi,V)\right] \tag{63}$$

$$Z^{*}(\xi,R^{*}(\xi,Y)V)=\overline{C}\left[\eta(V)Z^{*}(\xi,Y)-g(Y,V)Z^{*}(\xi,\xi)\right] \tag{64}$$

Using (63), (64) in (62) we obtain

$$0=\overline{C}\left[Z^{*}(Y,V)-\eta(Y)Z^{*}(\xi,V)+\eta(V)Z^{*}(\xi,Y)-g(Y,V)Z^{*}(\xi,\xi)\right] \tag{65}$$

Taking account of (4), (6) and (25), we get from (65)

$$S(Y,V)=\left[(n-1)+\overline{B}\right]g(Y,V)-\overline{B}\eta(Y)\eta(V). \tag{66}$$

This leads the following theorem.

Theorem 9. *A n-dimensional Sasakian manifold M admitting the general connction satisfying $R^{*}(X,Y).Z^{*}=0$, is an η- Einstein.*

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