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ON THE DISTRIBUTION OF ZEROS OF BICOMPLEX VALUED ENTIRE FUNCTIONS IN A CERTAIN DOMAIN

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Abstract. Bicomplex algebra is a modern developed area which is a generalization of the field of complex numbers. In this paper we derive some results related to the distribution of zeros of bicomplex valued entire functions in a certain domain. A few examples with related figures are given here to justify the results obtained.

1. Introduction

Bicomplex numbers which are the commutative generalization of complex numbers were first introduced by Segre (cf, [5]). Standard definitions, notations and many more properties of bicomplex numbers are available in [2] and [6]. A bicomplex entire function $f(z)$ is also represented by an everywhere convergent power series as $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$, where α_j's and z are bicomplex numbers. Thus, bicomplex entire functions can be thought of the natural generalization of bicomplex polynomials. The aim of the paper is to establish some results concerning the distribution of zeros of bicomplex entire functions in a certain domain.

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2. Preliminary Definitions and Notations

In this section we give some basic idea about bicomplex numbers.

The set of bicomplex numbers $\mathbb{C}_2$ is defined by $\mathbb{C}_2 = \{z : z = a_0 + ia_1 + ja_2 + ka_3, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$ or equivalently $\mathbb{C}_2 = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}_1\}$, where $\mathbb{C}_1$ is the set of complex numbers with imaginary unit i such that $i^2 = j^2 = -k^2 = -1$ and $ij = ji = k$.

2.1. **Idempotent Representation.** One of the important features of a bicomplex number is its idempotent representation . The bicomplex numbers $e_1 := \frac{1+ij}{2}, e_2 := \frac{1-ij}{2}$ are linearly independent in the $\mathbb{C}_1$-linear space $\mathbb{C}_2$ and $e_1 + e_2 = 1, e_1 - e_2 = ij, e_1.e_2 = 0, e_1^2 = e_1, e_2^2 = e_2$. Any $z \in \mathbb{C}_2$ can be uniqely expressed as $z = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$, which is known as the idempotent representation of z.

2.2. **Norm.** The norm $\|\| : \mathbb{C}_2 \to \mathbb{R}^+$ ($\mathbb{R}^+$denote the set of all non negative real numbers) is defined as follows:
If $z = z_1 + jz_2 = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2$, then

$$\|z\| = \{|z_1|^2 + |z_2|^2\}^{\frac{1}{2}} = \left\{\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right\}^{\frac{1}{2}}.$$

2.3. **Auxiliary Complex Spaces.** The spaces $\boldsymbol{A_1} = \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}_1\}$ and $\boldsymbol{A_2} = \{z_1 + iz_2 : z_1, z_2 \in \mathbb{C}_1\}$ are called the auxiliary complex spaces. Each point $z_1 + jz_2 = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2$ in $\mathbb{C}_2$ associates the points $z_1 - iz_2 \in A_1$ and $z_1 + iz_2 \in A_2$. Also to each pair of points $(z_1 - iz_2, z_1 + iz_2) \in A_1 \times A_2$ there is a uniqe point in $\mathbb{C}_2$.

2.4. $\mathbb{C}_2$**-Open Discus.** An open discus $D(\xi; r_1, r_2)$ with centre $\xi = \xi_1 e_1 + \xi_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined by

$$D(\xi; r_1, r_2) = \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - \xi_1| < r_1, |\omega_2 - \xi_2| < r_2\}.$$

2.5. $\mathbb{C}_2$**-Closed Discus.** A closed discus $\bar{D}(\xi; r_1, r_2)$ with centre $\xi = \xi_1 e_1 + \xi_2 e_2$ and radii $r_1 > 0, r_2 > 0$ is defined as

$$\bar{D}(\xi; r_1, r_2) = \{\omega_1 e_1 + \omega_2 e_2 \in \mathbb{C}_2 : |\omega_1 - \xi_1| \leq r_1, |\omega_2 - \xi_2| \leq r_2\}.$$

2.6. $\mathbb{C}_2$**-Disc.** If $r_1 > 0$, $r_2 > 0$ and $r_1 = r_2 = r$, then the discus is called a disc in $\mathbb{C}_2$ and is denoted by $D(\xi; r, r) = D(\xi; r)$.

3. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.1. [1] *If $f(z)$ is holomorphic in $\mid z \mid\leq R$ in $\mathbb{C}_1, f(0) = 0, f'(0) = b$, and $\mid f(z) \mid\leq M$ for $\mid z \mid= R$, then for $\mid z \mid\leq R$,*

$$\mid f(z) \mid\leq \frac{M \mid z \mid}{R^2} . \frac{M \mid z \mid +R^2 \mid b \mid}{M+ \mid b \mid\mid z \mid}.$$

The following lemma is due to Schwarz in $\mathbb{C}_1$.

Lemma 3.2. [4] *If $g(z)$ is holomorphic in $\mid z \mid\leq R$ in $\mathbb{C}_1, g(0) = 0$ and $\mid g(z) \mid\leq M$ for $\mid z \mid= R$, then*

$$|g(z)| \leq \frac{M|z|}{R}.$$

Lemma 3.3. [3] *Let $f(z)$ be holomorphic for $\mid z \mid< R$ in $\mathbb{C}_1$. Suppose $f(0) \neq 0$ and let $r_1, r_2, ..., r_n, ...$ be the moduli of the zeros of $f(z)$ in $\mid z \mid< R$ arranged as a non decreasing sequence. If $r_n \leq r \leq r_{n+1}$, then*

$$\log \frac{r^n \mid f(0) \mid}{r_1 r_2 ... r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log \mid f(re^{i\theta}) \mid d\theta,$$

where a zero of order p is counted p times.

Remark 3.1. *Lemma 3.3 is known as Jensen's Theorem in $\mathbb{C}_1$.*

Lemma 3.4. [5] *Let $X = X_1e_1 + X_2e_2 := \{\xi_1 e_1 + \xi_2 e_2 : \xi_1 \in X_1, \xi_2 \in X_2\}$ be a domain in $\mathbb{C}_2$. A bicomplex function $F = G_1e_1 + G_2e_2 : X \to \mathbb{C}_2$ is holomorphic if and only if both the component function G_1 and G_2 are holomorphic in X_1 and X_2 respectively .*

Lemma 3.5. [5] *Let F be a bicomplex holomorphic function defined in a domain $X = X_1e_1 + X_2e_2 := \{\xi_1 e_1 + \xi_2 e_2 : \xi_1 \in X_1, \xi_2 \in X_2\}$ such that $F(z) = G_1(\xi_1)e_1 + G_2(\xi_2)e_2$, for all $z = \xi_1 e_1 + \xi_2 e_2 \in X$. Then, $F(z)$ has zero in X if and only if $G_1(\xi_1)$ and $G_2(\xi_2)$ both have zero at ξ_1 in X_1 and at ξ_2 in X_2 respectively.*

Lemma 3.6. *Let $F(z) = G_1(\xi_1)e_1 + G_2(\xi_2)e_2$ be a bicomplex holomorphic function with $\|F(\mathbf{0})\| \neq 0$ and $\|F(z)\| \leq M(r_1, r_2)$ for all $z \in \bar{D}(0; r_1, r_2)$. Then the number of zeros $N_1(\frac{r_1}{2})$ of $G_1(\xi_1)$ in the domain $\{\xi_1 \in A_1 : |\xi_1| < \frac{r_1}{2}\}$ and $N_2(\frac{r_1}{2})$ of $G_2(\xi_2)$ in $\{\xi_2 \in A_2 : |\xi_2| < \frac{r_2}{2}\}$*

do not exceed respectively

$$\frac{1}{\log 2}\left\{\log\frac{\sqrt{2}M(r_1,r_2)}{|G_1(0)|}\right\} \;\; \textit{and} \; \frac{1}{\log 2}\left\{\log\frac{\sqrt{2}M(r_1,r_2)}{|G_2(0)|}\right\}.$$

Proof. In view of Lemma 3.4, $G_1(\xi_1)$ and $G_2(\xi_2)$ are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 : |\xi_1| \leq r_1\}$ and $X_2 = \{\xi_2 \in A_2 : |\xi_2| \leq r_2\}$.

Since

$$\frac{G_1(\xi_1)}{\sqrt{2}} \leq \left\{\frac{G_1(\xi_1)^2 + G_2(\xi_2)^2}{2}\right\}^{\frac{1}{2}} = \|F(z)\| \leq M(r_1,r_2)$$

for $z = \xi_1 e_1 + \xi_2 e_2 \in \bar{D}(0; r_1, r_2)$,

$$|\; G_1(\xi_1)\; | \leq \sqrt{2}M(r_1,r_2) \text{ for } \xi_1 \in X_1.$$

Let $\xi_{11}, \xi_{12}, ..., \xi_{1n}$ be n zeros of $G_1(\xi_1)$ such that $|\xi_{11}| \leq |\xi_{12}| \leq ... \leq |\xi_{1n}| < r_1$.

Then by Lemma 3.3,

$$\begin{aligned} N_1(\frac{r_1}{2})\log 2 &\leq \sum_{i=1}^{N_1(\frac{r_1}{2})} \log\frac{r_1}{|\;\xi_{1i}\;|} \\ &\leq \sum_{i=1}^{N_1(r_1)} \log\frac{r_1}{|\;\xi_{1i}\;|} \\ &\leq \log\sqrt{2}M(r_1,r_2) - \log|G_1(0)|, \end{aligned}$$

i.e,

$$N_1(\frac{r_1}{2}) \leq \frac{1}{\log 2}\left\{\log\frac{\sqrt{2}M(r_1,r_2)}{|G_1(0)|}\right\}.$$

Similarly,

$$N_2(\frac{r_2}{2}) \leq \frac{1}{\log 2}\left\{\log\frac{\sqrt{2}M(r_1,r_2)}{|G_2(0)|}\right\}.$$

This completes the proof of the lemma.

$\square$

4. Theorems

In this section we present the main results of the paper.

Theorem 4.1. *Let* $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *be an entire function with* $\| f(\mathbf{0}) \| \neq 0$, $\alpha_k = a_k e_1 + b_k e_2, k = 0, 1, 2, ..$ *and* $z = \xi_1 e_1 + \xi_2 e_2$. *Let* N_1, N_2 *be the respective highest positive integers less than or equal to* $N_1(r_1)$ *in* $\{\xi_1 \in A_1 :\mid \xi_1 \mid < r_1\}$ *and* $N_2(r_2)$ *in* $\{\xi_2 \in A_2 :\mid \xi_2 \mid < r_2\}$ *such that* $a_{N_1} \neq 0$, $a_{N_2} \neq 0$. *Then within the open discus* $D(0; r_1, r_2)$, $f(z)$ *does not vanish in the open discus* $D(0; t_1, t_2)$ *where* t_1 , t_2 *are respectively the least positive roots of the equations*

$$g_1(t) \equiv \mid a_0 \mid r_1^{N_1+1} - (\mid a_0 \mid + \sqrt{2} N_1 M(r_1, r_2)) r_1^{N_1} t + \\ + \sqrt{2} N_1 M(r_1, r_2) r_1^{N_1 - 1} t^2 - \sqrt{2} M(r_1, r_2) t^{N_1+1} = 0$$

and

$$g_2(t) \equiv \mid b_0 \mid r_2^{N_2+1} - (\mid b_0 \mid + \sqrt{2} N_2 M(r_1, r_2)) r_2^{N_2} t + \\ + \sqrt{2} N_2 M(r_1, r_2) r_2^{N_2 - 1} t^2 - \sqrt{2} M(r_1, r_2) t^{N_2+1} = 0,$$

$$\max_{z \in \bar{D}(0; r_1, r_2)} \| f(z) \| \leq M(r_1, r_2).$$

Proof. As $f(z)$ can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k \xi_1^k e_1 + \sum_{k=0}^{\infty} b_k \xi_2^k e_2 = f_1(\xi_1) e_1 + f_2(\xi_2) e_2,$$

clearly $f(z)$ is holomorphic in the closed discus $\bar{D}(0; r_1, r_2), 0 < r_1 < \infty, 0 < r_2 < \infty$. Hence in view of Lemma 3.4 , $f_1(\xi_1)$ and $f_2(\xi_1)$ are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :\mid \xi_1 \mid \leq r_1\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :\mid \xi_2 \mid \leq r_2\} \subset \mathbb{C}_1$.

Now for $\mid \xi_1 \mid < r_1$,

$$\begin{aligned} f_1(\xi_1) &= \sum_{k=0}^{\infty} a_k \xi_1^k \\ &= a_0 + a_1 \xi_1 + a_2 \xi_1^2 + ... + a_{N_1} \xi_1^{N_1} + \sum_{k=N_1+1}^{\infty} a_k \xi_1^k \\ &= a_0 + G(\xi_1) + H(\xi_1) \qquad (1) \end{aligned}$$

where $G(\xi_1) = a_1 \xi_1 + a_2 \xi_1^2 + ... + a_{N_1} \xi_1^{N_1}$ and $H(\xi_1) = \sum_{k=N_1+1}^{\infty} a_k \xi_1^k$.

Since

$$\|f(z)\| = \left\{\frac{f_1(\xi_1)^2 + f_2(\xi_2)^2}{2}\right\}^{\frac{1}{2}} \geq \frac{f_1(\xi_1)}{\sqrt{2}},$$

we have for the coefficients of the power series $\sum_{k=0}^{\infty} a_k \xi_1^k$ in $\mid \xi_1 \mid \leq r_1$,

$$\mid a_k \mid \leq \frac{\max\limits_{|\xi_1|=r_1} \mid f_1(\xi_1) \mid}{r_1^k} \leq \frac{\sqrt{2}M(r_1, r_2)}{r_1^k}\ . \tag{2}$$

Hence for $\mid \xi_1 \mid = r_1$, by using (2) it follows that

$$\begin{aligned}
&\mid G(\xi_1) \mid \leq \mid a_1 \mid\mid \xi_1 \mid + \mid a_2 \mid\mid \xi_1 \mid^2 + ... + \mid a_{N_1} \mid\mid \xi_1 \mid^{N_1} \\
&\leq \mid a_1 \mid r_1 + \mid a_2 \mid r_1^2 + ... + \mid a_{N_1} \mid r_1^{N_1} \\
&\leq \sqrt{2}N_1 M(r_1, r_2).
\end{aligned}$$

Since $G(\xi_1)$ is holomorphic in $\mid \xi_1 \mid \leq r_1$, $G(0) = 0$ and $\mid G(\xi_1) \mid \leq \sqrt{2}N_1 M(r_1, r_2)$ for $\mid \xi_1 \mid = r_1$, by Lemma 3.2, we get that

$$\mid G(\xi_1) \mid \leq \frac{\sqrt{2}N_1 M(r_1, r_2) \mid \xi_1 \mid}{r_1}\ . \tag{3}$$

Also for $\mid \xi_1 \mid < r_1$, by using (2) we have

$$\begin{aligned}
\mid H(\xi_1) \mid &= \mid \sum_{k=N_1+1}^{\infty} a_k \xi_1^k \mid \\
&\leq \sum_{k=N_1+1}^{\infty} \mid a_k \mid\mid \xi_1 \mid^k \\
&\leq \sqrt{2}M(r_1, r_2) \sum_{k=N_1+1}^{\infty} \left(\frac{\mid \xi_1 \mid}{r_1}\right)^k \\
&\leq \sqrt{2}M(r_1, r_2) \frac{\left(\frac{\xi_1}{r_1}\right)^{N_1+1}}{1 - \frac{\xi_1}{r_1}} \\
&= \frac{\sqrt{2}M(r_1, r_2) \mid \xi_1 \mid^{N_1+1}}{r_1^{N_1}(r_1 - \mid \xi_1 \mid)}\ . \qquad (4)
\end{aligned}$$

Hence for $|\xi_1| < r_1$, it follows from (1) by (3) and (4) that

$$
\begin{aligned}
f_1(\xi_1) &\geq |a_0| - |G(\xi_1)| - |H(\xi_1)| \\
&\geq |a_0| - \frac{\sqrt{2}N_1 M(r_1,r_2)|\xi_1|}{r_1} - \frac{\sqrt{2}M(r_1,r_2)|\xi_1|^{N_1+1}}{r_1^{N_1}(r_1 - |\xi_1|)}
\end{aligned}
$$

$$
= \frac{\begin{array}{c}|a_0| r_1^{N_1+1} - (|a_0| + \sqrt{2}N_1 M(r_1,r_2)) r_1^{N_1} |\xi_1| \\ + \sqrt{2}N_1 M(r_1,r_2) r_1^{N_1-1} |\xi_1|^2 - \sqrt{2}M(r_1,r_2)|\xi_1|^{N_1+1}\end{array}}{r_1^{N_1}(r_1 - |\xi_1|)}.
$$

Let

$$
\begin{aligned}
g_1(t) \equiv{}& |a_0| r_1^{N_1+1} - (|a_0| + \sqrt{2}N_1 M(r_1,r_2)) r_1^{N_1} t + \\
&+ \sqrt{2}N_1 M(r_1,r_2) r_1^{N_1-1} t^2 - \sqrt{2}M(r_1,r_2) t^{N_1+1}
\end{aligned}
$$

We see that the number changes in sign in the coefficients of $g_1(t)$ is 3. Hence by Descarte's rule of sign, the number of positive real roots of $g_1(t) = 0$ will be either 3 or 1.

Let us consider t_1 be the least positive root of the equation $g_1(t) = 0$.

Since $g_1(0) = |a_0| r^{N_1+1} > 0$ and $g_1(\infty) = -\infty < 0$, $g_1(t) > 0$ if $t < t_1$, otherwise there will be another positive root in $(0, t_1)$, which makes a contradiction. Hence for $|\xi_1| < r_1$, $|f_1(\xi_1)| > 0$ if $|\xi_1| < t_1$.

Similarly for $|\xi_2| < r_2$, $|f_2(\xi_2)| > 0$ if $|\xi_2| < t_2$ where t_2 is the least positive root of the equation

$$
\begin{aligned}
g_2(t) \equiv{}& |b_0| r_2^{N_2+1} - (|b_0| + \sqrt{2}N_2 M(r_1,r_2)) r_2^{N_2} t + \\
&+ \sqrt{2}N_2 M(r_1,r_2) r_2^{N_2-1} t^2 - \sqrt{2}M(r_1,r_2) t^{N_2+1} = 0.
\end{aligned}
$$

Therefore both $f_1(\xi_1)$ and $f_2(\xi_2)$ have no zeros respectively in $X_1' = \{\xi_1 \in X_1 : |\xi_1| < t_1\}$ and $X_2' = \{\xi_2 \in X_2 : |\xi_2| < t_2\}$.

Hence by Lemma 3.5, $f(z)$ has no zero in $X_1'e_1 + X_2'e_2 = D(0; t_1, t_2)$.

This proves the theorem.

□

Remark 4.1. *The following example with related figure justifies the validity of Theorem 4.1.*

Example 4.1. *Let* $f(z) = cos(z) = cos(\xi_1)e_1 + cos(\xi_2)e_2$.

Here, $f_1(\xi_1) = cos(\xi_1) = 1 - \frac{1}{2!}\xi_1^2 + \frac{1}{4!}\xi_1^4 - ...$ *and* $f_2(\xi_2) = cos(\xi_2) = 1 - \frac{1}{2!}\xi_2^2 + \frac{1}{4!}\xi_2^4 - ....$

For $r_1 = r_2 = 1$, *by Lemma 3.6* , $N_1(r_1) \leq 2.41, N_2(r_2) \leq 2.41.$

Since $a_2 \neq 0, b_2 \neq 0$ *, we may take* $N_1 = N_2 = 2$.
Now, $\|f(z)\| = \left\{\frac{|cos(\xi_1)|^2+|cos(\xi_2)|^2}{2}\right\}^{\frac{1}{2}} \leq \frac{e+e^{-1}}{2}$, *for all* $z \in \bar{D}(0;1,1)$.
Hence,

$$\begin{aligned} g_1(t) \equiv & | a_0 | r_1^{N_1+1} - (| a_0 | + \sqrt{2}N_1M(r_1,r_2))r_1^{N_1}t + \\ & + \sqrt{2}N_1M(r_1,r_2)r_1^{N_1-1}t^2 - \sqrt{2}M(r_1,r_2)t^{N_1+1} \\ = & 1 - (1+\sqrt{2}(e+e^{-1}))t + \sqrt{2}(e+e^{-1})t^2 - \sqrt{2}\cdot\tfrac{e+e^{-1}}{2}t^3 \end{aligned}$$

and

$$\begin{aligned} g_2(t) \equiv & | b_0 | r_2^{N_2+1} - (| b_0 | + \sqrt{2}N_2M(r_1,r_2))r_2^{N_2}t + \\ & + \sqrt{2}N_2M(r_1,r_2)r_2^{N_2-1}t^2 - \sqrt{2}M(r_1,r_2)t^{N_2+1} \\ = & 1 - (1+\sqrt{2}(e+e^{-1}))t + \sqrt{2}(e+e^{-1})t^2 - \sqrt{2}\cdot\tfrac{e+e^{-1}}{2}t^3. \end{aligned}$$

We see that $g_1(.22) > 0$, $g_1(.23) < 0$ *and* $g_1(t) > 0$ *for* $0 \leq t \leq .22$. *Hence the positive root of* $g_1(t)$ *lies between* .22 *and* .23.

Similarly the least positive root of $g_2(t)$ *lies between* .22 *and* .23.

Hence by Theorem 4.1, $f(z)$ *has no zeros in* $\bar{D}(0;.22,.22)$ *within the open discus* $D(0;1,1)$.

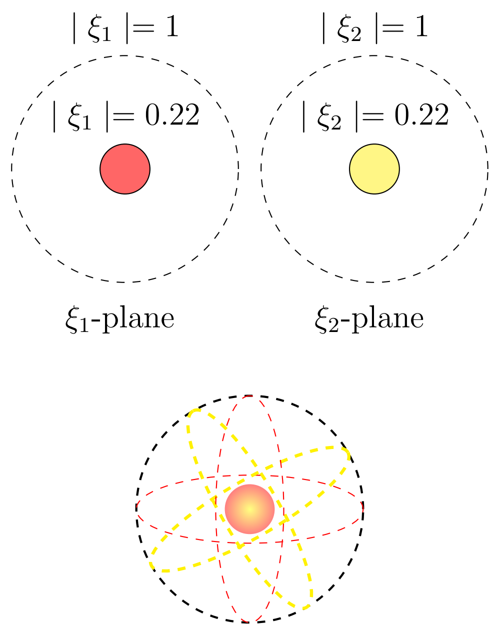

FIGURE 1. Zero free region of $f(z) = cos(z)$ in $D(0;1,1)$

Theorem 4.2. *Let* $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *be an entire function with* $\|f(\mathbf{0})\| \neq 0, \alpha_k = a_k e_1 + b_k e_2, k = 0, 1, 2, ..$ *and* $z = \xi_1 e_1 + \xi_2 e_2$. *Then within any open discus* $D(0; r_1, r_2)$, $f(z)$ *has no zero in the open discus* $D(0; t_1, t_2)$ *where* t_1 *and* t_2 *are respectively the positive roots of the equation*

$$|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$$

and

$$|b_0|(|b_0| + 2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0| + 2C)^2 t^2 = 0,$$

where

$$B = \sum_{k=1}^{\infty} |a_k| r_1^k \ , \ C = \sum_{k=1}^{\infty} |b_k| r_2^k .$$

Proof. As $f(z)$ can be expressed as

$$f(z) = \sum_{k=0}^{\infty} a_k \xi_1^k e_1 + \sum_{k=0}^{\infty} b_k \xi_2^k e_2 = f_1(\xi_1)e_1 + f_2(\xi_2)e_2,$$

$f(z)$ being holomorphic in $\mathbb{C}_2$, by Lemma 3.4, $f_1(\xi_1)$ and $f_2(\xi_2)$ both are holomorphic respectively in $X_1 = \{\xi_1 \in A_1 :| \xi_1 |\leq r_1\} \subset \mathbb{C}_1$ and $X_2 = \{\xi_2 \in A_2 :| \xi_2 |\leq r_2\} \subset \mathbb{C}_1$.

Clearly, $\lim_{k\to\infty} a_k r_1^k = 0$ and $\lim_{k\to\infty} b_k r_2^k = 0$.

Let us consider

$$F(\xi_1) = (\xi_1 - r_1) f_1(\xi_1),$$
$$\text{i.e, } F(\xi_1) = (\xi_1 - r_1) \textstyle\sum_{k=0}^{\infty} a_k \xi_1^k$$
$$\text{i.e, } F(\xi_1) = -a_0 r_1 + \textstyle\sum_{k=1}^{\infty} (a_{k-1} - r_1 a_k) \xi_1^k$$
$$\text{i.e, } F(\xi_1) = -a_0 r_1 + G(\xi_1).$$

Then for $|\xi_1| < r_1$,

$$|F(\xi_1)| \geq |a_0| r_1 - |G(\xi_1)| \tag{5}$$

For $|\xi_1| = r_1$ and as because the series $\sum_{k=1}^{\infty}|a_k|r_1^k$ converges for $|\xi_1| \leq r_1$ we have

$$\begin{aligned}
|G(\xi_1)| =& |\sum_{k=1}^{\infty}(a_{k-1} - r_1 a_k)\xi_1^k| \\
\leq& \sum_{k=1}^{\infty}|a_{k-1} - r_1 a_k||\xi_1|^k \\
\leq& \sum_{k=1}^{\infty}(|a_{k-1}| + r_1|a_k|)r_1^k \\
=& |a_0|r_1 + 2r_1\sum_{k=1}^{\infty}|a_k|r_1^k \\
=& (|a_0| + 2B)r_1,
\end{aligned}$$

where

$$B = \sum_{k=1}^{\infty}|a_k|r_1^k.$$

Since $G(\xi_1)$ is analytic in $|\ \xi_1\ | \leq r_1$, $G(0) = 0, G'(0) = (a_0 - r_1 a_1)$ and $|\ G(\xi_1)\ | \leq (|a_0| + 2B)r_1$ for $|\ \xi_1\ | = r_1$ and in view of Lemma 3.1, it follows for $|\xi_1| \leq r_1$ that

$$\begin{aligned}
|\ G(\xi_1)\ | \leq& \frac{(|a_0| + 2B)r_1|\xi_1|}{r_1^2}\cdot\frac{(|a_0| + 2B)r_1|\xi_1| + r_1^2|a_0 - r_1 a_1|}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \\
=& \frac{(|a_0| + 2B)|\xi_1|\{(|a_0| + 2B)|\xi_1| + r_1|a_0 - r_1 a_1|\}}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} .
\end{aligned}$$

Hence for $|\xi_1| < r_1$, we obtain from (5) that

$$\begin{aligned}
|F(\xi_1)| \geq& |a_0|r_1 - \frac{(|a_0| + 2B)|\xi_1|\{(|a_0| + 2B)|\xi_1| + r_1|a_0 - r_1 a_1|\}}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} \\
=& \frac{|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1||\xi_1| - (|a_0| + 2B)^2|\xi_1|^2}{(|a_0| + 2B)r_1 + |a_0 - r_1 a_1||\xi_1|} .
\end{aligned}$$

Clearly the equation

$$g(t) \equiv |a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$$

has exactly one positive root.

Let the positive root of the equation be t_1.

Since $g(0) = |a_0|(|a_0| + 2B)r_1^2 > 0$, $g(t) > 0$ for $t < t_1$.

Hence for $|\xi_1| < r_1$, $|F(\xi_1)| > 0$, i.e. $|f_1(\xi_1)| > 0$ if $|\xi_1| < t_1$.

Similarly for $|\xi_2| < r_2$, $|f_2(\xi_2)| > 0$ if $|\xi_2| < t_2$ where t_2 is the positive root of the equation

$$|b_0|(|b_0| + 2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0| + 2C)^2 t^2 = 0$$

where

$$C = \sum_{k=1}^{\infty} |b_k| r_2^k.$$

Therefore both $f_1(\xi_1)$ and $f_2(\xi_2)$ have no zeros respectively in

$$X_1' = \{\xi_1 \in X_1 :| \xi_1 |< t_1\} \subset A_1$$

and

$$X_2' = \{\xi_2 \in X_2 :| \xi_2 |< t_2\} \subset A_2.$$

Consequently by Lemma 3.5, $f(z)$ has no zeros in

$$X_1' e_1 + X_2' e_2 = D(0; t_1, t_2).$$

Thus the theorem is established.

□

Remark 4.2. *The following example with related figure ensures the validity of Theorem 4.2.*

Example 4.2. *Let* $f(z) = -(1-j) + z + (2+ji)z^2$.

Here,

$$\begin{aligned} \alpha_0 &= -(1-j) = (-1-i)e_1 + (-1+i)e_2 = a_0 e_1 + b_0 e_2, \\ \alpha_1 &= 1 = 1e_1 + 1e_2 = a_1 e_1 + b_1 e_2, \\ \alpha_2 &= (2+ji) = (2 - i.i)e_1 + (2 + i.i)e_2 = 3e_1 + 1e_2 = a_2 e_1 + b_2 e_2. \end{aligned}$$

Now, $f(z)$ *can be written as*

$$f(z) = (3\xi_1^2 + \xi_1 - 1 - i)e_1 + (\xi_2^2 + \xi_2 - 1 + i)e_2$$

For $r_1 = r_2 = 1$, *we get* $B = 4, C = 2$.

Hence the equation

$$|a_0|(|a_0| + 2B)r_1^2 - 2B|a_0 - r_1 a_1|t - (|a_0| + 2B)^2 t^2 = 0$$

becomes

$$\sqrt{2}(\sqrt{2} + 8) - 8\sqrt{5}t - (\sqrt{2} + 8)^2 t^2 = 0$$

and the positive root $t_1 \approx 0.30$.

Again the equation

$$|b_0|(|b_0|+2C)r_2^2 - 2C|b_0 - r_2 b_1|t - (|b_0|+2C)^2 t^2 = 0$$

reduces to

$$\sqrt{2}(\sqrt{2}+4) - 4\sqrt{5}t - (\sqrt{2}+4)^2 t^2 = 0$$

and the positive root $t_2 \approx 0.38$.

Hence by Theorem 4.2, $f(z)$ *has no zeros in* $D(0; .30, .38)$ *within the open discus* $D(0; 1, 1)$.

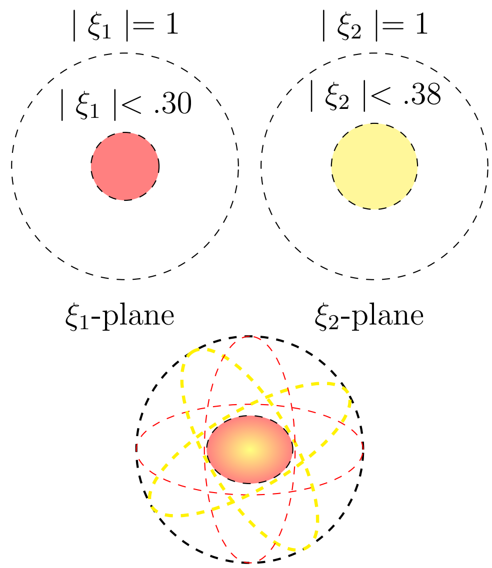

FIGURE 2. Zero free region of $f(z) = -(1-j) + z + (2+ji)z^2$ in $D(0; 1, 1)$

Future prospect. In the line of the works as carried out in the paper one may think of the extension of the results obtained dealing with n-dimensional bicomplex numbers with the help of the idempotents $0, 1, \dfrac{1 \pm i_1 i_2}{2}, \dfrac{1 \pm i_2 i_3}{2}, ..., \dfrac{1 \pm i_{n-1} i_n}{2}$ in $\mathbb{C}_n$. As a consequence, the problem of taking the coeffiecients of the power series in $\mathbb{C}_n$ is still virgin and may be considered as an open problem to the future workers of this branch.

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