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# ON THE GROWTH OF COMPOSITE ENTIRE FUNCTIONS WITH FINITE ITERATED LOGARITHMIC ORDER

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**Abstract.**In this article we studied some growth properties of composite entire functions with finite iterated logarithmic order. Also we defined iterated logarithmic order of an entire function by using their maximum term. Further, we proved some results on the growth of composite entire functions of finite iterated logarithmic order in terms of their maximum terms.

## 1. Introduction

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the maximum modulus of $f(z)$ is defined by $M_f(r) = \max\{|f(z)| : |z| \leq r\}$ for $r > 0$. It follows immediately that $M_f(r)$ is nondecreasing function of $r$. The maximum term $\mu_f(r)$ of the function $f(z)$ on $|z| = r$ is defined as $\mu_f(r) = \max\limits_{n\geq 0} |a_n| r^n$.

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We recall the order $\rho(f)$ and lower order $\lambda(f)$ of an entire function $f(z)$ which are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

respectively.

Also by Nevanlinna theory [4] , one get the order $\rho(f)$ and lower order $\lambda(f)$ of $f(z)$ as

$$\begin{aligned} \rho(f) &= \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \\ \lambda(f) &= \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} \end{aligned}$$

where $T_f(r)$ is the Nevanlinna's characteristic function.

Now it is already known [2] that for any two transcendental entire functions $f(z)$ and $g(z)$,

$$\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_f(r)} = \infty$$

and

$$\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_g(r)} = \infty.$$

There are so many results that have been proved on the composition of two entire functions with finite order ([2],[5],[6],[8],[9],[13]).

**Definition 1.** [1] *Let $S(r)$ $(r > 0)$ be a nonnegative increasing function of order zero is said to have finite logarithmic order $\rho_{\log}$ if*

$$\rho_{\log} = \limsup_{r\to\infty} \frac{\log S(r)}{\log\log r}.$$

If $f(z)$ is an entire series in the complex plane $\mathbb{C}$ then the logarithmic order of $\log^+ M_f(r)$ is equal to the logarithmic order of $f$.

If $f$ is a transcendental with finite logarithmic order $\rho_{\log}$ then its lower logarithmic order

$$\lambda_{\log} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\log r}.$$

One can easily check that $\rho_{\log} < \lambda_{\log} + 1$ and there is a constant $c$ satisfying $0 \leq c < \rho_{\log} - \lambda_{\log}$.

In other words for a transcendental entire function $f$ with order zero we can define $\rho_{\log}(f)$ and $\lambda_{\log}(f)$ as follows:

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\log r}, \tag{1}$$

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log\log r}. \tag{2}$$

**Definition 2.** [10] *For $0 \leq r < R$,*

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R). \tag{3}$$

*Using this result we get*

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r} \tag{4}$$

*and*

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log\log \mu_f(r)}{\log\log r}. \tag{5}$$

**Definition 3.** [12] *The iterated $p$ order $\rho_p(f)$ of an entire function $f$ as*

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log_p T_f(r)}{\log r} (p \in \mathbb{N}). \tag{6}$$

*Similarly, the iterated $p$ lower order $\lambda_p(f)$ of an entire function $f$ as*

$$\lambda_p(f) = \liminf_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log r} = \liminf_{r\to\infty} \frac{\log_p T_f(r)}{\log r} (p \in \mathbb{N}). \tag{7}$$

**Definition 4.** [12] *The finiteness degree of the order of an entire function $f$ is defined by*

$$i(f) = \begin{cases} 0 & \text{when } f \text{ is a polynomial,} \\ \min\{q \in \mathbb{N} : \rho_q(f) < \infty\} & \text{for } f \text{ transcendental for which some } q \in \mathbb{N} \text{ with } \rho_q(f) < \infty \text{ exists.} \\ \infty & \text{for } f \text{ with } \rho_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

It is easily seen that $i(f)$ and $i(g)$ are positive integers.

We use the notations $\exp_1 r = e^r, \exp_{i+1} r = \exp(\exp_i r)$ for $0 \leq r < \infty$ and $i = 1, 2, \ldots$ . Also for sufficiently large $r$, we use the notations $\log_1 r = \log r, \log_{i+1} r = \log(\log_i r)$ for $i = 1, 2, \ldots$ .

In this paper we established some results of composite entire functions on the basis of iterated logarithmic order.To prove these results we use some known lemmas which are stated in the following section .

## 2. Preliminary Lemmas

In this section we shall present first the following known lemmas.

**Lemma 5.** [11] *If $f(z)$ and $g(z)$ are two entire functions with $M_g(r) > \frac{2+\varepsilon}{\varepsilon}|g(0)|$ for any $\varepsilon > 0$, then*

$$T_{f \circ g}(r) \leq (1+\varepsilon) T_f(M_g(r)). \tag{8}$$

In particular if $g(0) = 0$, then for all $r > 0$

$$T_{f \circ g}(r) \leq T_f(M_g(r)). \tag{9}$$

**Lemma 6.** [11] *Let $\lambda(g) < \infty$. Then for any $\varepsilon > 0$ and sufficiently large $r$,*

$$M_{f \circ g}\left(r^{1+\varepsilon}\right) \geq M_f(M_g(r)). \tag{10}$$

**Lemma 7.** [2] *If $f(z)$ and $g(z)$ are two entire functions with $g(0) = 0$, then*

$$M_{f \circ g}(r) \geq M_f(c(\alpha) M_g(\alpha r)). \tag{11}$$

where $\alpha$ satisfy $0 < \alpha < 1$ and take $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Then for $r > 0$

Further if $g(z)$ is any entire function then with $\alpha = \frac{1}{2}$, for sufficiently large values of $r$,

$$M_{f \circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right) - |g(0)|\right). \tag{12}$$

Also from the definition it follows immediately that

$$M_{f \circ g}(r) \leq M_f(M_g(r)) \tag{13}$$

**Lemma 8.** [10] *Let $f(z)$ and $g(z)$ be entire functions, then for $\alpha > 1$, and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R - r} \mu_g(r)\right).$$

*In particular taking $\alpha = 2$ and $R = 2r$,*

$$\mu_{f\circ g}(r) \leq 2\mu_f(4\mu_g(2r)) \tag{14}$$

**Lemma 9.** [10] *Let $f(z)$ and $g(z)$ be entire functions with $g(0) = 0$. Let $\alpha$ satisfy $0 < \alpha < 1$ and let $c(\alpha) = \frac{(1-\alpha)^2}{4\alpha}$. Also let $0 < \delta < 1$ then*

$$\mu_{f\circ g}(r) \geq (1-\delta)\,\mu_f(c(\alpha)\mu_g(\alpha\delta r)).$$

*And if $g(z)$ is any entire function, then with $\alpha = \delta = \frac{1}{2}$, for sufficiently large values of $r$,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right). \tag{15}$$

**Lemma 10.** [3] *Suppose that $f(z)$ and $g(z)$ are entire functions of finite iterated order. Then for all sufficiently large values of $r$ and for any $\varepsilon > 0$, we have*

$$\log_{p+q+1} M_{f\circ g}\left(r^{1+\varepsilon}\right) \geq \log_p \phi_f(M_g(r)) \log_{q+1} M_g(r) \tag{16}$$

*and*

$$\log_{p+q+1} M_{f\circ g}(r) \leq \log_p \phi_f(M_g(r)) \log_{q+1} M_g(r) \tag{17}$$

*where $\varphi(r) = \varphi_f(r)$ is defined by*

$$\varphi_f(r) = \frac{\log_{p+1} M_f(r)}{\log\log r}\,(r \geq r_0).$$

## 3. Main Results

In this section we first introduce the following definitions.

**Definition 11.** *The iterated logarithmic $p$ order $\rho^p_{\log}(f)$ of an entire function $f$ as follows*

$$\rho^p_{\log}(f) = \limsup_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log\log r} = \limsup_{r\to\infty} \frac{\log_p T_f(r)}{\log\log r}(p \in \mathbb{N}) \tag{18}$$

*and iterated logarithmic $p$ lower order $\lambda^p_{\log}(f)$ as*

$$\lambda^p_{\log}(f) = \liminf_{r\to\infty} \frac{\log_{p+1} M_f(r)}{\log\log r} = \liminf_{r\to\infty} \frac{\log_p T_f(r)}{\log\log r}(p \in \mathbb{N}). \tag{19}$$

**Theorem 12.** *For any two entire functions $f(z)$ and $g(z)$ of finite iterated logarithmic order with $i(f) = p, i(g) = q$ and if $\lambda^p_{\log}(f) > 0$, then*

$$\rho^{p+q-1}_{\log}(f\circ g) = \rho^q_{\log}(g).$$

*Proof.* By the definition of iterated logarithmic order we have,

$$\rho_{\log}^{p}(f)=\limsup_{r\to\infty}\frac{\log_p T_f(r)}{\log\log r},\ \rho_{\log}^{q}(g)=\limsup_{r\to\infty}\frac{\log_{q+1} M_g(r)}{\log\log r}.$$

Therefore for all sufficiently large $r$ and for given any $\varepsilon>0$ we get

$$\log_p T_f(r)\leq\log(\log r)^{\left(\rho_{\log}^{p}(f)+\varepsilon\right)}$$

i.e,

$$T_f(r)\leq\exp_{p-1}\left\{(\log r)^{\left(\rho_{\log}^{p}(f)+\varepsilon\right)}\right\}$$

and

$$\log_{q+1} M_g(r)\leq\log(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}$$

i.e,

$$M_g(r)\leq\exp_q\left\{(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}\right\}.$$

Now by lemma 5 we get

$$\begin{aligned}
T_{f\circ g}(r) &\leq 2T_f\left(M_g(r)\right)\\
&\leq 2\exp_{p-1}\left\{(\log M_g(r))^{\left(\rho_{\log}^{p}(f)+\varepsilon\right)}\right\}\\
&\leq 2\exp_{p-1}\left[\left\{\exp_{q-1}\left\{(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}\right\}\right\}^{\left(\rho_{\log}^{p}(f)+\varepsilon\right)}\right]\\
&\leq 2\exp_p\left[\left(\rho_{\log}^{p}(f)+\varepsilon\right)\log\left\{\exp_{q-1}\left\{(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}\right\}\right\}\right]\\
&\leq 2\exp_p\left[c\exp_{q-2}\left\{d\,(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}\right\}\right] \qquad (20)
\end{aligned}$$

where we take $c>\rho_{\log}^{p}(f)$ and $d\geq 1$ are some constants not necessarily same at each occurrence.

Therefore by (20) and from definition we get

$$\limsup_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log\log r}\leq\rho_{\log}^{q}(g). \qquad (21)$$

Next, since $i(g)=q$, we have

$$\rho_{\log}^{q}(g)=\limsup_{r\to\infty}\frac{\log_{q+1} M_g(r)}{\log\log r}.$$

If $\rho_{\log}^{q}(g)>0$, there exist a sequence $\{r_n\}\to\infty$ such that for any given $\varepsilon$ ,where $\left(0<\varepsilon<\rho_{\log}^{q}(g)\right)$ and for $r_n$ sufficiently large we have

$$M_g(r_n)\geq\exp_q\left\{(\log r_n)^{\left(\rho_{\log}^{q}(g)-\varepsilon\right)}\right\}. \qquad (22)$$

Since $\{r_n\}$ is a sequence tending to infinity, not necessarily same at each occurence and $\lambda_{\log}^{p}(f) > 0$, therefore from Lemma 7 for sufficiently large $r_n$ we have

$$\begin{aligned}
T_{f\circ g}(r_n) &\geq \frac{1}{3}\log M_f\left(\frac{1}{8}M_g\left(\frac{r_n}{4}\right)+O(1)\right)\\
&\geq \frac{1}{3}\log M_f\left(\frac{1}{9}M_g\left(\frac{r_n}{4}\right)\right)\\
&\geq \frac{1}{3}\exp_{p-1}\left[\left\{\log\frac{1}{9}M_g\left(\frac{r_n}{4}\right)\right\}^{\left(\lambda_{\log}^{p}(f)-\varepsilon\right)}\right]\\
&\geq \frac{1}{3}\exp_{p-1}\left[c_1\exp_{q-1}\left\{c_2\left(\log r_n\right)^{\left(\rho_{\log}^{q}(g)-\varepsilon\right)}\right\}\right]\\
&\geq \frac{1}{3}\exp_{p-1}\left[c_1\exp_{q}\left\{\left(\rho_{\log}^{q}(g)-\varepsilon\right)\log\left\{c_2\left(\log r_n\right)\right\}\right\}\right] \qquad (23)
\end{aligned}$$

where $c_1, c_2$ are positive constants.

Hence by (22) and (23) we get,

$$\limsup_{r\to\infty}\frac{\log_{p+q-1}T_{f\circ g}(r_n)}{\log\log r_n}\geq \rho_{\log}^{q}(g)\,. \qquad (24)$$

Thus combining (21) and (24) we have

$$\limsup_{r\to\infty}\frac{\log_{p+q-1}T_{f\circ g}(r_n)}{\log\log r_n}= \rho_{\log}^{q}(g)\,.$$

Therefore we have,

$$\rho_{\log}^{p+q-1}(f\circ g)=\rho_{\log}^{q}(g)$$

for $\rho_{\log}^{q}(g)>0$.

Next consider $\rho_{\log}^{q}(g)=0$.

Hence by definition

$$\limsup_{r\to\infty}\frac{\log_q M_g(r)}{\log\log r}=\infty.$$

So there exist a sequence $\{r_n\}\to\infty$ such that for any arbitrary $A>0$

$$\limsup_{r\to\infty}\frac{\log_q M_g(r_n)}{\log\log r_n}\geq A\Rightarrow M_g(r_n)\geq \exp_{q-1}\left(\log r_n\right)^{A}. \qquad (25)$$

Thus from (23) and (25) we have

$$\limsup_{r\to\infty} \frac{\log_{p+q-2} T_{f\circ g}(r_n)}{\log\log r_n} \geq A.$$

Since $A$ is arbitrarily large, thus get

$$\limsup_{r\to\infty} \frac{\log_{p+q-2} T_{f\circ g}(r_n)}{\log\log r_n} = \infty. \tag{26}$$

Therefore by (21) and (26) we have

$$\rho_{\log}^{p+q-1}(f\circ g) = \rho_{\log}^{q}(g) = 0.$$

□

**Corollary 13.** *For any two entire functions $f(z)$ and $g(z)$ with $i(g) = 1$, if $i(f\circ g) = p$, then $p-1 \leq i(f) \leq p$ and $\rho_{\log}^{p}(f) = 0$.*

*Proof.* Given $i(f\circ g) = p$, which implies $\rho_{\log}^{p}(f\circ g) = \alpha < \infty$.

Therefore for any sufficiently large $r$ and given $\varepsilon > 0$, we have

$$\limsup_{r\to\infty} \frac{\log_{p+1} M_{f\circ g}(r)}{\log\log r} = \alpha \Rightarrow M_{f\circ g}(r) \leq \exp_p (\log r)^{\alpha+\varepsilon}. \tag{27}$$

Again since $i(g) = 1$, then $g(z)$ is transcendental and for any sufficiently large $r$ and $m$ arbitrarily large we get

$$\frac{1}{9} M_g(\frac{r}{2}) \geq r^m. \tag{28}$$

Now by Lemma 7 and from (28)

$$M_f(r^m) \leq M_f\left(\frac{1}{9} M_g(\frac{r}{2})\right) \leq M_{f\circ g}(r) \leq \exp_p (\log r)^{\alpha+\varepsilon}, \tag{29}$$

which implies from (29)

(30)

$$M_f(r) \leq \exp_p (\log r)^{\left(\frac{\alpha}{m}+\varepsilon\right)} \Rightarrow \log_{p+1} M_f(r) \leq \left(\frac{\alpha}{m}+\varepsilon\right) \log\log r$$

i.e from(30)

$$\rho_{\log}^{p}(f) \leq \frac{\alpha}{m}.$$

Since $m$ is arbitrarily large, we get

$$\rho_{\log}^{p}(f) = 0.$$

□

**Theorem 14.** *Suppose $f(z)$ and $g(z)$ are two entire functions of finite iterated logarithmic order with $0<\rho_{\log}^{p}(f)<\infty$ and $0<\lambda_{\log}^{q}(g)\leq\rho_{\log}^{q}(g)<\infty$, then*

$$\lambda_{\log}^{q}(g)\leq\rho_{\log}^{p+q-1}(f\circ g)\leq\rho_{\log}^{q}(g).$$

*Proof.* It is given $\rho_{\log}^{p}(f)>0$, thus there exists a sequence $\{R_n\}\to\infty$ such that for any given $\varepsilon$ ,where $\left(0<\varepsilon<\rho_{\log}^{p}(f)\right)$ and for $R_n$ sufficiently large, we have

$$M_f(R_n)\geq\exp_p\left\{(\log R_n)^{\left(\rho_{\log}^{p}(f)-\varepsilon\right)}\right\}. \tag{31}$$

Now $M_g(r)$ is an increasing, continuous function, there exists a sequence $\{r_n\}\to\infty$ satisfying $R_n=\frac{1}{9}M_g(\frac{r_n}{2})$ for $r_n$ sufficiently large, we have from Lemma 7

$$\begin{aligned}M_{f\circ g}(r_n) &\geq M_f\left(\frac{1}{9}M_g(\frac{r_n}{2})\right)=M_f(R_n)\\ &\geq \exp_p\left\{(\log R_n)^{\left(\rho_{\log}^{p}(f)-\varepsilon\right)}\right\}\geq\exp_{p+1}\{c\log\log R_n\}\end{aligned}$$

i.e;

$$\begin{aligned}M_{f\circ g}(r_n) &\geq \exp_{p+1}\left\{c\log\log\frac{1}{9}M_g(\frac{r_n}{2})\right\}\\ &\geq \exp_{p+1}\left\{c\exp_{q-2}\left\{d(\log r_n)^{\left(\lambda_{\log}^{q}(g)-\varepsilon\right)}\right\}\right\}, \end{aligned}\tag{32}$$

where $c,d$ are positive constants.

Therefore we have

$$\limsup_{r\to\infty}\frac{\log_{p+q}M_{f\circ g}(r)}{\log\log r}\geq\lambda_{\log}^{q}(g)\Rightarrow\rho_{\log}^{p+q-1}(f\circ g)\geq\lambda_{\log}^{q}(g). \tag{33}$$

or the second part of the inequality,

$$M_{f\circ g}(r)\leq M_f(M_g(r))\leq\exp_p(\log M_g(r))^{\left(\rho_{\log}^{p}(f)+\varepsilon\right)}\leq\exp_{p+1}\{c_1\log\log M_g(r)\}$$

i.e;

$$\begin{aligned}M_{f\circ g}(r) &\leq \exp_{p+1}\left\{c_1\exp_{q-2}(\log r)^{\left(\rho_{\log}^{q}(g)+\varepsilon\right)}\right\}\\ &\leq \exp_{p+1}\left\{c_1\exp_{q-1}\left(\left(\rho_{\log}^{q}(g)+\varepsilon\right)\log\log r\right)\right\}\end{aligned}\tag{34}$$

where we take $c_1>\rho_{\log}^{p}(f)$.

From (34) we thus have

$$\rho_{\log}^{p+q-1}(f \circ g) \le \rho_{\log}^{q}(g) . \tag{35}$$

Therefore combining (33) and (35) we get,

$$\lambda_{\log}^{q}(g) \le \rho_{\log}^{p+q-1}(f \circ g) \le \rho_{\log}^{q}(g) .$$

□

**Theorem 15.** *Let $f(z)$ and $g(z)$ be two entire functions of finite logarithmic order with the condition that $g(0) = 0$ and $\rho_{\log}(g) < \lambda_{\log}(f) < \rho_{\log}(f)$ ,then*

$$\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_f(r)} = 0.$$

*Proof.* By definition there exists a sequence $\{r_n\} \to \infty$ such that for any given $\varepsilon$ $(> 0)$ and for $r_n$ sufficiently large, we have

$$\lambda_{\log}(f) = \liminf_{r\to\infty} \frac{\log T_f(r_n)}{\log\log r_n} \Rightarrow T_f(r_n) \ge (\log r_n)^{\left(\lambda_{\log}(f)-\varepsilon\right)}$$

and

$$\rho_{\log}(f) = \limsup_{r\to\infty} \frac{\log T_f(r_n)}{\log\log r_n} \Rightarrow T_f(r_n) \le (\log r_n)^{\left(\rho_{\log}(f)+\varepsilon\right)} .$$

Also,

$$\rho_{\log}(g) = \limsup_{r\to\infty} \frac{\log\log M_g(r_n)}{\log\log r_n} \Rightarrow \log\log M_g(r_n) \le (\rho_{\log}(g) + \varepsilon) \log\log r_n. \tag{36}$$

Combining the above two,

$$(\log r_n)^{\left(\lambda_{\log}(f)-\varepsilon\right)} \le T_f(r_n) \le (\log r_n)^{\left(\rho_{\log}(f)+\varepsilon\right)} .$$

Now,

$$T_{f\circ g}(r_n) \le T_f(M_g(r_n)) \le \{\log M_g(r_n)\}^{\left(\rho_{\log}(f)+\varepsilon\right)} \le \exp\{c \log\log M_g(r_n)\}$$

which implies by(36)

$$T_{f\circ g}(r_n) \le \exp\{c(\rho_{\log}(g) + \varepsilon) \log\log r_n\} ,$$

where we take $c > \rho_{\log}(f)$ .

Therefore

$$\frac{\log T_{f\circ g}(r_n)}{T_f(r_n)} \le \frac{\{c(\rho_{\log}(g) + \varepsilon) \log\log r_n\}}{(\log r_n)^{\left(\lambda_{\log}(f)-\varepsilon\right)}}.$$

Since $\lambda_{\log}(f) > \rho_{\log}(g)$, then for any given $\varepsilon > 0$ we have $\lambda_{\log}(f) - \varepsilon > \rho_{\log}(g) + \varepsilon$.

Hence for sufficiently large $r_n$ ,we have

$$\lim_{r\to\infty} \frac{\log T_{f\circ g}(r)}{T_f(r)} = 0.$$

This completes the proof. □

We prove following theorems which improves the above one on composite entire functions with finite iterated logarithmic order.

**Theorem 16.** *Let $f(z)$ and $g(z)$ be two entire functions of iterated logarithmic order with the condition that $i(f) = p, i(g) = q$ and $\rho^q_{\log}(g) < \lambda^p_{\log}(f) < \rho^p_{\log}(f)$, then*

$$\lim_{r\to\infty} \frac{\log_q T_{f\circ g}(r)}{T_f(r)} = 0, \lim_{r\to\infty} \frac{\log_{q+1} M_{f\circ g}(r)}{\log M_f(r)} = 0.$$

*Proof.* For sufficiently large values of $r$ and given any $\varepsilon > 0$, we have

$$\lambda^p_{\log}(f) = \liminf_{r\to\infty} \frac{\log_p T_f(r)}{\log\log r} \Rightarrow \log_p T_f(r) \geq \left(\lambda^p_{\log}(f) - \varepsilon\right) \log\log r$$

i.e;

$$T_f(r) \geq \exp_{p-1}(\log r)^{\left(\lambda^p_{\log}(f)-\varepsilon\right)}. \tag{37}$$

and

$$\rho^p_{\log}(f) = \limsup_{r\to\infty} \frac{\log_p T_f(r)}{\log\log r} \Rightarrow T_f(r) \leq \exp_{p-1}(\log r)^{\left(\rho^p_{\log}(f)+\varepsilon\right)}. \tag{38}$$

Combining (37) and (38) we have

$$\exp_{p-1}(\log r)^{\left(\lambda^p_{\log}(f)-\varepsilon\right)} \leq T_f(r) \leq \exp_{p-1}(\log r)^{\left(\rho^p_{\log}(f)+\varepsilon\right)}. \tag{39}$$

Again

(40)

$$\rho^q_{\log}(g) = \limsup_{r\to\infty} \frac{\log_{q+1} M_g(r)}{\log\log r} \Rightarrow M_g(r) \leq \exp_q(\log r)^{\left(\rho^q_{\log}(g)+\varepsilon\right)}.$$

Now from Lemma 5 and using (39) and (40) we get

$$\begin{aligned} T_{f\circ g}(r) &\leq T_f(M_g(r)) \leq \exp_{p-1}\{\log M_g(r)\}^{\left(\rho^p_{\log}(f)+\varepsilon\right)} \\ &\leq \exp_p\left\{\left(\rho^p_{\log}(f)+\varepsilon\right)\log\log M_g(r)\right\} \end{aligned}$$

i.e;

$$\begin{aligned} T_{f \circ g}(r) &\leq \exp_p \left\{ c \exp_{q-2} (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)} \right\} \\ (41) \qquad &\Rightarrow \log_q T_{f \circ g}(r) \leq \exp_{p-q} \left\{ c \exp_{q-2} (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)} \right\}, \end{aligned}$$

where $c > \rho_{\log}^p(f)$.

Hence for sufficiently large values of $r$ and for given any $\varepsilon \left(0 < \varepsilon < \mu_{\log}^p(f) - \rho_{\log}^q(g)\right)$, we have

$$\frac{\log_q T_{f \circ g}(r)}{T_f(r)} \leq \frac{\exp_{p-q} \left\{ c \exp_{q-2} (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)} \right\}}{\exp_{p-1} (\log r)^{\left(\lambda_{\log}^p(f)-\varepsilon\right)}} \to 0.$$

In a similar way for sufficiently large values of $r$ and for given any $\varepsilon > 0$, we have

$$(42) \quad \exp_{p-1} (\log r)^{\left(\lambda_{\log}^p(f)-\varepsilon\right)} \leq \log M_f(r) \leq \exp_{p-1} (\log r)^{\left(\rho_{\log}^p(f)+\varepsilon\right)}$$

and

$$M_g(r) \leq \exp_q (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)}.$$

Now from Lemma 7 and using (42) we get

$$\begin{aligned} M_{f \circ g}(r) &\leq M_f(M_g(r)) \leq \exp_p \{\log M_g(r)\}^{\left(\rho_{\log}^p(f)+\varepsilon\right)} \\ &\leq \exp_{p+1} \{c_1 \log \log M_g(r)\} \end{aligned}$$

i.e;

$$M_{f \circ g}(r) \leq \exp_{p+1} \left\{ c_1 \exp_{q-2} (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)} \right\}$$

where $c_1 > \rho_{\log}^p(f)$.

Therefore

$$\frac{\log_{q+1} M_{f \circ g}(r)}{\log M_f(r)} \leq \frac{\exp_{p-q} \left\{ c_1 \exp_{q-2} (\log r)^{\left(\rho_{\log}^q(g)+\varepsilon\right)} \right\}}{\exp_{p-1} (\log r)^{\left(\lambda_{\log}^p(f)-\varepsilon\right)}} \to 0.$$

Hence the theorem is proved. □

**Theorem 17.** *Let $f(z)$ and $g(z)$ be two entire functions of finite iterated logarithmic order with the condition that $i(f) = p, i(g) = q$ and $\rho_{\log}^q(g) < \rho_{\log}^p(f)$, then*

$$\liminf_{r \to \infty} \frac{\log_q T_{f \circ g}(r)}{T_f(r)} = 0, \ \liminf_{r \to \infty} \frac{\log_{q+1} M_{f \circ g}(r)}{\log M_f(r)} = 0.$$

*Proof.* From definition we have a sequence $\{r_n\} \to \infty$ such that for any given $\varepsilon$ $(> 0)$ and for $r_n$ sufficiently large,

$$T_f(r_n) \geq \exp_{p-1}\left\{(\log r_n)^{\left(\rho^p_{\log}(f)-\varepsilon\right)}\right\}.$$

In the same line of the previous theorem we can easily obtain this result. □

The following result can also be deduced as above.

**Theorem 18.** *Let $f(z)$ and $g(z)$ be two entire functions of finite iterated logarithmic order with the condition that $i(f) = p, i(g) = q$ and $\lambda^q_{\log}(g) < \lambda^p_{\log}(f) \leq \rho^p_{\log}(f)$, then*

$$\liminf_{r\to\infty} \frac{\log_q T_{f\circ g}(r)}{T_f(r)} = 0, \ \liminf_{r\to\infty} \frac{\log_{q+1} M_{f\circ g}(r)}{\log M_f(r)} = 0.$$

**Theorem 19.** *Let $f(z)$, $g(z)$ be transcendental entire functions of finite logarithmic order. Let $g(0) = 0$ and let $\lambda_{\log}(g) > 0$. Then*

$$\limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_g(r)} \leq \frac{\rho_{\log}(g)}{\lambda_{\log}(g)}.$$

*Proof.* From definition (1) and (2) we get,

$$\begin{aligned} T_f(r) &< (\log r)^{\rho_{\log}(f)+\varepsilon}, \text{ for all } r \geq r_0, \\ T_f(r) &> (\log r)^{\lambda_{\log}(f)-\varepsilon}, \text{ for all } r \geq r_0. \end{aligned}$$

Now by Theorem12,

$$\limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log\log r} = \rho_{\log}(g).$$

Thus for sufficiently large $r$ and for $\varepsilon > 0$,we obtain,

$$\log T_{f\circ g}(r) \leq (\rho_{\log}(g) + \varepsilon)\log\log r. \tag{43}$$

Again for large $r$,

$$\log T_g(r) > (\lambda_{\log}(g) - \varepsilon)\log\log r.$$

Since $\varepsilon > 0$ is arbitrary hence we have,

$$\limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_g(r)} \leq \frac{\rho_{\log}(g)}{\lambda_{\log}(g)}.$$

□

**Theorem 20.** *Let $f(z)$ and $g(z)$ be entire functions such that $0 < \lambda_{\log}(f) \leq \rho_{\log}(f) < \infty$ and $0 < \lambda_{\log}(g) \leq \rho_{\log}(g) < \infty$, then*

$$\frac{\lambda_{\log}(g)}{\rho_{\log}(f)} \leq \liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} \leq \limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} \leq \frac{\rho_{\log}(g)}{\lambda_{\log}(f)}.$$

*Proof.* From Lemma 7,

$$\begin{aligned} T_{f\circ g}(r) &\geq \frac{1}{3}\log M_f\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right) \\ &\geq \frac{1}{3}\left(\log\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)\right)^{\lambda_{\log}(f)-\varepsilon} \\ &\geq \frac{1}{3}\exp\left\{(\lambda_{\log}(f)-\varepsilon)\log\log\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)\right\} \\ &\geq \frac{1}{3}\exp\left\{c_1\log\log\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right)\right\} \\ &\geq \frac{1}{3}\exp\left[c_1\log\left\{d_1\left((\log r)^{\lambda_{\log}(g)-\varepsilon}\right)\right\}\right] \end{aligned}$$

i.e;

$$\begin{aligned} \log T_{f\circ g}(r) &\geq c_1\log\left\{d_1\left((\log r)^{\lambda_{\log}(g)-\varepsilon}\right)\right\} + O(1) \\ &\geq c_1\left(\lambda_{\log}(g)-\varepsilon\right)\log\log r + O(1) \end{aligned}$$

where $c_1, d_1$ are positive constants.

Also

$$\log T_f(r) \leq (\rho_{\log}(f)+\varepsilon)\log\log r.$$

Hence we have,

$$\liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} \geq \frac{\lambda_{\log}(g)-\varepsilon}{\rho_{\log}(f)+\varepsilon}. \tag{44}$$

Again from (43) and

$$\log T_f(r) \geq (\lambda_{\log}(f)-\varepsilon)\log\log r$$

we have

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} &\leq \limsup_{r\to\infty} \frac{(\rho_{\log}(g)+\varepsilon)\log\log r}{(\lambda_{\log}(f)-\varepsilon)\log\log r} \\ &= \frac{\rho_{\log}(g)+\varepsilon}{\lambda_{\log}(f)-\varepsilon}. \end{aligned} \tag{45}$$

Since $0 < \lambda_{\log}(f) \leq \rho_{\log}(f) < \infty$ and $0 < \lambda_{\log}(g) \leq \rho_{\log}(g) < \infty$, combining(44) and (45) we have

$$\frac{\lambda_{\log}(g)}{\rho_{\log}(f)} \leq \liminf_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} \leq \limsup_{r\to\infty} \frac{\log T_{f\circ g}(r)}{\log T_f(r)} \leq \frac{\rho_{\log}(g)}{\lambda_{\log}(f)}.$$

□

**Theorem 21.** *Let $f(z)$ and $g(z)$ be entire functions of finite iterated logarithmic order such that $0 < \lambda^p_{\log}(f) \leq \rho^p_{\log}(f) < \infty, 0 < \lambda^q_{\log}(g) \leq \rho^q_{\log}(g) < \infty$, then*

$$\begin{aligned}\frac{\lambda^q_{\log}(g)}{\rho^p_{\log}(f)} &\leq \liminf_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} \leq \min\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \\ &\leq \max\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \leq \limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^p_{\log}(f)}.\end{aligned}$$

*Proof.* For given any $\varepsilon > 0$ and for sufficiently large $r$, we have from the definition

$$\left(\lambda^p_{\log}(f) - \varepsilon\right) \log\log r \leq \log_p T_f(r) \leq \left(\rho^p_{\log}(f) + \varepsilon\right) \log\log r. \tag{46}$$

Again from (21) and (23) we get

$$\left(\lambda^q_{\log}(g) - \varepsilon\right) \log\log r \leq \log_{p+q-1} T_{f\circ g}(r) \leq \left(\rho^q_{\log}(g) + \varepsilon\right) \log\log r. \tag{47}$$

By (46) and (47) we get for sufficiently large $r$.

$$\frac{\rho^q_{\log}(g) + \varepsilon}{\lambda^p_{\log}(f) - \varepsilon} \geq \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} = \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log\log r}.\frac{\log\log r}{\log_p T_f(r)} \geq \frac{\lambda^q_{\log}(g) - \varepsilon}{\rho^p_{\log}(f) + \varepsilon}. \tag{48}$$

As $\varepsilon > 0$ is arbitrary, we get from(48)

$$\limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} \geq \frac{\lambda^q_{\log}(g)}{\rho^p_{\log}(f)}, \tag{49}$$

$$\limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^p_{\log}(f)}. \tag{50}$$

Now for sufficiently large $r_n$ and $R_m$, there exist two sequences $\{r_n\}$ and $\{R_m\}$ tending to infinity, then we have

$$\left.\begin{aligned}\log_p T_f(r_n) &\geq \left(\rho^p_{\log}(f) - \varepsilon\right) \log\log r_n, \\ \log_p T_f(R_m) &\leq \left(\lambda^p_{\log}(f) + \varepsilon\right) \log\log R_m.\end{aligned}\right\} \tag{51}$$

Similarly in (23) and (20), there exist two sequences $\{r'_n\}$ and $\{R'_m\}$ tending to infinity for sufficiently large $r'_n$ and $R'_m$, we obtain

$$\left.\begin{array}{r}\log_{p+q-1} T_{f\circ g}(r'_n) \geq \left(\rho^q_{\log}(g) - \varepsilon\right) \log\log r'_n, \\ \log_{p+q-1} T_{f\circ g}(R'_m) \leq \left(\lambda^q_{\log}(g) + \varepsilon\right) \log\log R'_m.\end{array}\right\} \tag{52}$$

From (46), (50) and (51), (52) we have

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)} &\leq \min\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \\ &\leq \max\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_f(r)}.\end{aligned}$$

Hence proves the theorem. □

**Corollary 22.** *Let $f(z)$, $g(z)$ satisfy Theorem 21, then*

$$\begin{aligned}\frac{\lambda^q_{\log}(g)}{\rho^p_{\log}(f)} &\leq \liminf_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_{f^{(k)}}(r)} \leq \min\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \\ &\leq \max\left\{\frac{\lambda^q_{\log}(g)}{\lambda^p_{\log}(f)}, \frac{\rho^q_{\log}(g)}{\rho^p_{\log}(f)}\right\} \\ &\leq \limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_p T_{f^{(k)}}(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^p_{\log}(f)}\end{aligned}$$

*for $k = 1, 2, \ldots$ .*

**Remark 23.** *One can get the same result by replacing $\log M_{f\circ g}(r), \log M_f(r)$ by $T_{f\circ g}(r), T_f(r)$ respectively in Theorem 21 and Corollary 22.*

**Theorem 24.** *Let $f(z)$ and $g(z)$ be entire functions of finite iterated logarithmic order such that $0 < \lambda^p_{\log}(f) \leq \rho^p_{\log}(f) < \infty, 0 < \lambda^q_{\log}(g) \leq$*

$\rho^q_{\log}(g) < \infty$, *then*

$$\begin{aligned}
\frac{\lambda^q_{\log}(g)}{\rho^q_{\log}(g)} &\leq \liminf_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_g(r)} \leq 1 \\
&\leq \limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_g(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^q_{\log}(g)}, \\
\frac{\lambda^q_{\log}(g)}{\rho^q_{\log}(g)} &\leq \liminf_{r\to\infty} \frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_g(r)} \leq 1 \\
&\leq \limsup_{r\to\infty} \frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_g(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^q_{\log}(g)}.
\end{aligned}$$

*Proof.* For sufficiently large $r$ and for given $\varepsilon > 0$, we have

$$\log_q T_g(r) \leq \left(\rho^q_{\log}(g) + \varepsilon\right) \log\log r. \tag{53}$$

For sufficiently large $r$, we have from (23)

$$\begin{aligned}
T_{f\circ g}(r) &\geq \frac{1}{3}\log M_f\left(\frac{1}{9}M_g\left(\frac{r}{4}\right)\right) \\
&\geq \frac{1}{3}\exp_{p-1}\left[\left\{\log\frac{1}{9}M_g\left(\frac{r}{4}\right)\right\}^{\left(\lambda^p_{\log}(f)-\varepsilon\right)}\right] \\
&\geq \frac{1}{3}\exp_{p-1}\left[c_1\exp_{q-1}\left\{c_2\left(\log r\right)^{\left(\lambda^q_{\log}(g)-\varepsilon\right)}\right\}\right] \\
&\geq \frac{1}{3}\exp_{p-1}\left[c_1\exp_q\left\{\left(\lambda^q_{\log}(g)-\varepsilon\right)\log\left\{c_2\left(\log r\right)\right\}\right\}\right] \qquad (54)
\end{aligned}$$

where $c_1, c_2$ are positive constants.

From (53) and (54) we have

$$\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_g(r)} = \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log\log r}.\frac{\log\log r}{\log_q T_g(r)} \geq \frac{\lambda^q_{\log}(g)-\varepsilon}{\rho^q_{\log}(g)+\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, then we get

$$\liminf_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_g(r)} \geq \frac{\lambda^q_{\log}(g)}{\rho^q_{\log}(g)}. \tag{55}$$

Again by definition, there exists a sequence $r_n$ tending to infinity such that for sufficiently large $r_n$,

$$\log_q T_g(r_n) \geq \left(\rho^q_{\log}(g) - \varepsilon\right)\log\log r_n. \tag{56}$$

From Theorem 12 and also for any $\varepsilon > 0$ and for sufficiently large $r$, we have

$$\log_{p+q-1} T_{f\circ g}(r) \leq \left(\rho_{\log}^{q}(g)+\varepsilon\right)\log\log r, \tag{57}$$
$$\log_{q} T_{g}(r) \leq \left(\rho_{\log}^{q}(g)+\varepsilon\right)\log\log r, \tag{58}$$
$$\log_{q} T_{g}(r) \geq \left(\lambda_{\log}^{q}(g)-\varepsilon\right)\log\log r. \tag{59}$$

From (56) and (57) we have,

$$\liminf_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_{q} T_{g}(r)} \leq 1. \tag{60}$$

From (57) and (59) we have,

$$\limsup_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_{q} T_{g}(r)} \leq \frac{\rho_{\log}^{q}(g)}{\lambda_{\log}^{q}(g)}. \tag{61}$$

Similarly from (54), for suffuciently large $r_m$, there exists a sequence $\{r_m\}$ tending to infinity then we get

$$\begin{aligned} T_{f\circ g}(r_m) &\geq \frac{1}{3}\exp_{p-1}\left[\left\{\log\frac{1}{9}M_g\left(\frac{r_m}{4}\right)\right\}^{\left(\lambda_{\log}^{p}(f)-\varepsilon\right)}\right] \\ &\geq \frac{1}{3}\exp_{p-1}\left[c_1\exp_q\left\{\left(\rho_{\log}^{q}(g)-\varepsilon\right)\log\left\{c_2\left(\log r_m\right)\right\}\right\}\right]. \end{aligned} \tag{62}$$

Thus from (58) and (62) we have,

$$\limsup_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_{q} T_{g}(r)} \geq 1. \tag{63}$$

Hence from $(55)$, $(60)$, $(61)$ and $(63)$, we get the proof of the theorem.

In similar way we have

$$\frac{\lambda_{\log}^{q}(g)}{\rho_{\log}^{q}(g)} \leq \liminf_{r\to\infty}\frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_{g}(r)} \leq 1 \leq \limsup_{r\to\infty}\frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_{g}(r)} \leq \frac{\rho_{\log}^{q}(g)}{\lambda_{\log}^{q}(g)}.$$

$\square$

**Corollary 25.** *Let $f(z), g(z)$ satisfy Theorem 24 and if $\lambda_{\log}^{q}(g) = \rho_{\log}^{q}(g)$, then*

$$\lim_{r\to\infty}\frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_{q} T_{g}(r)} = \lim_{r\to\infty}\frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_{g}(r)} = 1.$$

**Corollary 26.** *Let $f(z), g(z)$ satisfy Theorem 24 then*

$$\frac{\lambda^q_{\log}(g)}{\rho^q_{\log}(g)} \leq \liminf_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_{g^{(k)}}(r)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log_{p+q-1} T_{f\circ g}(r)}{\log_q T_{g^{(k)}}(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^q_{\log}(g)},$$
$$\frac{\lambda^q_{\log}(g)}{\rho^q_{\log}(g)} \leq \liminf_{r\to\infty} \frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_{g^{(k)}}(r)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log_{p+q} M_{f\circ g}(r)}{\log_{q+1} M_{g^{(k)}}(r)} \leq \frac{\rho^q_{\log}(g)}{\lambda^q_{\log}(g)}$$

*for* $k = 1, 2, ...$

**Theorem 27.** *If* $\rho^p_{\log}(f) = 0$ *and* $0 < \rho^q_{\log}(g) < \infty$, *then*

$$\rho^{p+q}_{\log}(f \circ g) = \infty$$

*provided*

***(a)*** $\lambda^q_{\log}(g) > 0$ *and* $\limsup\limits_{r\to\infty} \log_p \varphi(r) = \infty$ *or*

***(b)*** $\lambda^q_{\log}(g) = 0$ *and* $\limsup\limits_{r\to\infty} \log_p \varphi(r) = \infty$,

*where* $\varphi(r) = \varphi_f(r)$ *is defined by* $\varphi_f(r) = \frac{\log_{p+1} M_f(r)}{\log\log r}\,(r \geq r_0)$.

*Proof.* For any $\varepsilon > 0$, we have from Lemma 10

$$\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon}) \geq \log_p \varphi\left(M_g(r)\right) \log_{q+1} M_g(r).$$

Hence,

$$\rho^{p+q}_{\log}(f \circ g) = \limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}} \geq \limsup_{r\to\infty} \frac{\log_p \varphi\left(M_g(r)\right) \log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}.$$

Again if $g(z)$ has positive iterated lower logarithmic order $\lambda^q_{\log}(g)$, then

$$\liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{\log\log r} = \lambda^q_{\log}(g).$$

This implies

$$\log_{q+1} M_g(r) > \left(\lambda^q_{\log}(g) - \varepsilon\right) \log\log r$$

for all sufficiently large values of $r$.Thus

$$\begin{aligned}\rho^{p+q}_{\log}(f \circ g) &\geq \limsup_{r\to\infty} \frac{\log_p \varphi\left(M_g(r)\right) \log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}} \\ &\geq \limsup_{r\to\infty} \frac{\log_p \varphi\left(M_g(r)\right) \log\left(\log r\right)^{\left(\lambda^q_{\log}(g)-\varepsilon\right)}}{\log\log r^{1+\varepsilon}}.\end{aligned}$$

Now for $\varepsilon > 0$,

$$\log\left(\log r\right)^{\left(\lambda_{\log}^{q}(g)-\varepsilon\right)} > \log\log r^{1+\varepsilon}.$$

Therefore as $r \to \infty$,

$$\frac{\log\left(\log r\right)^{\left(\lambda_{\log}^{q}(g)-\varepsilon\right)}}{\log\log r^{1+\varepsilon}} \to \infty$$

and $\varphi\left(M_g\left(r\right)\right)$ is increasing, continuous and unbounded in $r$, which implies

$$\rho_{\log}^{p+q}\left(f \circ g\right) = \infty.$$

Next if $g(z)$ is of finite positive iterated logarithmic order $\rho_{\log}^{q}\left(g\right)$ and of zero lower iterated logarithmic order $\lambda_{\log}^{q}\left(g\right)$ and if $\lim\limits_{r\to\infty} \log_p \varphi(r) = \infty$, then for any $\varepsilon' > 0$ we have

$$\begin{aligned}
\rho_{\log}^{p+q}\left(f \circ g\right) &= \limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon'})}{\log\log r^{1+\varepsilon\prime}} \\
&\geq \limsup_{r\to\infty} \frac{\log_p \varphi\left(M_g(r)\right) \log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon'}} \\
&\geq \limsup_{r\to\infty} \log_p \varphi\left(M_g(r)\right) . \limsup_{r\to\infty} \frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon'}} \\
&\geq \limsup_{r\to\infty} \log_p \varphi\left(M_g(r)\right) .\rho_{\log}^{q}\left(g\right), \ \text{[since, } \varepsilon' > 0\text{]} \\
&= \infty.
\end{aligned}$$

This proves the theorem. ☐

**Theorem 28.** *Suppose that* $\rho_{\log}^{q}\left(g\right) > 0, \rho_{\log}^{p}\left(f\right) = 0$. *Let* $\limsup\limits_{r\to\infty} \log_p \varphi(r) = \tau$. *If* $\tau$ *is finite, then*

$$\rho_{\log}^{p+q}\left(f \circ g\right) \leq \tau \rho_{\log}^{q}\left(g\right).$$

*Furthermore, if* $\lim\limits_{r\to\infty} \log_p \varphi(r) = \tau$ *then the above inequality becomes an equality.*

*Proof.* Since, $\limsup\limits_{r\to\infty} \log_p \varphi(r) = \tau$, then for any $\varepsilon > 0$, we have

$$\log_p \varphi(r) < \tau + \varepsilon.$$

Also if $g(z)$ is of iterated logarithmic order $\rho_{\log}^{q}\left(g\right)$, then for all sufficiently large $r$,

$$\log_{q+1} M_g(r) < \left(\rho_{\log}^{q}\left(g\right) + \varepsilon\right) \log\log r.$$

Hence

$$\begin{aligned}
\rho_{\log}^{p+q}(f \circ g) &= \limsup_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r)}{\log\log r} \\
&\leq \limsup_{r\to\infty} \frac{\log_p \varphi(M_g(r)) \log_{q+1} M_g(r)}{\log\log r} \\
&\leq \limsup_{r\to\infty} \frac{(\tau+\varepsilon)\left(\rho_{\log}^{q}(g)+\varepsilon\right)\log\log r}{\log\log r} \\
&= (\tau+\varepsilon)\left(\rho_{\log}^{q}(g)+\varepsilon\right).
\end{aligned}$$

Since, $\varepsilon > 0$ is arbitrary, thus

$$\rho_{\log}^{p+q}(f \circ g) \leq \tau \rho_{\log}^{q}(g).$$

Similarly, equality part follows.
Hence the theorem is proved. □

**Theorem 29.** *If $f$ and $g$ are transcendental entire functions of iterated logarithmic order with (i) $\lambda_{\log}^{q}(g) = \infty$ or (ii) $\lambda_{\log}^{p}(f) > 0$ then*

$$\lambda_{\log}^{p+q}(f \circ g) = \infty.$$

*Proof.* (i) Let $\lambda_{\log}^{q}(g) = \infty$.
From Lemma 7

$$\begin{aligned}
\log_{p+q} M_{f\circ g}(r) &\geq \log_{p+q} M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \\
&\geq \frac{\log_{p+q} M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right)}{\log_q\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)| - |g(0)|\right)}.\log_q\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \\
&\geq \frac{\log_{p+q} M_f(r)}{\log_q(r)}.\left(\log_q M_g\left(\frac{r}{2}\right) + O(1)\right).
\end{aligned}$$

Since $\frac{\log_{p+q} M_f(r)}{\log_q(r)}$ is an increasing function of $r$ for large $r$ and $\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)| > r$ , we get

$$\log_{p+q} M_{f\circ g}(r) \geq \log_q M_g\left(\frac{r}{2}\right)$$

for large $r$.
Hence

$$\log_{p+q+1} M_{f\circ g}(r) \geq \log_{q+1} M_g\left(\frac{r}{2}\right)$$

i.e.,

$$\frac{\log_{p+q+1} M_{f\circ g}(r)}{\log\log r} \geq \frac{\log_{q+1} M_g\left(\frac{r}{2}\right)}{\log\log r}$$

i.e.,

$$\lambda_{\log}^{p+q}(f\circ g) \geq \lambda_{\log}^{q}(g) = \infty.$$

Hence first part of theorem is proved.

(ii) If $\lambda_{\log}^{p}(f) > 0$ and also let $\lambda_{\log}^{q}(g) < \infty$.Then by Lemma 6, for any $\varepsilon > 0$,we have

$$\begin{aligned}
\lambda_{\log}^{p+q}(f\circ g) &= \liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}} \\
&\geq \liminf_{r\to\infty} \left[\frac{\log_{p+q+1} M_f\left(M_g(r)\right)}{\log_{q+1} M_g\left(r\right)} \times \frac{\log_{q+1} M_g\left(r\right)}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty} \left[\frac{\log_{p+q+1} M_f\left(M_g(r)\right)}{\log_{q+1} M_g\left(r\right)} \times \frac{\log_{q+1} M_g\left(r\right)}{\log\log r}\right],
\end{aligned}$$

since $\varepsilon > 0$. Since, $g(z)$ is transcendental, for arbitrarily large $K > 0$ we have,

$$\frac{\log_{q+1} M_g\left(r\right)}{\log\log r} > K \ \ \left(r \geq r_0\right).$$

As $M_g\left(r\right)$ is increasing,continuous and unbounded in $r$, thus we obtain

$$\lambda_{\log}^{p+q}(f\circ g) \geq \lambda_{\log}^{p}(f)\,K.$$

Since $\lambda_{\log}^{p}(f) > 0$,

$$\lambda_{\log}^{p+q}(f\circ g) = \infty.$$

Hence the theorem follows. □

**Theorem 30.** *If $f$ and $g$ are transcendental entire functions of iterated logarithmic order with $\lambda_{\log}^{q}(g) < \infty$ and $\limsup_{r\to\infty} \log_p \phi\left(r\right) = \tau < \infty$, then*

$$\lambda_{\log}^{p+q}(f\circ g) \leq \tau.\lambda_{\log}^{q}(g) \leq \rho_{\log}^{p+q}(f\circ g)\,. \tag{64}$$

*Furthermore, in the above result the first inequaity becomes equality if*

$$\lim_{r\to\infty} \log_p \phi\left(r\right) = \tau < \infty.$$

*Proof.* We have from maximum modulus principle

$$M_{f\circ g}(r) \leq M_f\left(M_g\left(r\right)\right)$$

Hence

$$\begin{aligned}
\lambda_{\log}^{p+q}\left(f\circ g\right) &= \liminf_{r\to\infty}\frac{\log_{p+q+1}M_{f\circ g}\left(r\right)}{\log\log r}\\
&\leq \liminf_{r\to\infty}\frac{\log_{p+q+1}M_f\left(M_g\left(r\right)\right)}{\log\log r}\\
&\leq \limsup_{r\to\infty}\frac{\log_{p+q+1}M_f\left(M_g\left(r\right)\right)}{\log_{q+1}M_g(r)}.\liminf_{r\to\infty}\frac{\log_{q+1}M_g(r)}{\log\log r}\\
&= \tau.\lambda_{\log}^{q}\left(g\right)
\end{aligned}$$

which proves the first inequality of (64) .
Again by Lemma 6, we have

$$\begin{aligned}
\rho_{\log}^{p+q}\left(f\circ g\right) &= \limsup_{r\to\infty}\frac{\log_{p+q+1}M_{f\circ g}\left(r^{1+\varepsilon}\right)}{\log\log r^{1+\varepsilon}}\\
&\geq \limsup_{r\to\infty}\left[\frac{\log_{p+q+1}M_f\left(M_g\left(r\right)\right)}{\log_{q+1}M_g(r)}.\frac{\log_{q+1}M_g(r)}{\log\log r^{1+\varepsilon}}\right]\\
&\geq \limsup_{r\to\infty}\left[\frac{\log_{p+q+1}M_f\left(M_g\left(r\right)\right)}{\log_{q+1}M_g(r)}.\frac{\log_{q+1}M_g(r)}{\log\log r}\right] \text{ since } \varepsilon>0,\\
&\geq \limsup_{r\to\infty}\frac{\log_{p+q+1}M_f\left(M_g\left(r\right)\right)}{\log_{q+1}M_g(r)}.\liminf_{r\to\infty}\frac{\log_{q+1}M_g(r)}{\log\log r}\\
&= \tau.\lambda_{\log}^{q}\left(g\right).
\end{aligned}$$

Thus we get

$$\tau.\lambda_{\log}^{q}\left(g\right) \leq \rho_{\log}^{p+q}\left(f\circ g\right).$$

This proves the second inequality of (64).
Finally, if the limit

$$\lim_{r\to\infty}\log_p\phi\left(r\right) = \tau$$

exists, then we have

$$\begin{aligned}
\lambda_{\log}^{p+q}(f \circ g) &\geq \liminf_{r \to \infty} \left[ \frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)} \cdot \frac{\log_{q+1} M_g(r)}{\log \log r^{1+\varepsilon}} \right] \\
&\geq \liminf_{r \to \infty} \left[ \frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)} \cdot \frac{\log_{q+1} M_g(r)}{\log \log r} \right] \quad (\text{since } \varepsilon > 0) \\
&= \tau \lambda_{\log}^{q}(g)
\end{aligned}$$

which gives

$$\lambda_{\log}^{p+q}(f \circ g) = \tau \lambda_{\log}^{q}(g).$$

Hence the theorem is proved. □

**Remark 31.** *If $\lambda_{\log}^{q}(g) = \infty$, then by Theorem 29, $\rho_{\log}^{p+q}(f \circ g) = \lambda_{\log}^{p+q}(f \circ g) = \infty$, and the inequalities in (64) become trivial. If $\lambda_{\log}^{q}(g) > 0$ and $\tau = \infty$, then by Theorem 27, $\rho_{\log}^{p+q}(f \circ g) = \infty$. Hence the inequality is trivially true.*

**Theorem 32.** *Suppose that $\lambda_{\log}^{p}(f) = \lambda_{\log}^{q}(g) = 0$ and that $\liminf_{r \to \infty} \frac{\log_{q+1} M_g(r)}{(\log r)^{\alpha}} = a > 0$, $\liminf_{r \to \infty} \frac{\log_p \phi_f(r)}{\left(\log_{q+1} r\right)^{\beta}} = b > 0$ for any positive numbers $\alpha$ and $\beta$ with $\alpha < 1$ and $\alpha(\beta + 1) > 1$. Then*

$$\lambda_{\log}^{p+q}(f \circ g) = \infty.$$

*Proof.* It is given that,

$$\begin{aligned}
\liminf_{r \to \infty} \frac{\log_{q+1} M_g(r)}{(\log r)^{\alpha}} &= a \Rightarrow \frac{\log_{q+1} M_g(r)}{(\log r)^{\alpha}} \geq a - \varepsilon \text{ for some } \varepsilon > 0, \\
&\Rightarrow \log_{q+1} M_g(r) \geq (a - \varepsilon)(\log r)^{\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{r \to \infty} \frac{\log_p \varphi(r)}{\left(\log_{q+1} r\right)^{\beta}} &= b \Rightarrow \log_p \varphi(r) \geq (b - \varepsilon)\left(\log_{q+1} r\right)^{\beta} \text{ for some } \varepsilon > 0, \\
&\Rightarrow \log_p \varphi(M_g(r)) \geq \left(\log_{q+1} M_g(r)\right)^{\beta}(b - \varepsilon) \\
&\Rightarrow \log_p \varphi(M_g(r)) \geq (a - \varepsilon)^{\beta}(\log r)^{\alpha\beta}(b - \varepsilon).
\end{aligned}$$

Then we have for $0 < \varepsilon < \min(a, b)$,

$$\begin{aligned}
\lambda_{\log}^{p+q}(f \circ g) &\geq \liminf_{r\to\infty}\left[\frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty}\left[\frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)}.\frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty}\left[\log_p \varphi(M_g(r)).\frac{\log_{q+1} M_g(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty}\left[\log_p \varphi(M_g(r)).\frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}\right] \text{ (since, } \varepsilon > 0) \\
&\geq \liminf_{r\to\infty}\frac{\left[(b-\varepsilon)(a-\varepsilon)^{\beta}(\log r)^{\alpha\beta}\right].\left[(a-\varepsilon)(\log r)^{\alpha}\right]}{\log\log r^{1+\varepsilon}}.
\end{aligned}$$

Taking $\log r = x, (b-\varepsilon)(a-\varepsilon)^{\beta} = c$ and $(a-\varepsilon) = c_1$, we deduce

$$\begin{aligned}
\lambda_{\log}^{p+q}(f \circ g) &\geq \liminf_{r\to\infty}\frac{\left(cx^{\alpha\beta}\right).\left(c_1 x^{\alpha}\right)}{\log\log r^{1+\varepsilon}} \\
&\geq \liminf_{r\to\infty}\frac{\left(cx^{\alpha\beta}\right).\left(c_1 x^{\alpha}\right)}{\log\log r} \\
&\geq \liminf_{r\to\infty}\frac{\left(cx^{\alpha\beta}\right).\left(c_1 x^{\alpha}\right)}{\log x} \quad \text{(since, } \log r = x, \log\log r = \log x) \\
&= \infty. \text{ (since } \alpha < 1 \text{ and } \alpha(\beta+1) > 1)
\end{aligned}$$

This completes the proof. □

**Theorem 33.** *Suppse that* $\lambda_{\log}^{p}(f) = \lambda_{\log}^{q}(g) = 0$ *and that* $\liminf_{r\to\infty}\frac{\log_{q+k+2} M_g(r)}{[\log_k(r)]^{\alpha}} = a > 0$, $\liminf_{r\to\infty}\frac{\log_{p+k-1}(\phi(r))}{\left[\log_{q+k+2}(r)\right]^{\beta}} = b > 0$ *for any positive integer* $k \geq s+1$ *and any positive numbers* $\alpha$ *and* $\beta$ *with* $\max(\alpha, \alpha\beta) > 1$. *Then* $\lambda_{\log}^{p+q}(f \circ g) = \infty$.

*Proof.* It is given that,

$$\liminf_{r\to\infty}\frac{\log_{q+k+2} M_g(r)}{[\log_k(r)]^{\alpha}} = a$$

$$\begin{aligned}
&\Rightarrow \log_{q+k+2} M_g(r) \geq (a-\varepsilon)[\log_k(r)]^{\alpha} \text{ for some } \varepsilon > 0, \\
&\Rightarrow \log_{q+1} M_g(r) \geq \exp_{k-1}\left[(a-\varepsilon)(\log_k(r))^{\alpha}\right]
\end{aligned}$$

and

$$\liminf_{r\to\infty} \frac{\log_{p+k-1}(\phi(r))}{\left[\log_{q+k+2}(r)\right]^{\beta}} = b$$

$$\begin{aligned}
&\Rightarrow \quad \log_{p+k-1}(\phi(r)) \geq (b-\varepsilon)\left[\log_{q+k+2}(r)\right]^{\beta} \\
&\Rightarrow \quad \log_{p+k-1}(\phi(M_g(r))) \geq (b-\varepsilon)\left[(a-\varepsilon)(\log_k(r))^{\alpha}\right]^{\beta} \\
&\Rightarrow \quad \log_p \varphi(M_g(r)) \geq \exp_{k-1}\left[(b-\varepsilon)(a-\varepsilon)^{\beta}(\log_k(r))^{\alpha\beta}\right].
\end{aligned}$$

For $0 < \varepsilon < \min(a, b)$

$$\begin{aligned}
\lambda_{\log}^{p+q}(f\circ g) &\geq \liminf_{r\to\infty}\left[\frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty}\left[\frac{\log_{p+q+1} M_f(M_g(r))}{\log_{q+1} M_g(r)} . \frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}\right]
\end{aligned}$$

i.e,

$$\liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}} \geq \liminf_{r\to\infty}\left[\log_p \phi_f(M_g(r)) \frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}\right]$$

$$\geq \liminf_{r\to\infty} \frac{\exp_{k-1}\left[(b-\varepsilon)(a-\varepsilon)^{\beta}(\log_k(r))^{\alpha\beta}\right]\exp_{k-1}\left[(a-\varepsilon)(\log_k(r))^{\alpha}\right]}{\log\log r^{1+\varepsilon}}$$

$$. \geq \liminf_{r\to\infty} \frac{\exp_{k-1}\left[(b-\varepsilon)(a-\varepsilon)^{\beta}(\log_k(r))^{\alpha\beta}\right]\exp_{k-1}\left[(a-\varepsilon)(\log_k(r))^{\alpha}\right]}{\log\log r}$$

Putting $\log_k(r) = x$, $(b-\varepsilon)(a-\varepsilon)^{\beta} = d_1$ and $(a-\varepsilon) = d_2$, thus we have

$$\lambda_{\log}^{p+q}(f\circ g) \geq \liminf_{r\to\infty} \frac{\exp_{k-1}\left(d_1 x^{\alpha\beta}\right).\log_k\left(d_2 x^{\alpha}\right)}{\exp_{k-2}(x)} = \infty.$$

Since $\max(\alpha, \alpha\beta) > 1$. This completes the proof. $\square$

**Theorem 34.** *Suppose for any two transcendental entire function with* $\lambda_{\log}^{p}(f) = \lambda_{\log}^{q}(g) = 0$ *and that one of the following conditions* $(I)$ *and* $(II)$ *is satisfied:*

$(I)$ $\liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{(\log r)^{\alpha_1}} = A_1 < \infty, \limsup_{r\to\infty} \frac{\log_p \phi(r)}{\left(\log_{q+1} r\right)^{\beta_1}} = B_1 < \infty$ *for any positive numbers* $\alpha_1$ *and* $\beta_1$ *with* $\alpha_1(\beta_1 + 1) > 1$;

*$(II)$ $\liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{(\log\log r)^{\alpha_2}} = A_2 < \infty, \limsup_{r\to\infty} \frac{\log_{p+1}\phi(r)}{\left(\log_{q+1} r\right)^{\beta_2}} = B_2 < \infty$ for any positive numbers $\alpha_2$ and $\beta_2$ with $\alpha_2\beta_2 > 1$.*

*Then*

$$\lambda_{\log}^{p+q}(f\circ g) = \infty.$$

*Proof.* Suppose first that condition $(I)$ hold, then

$$\liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{(\log r)^{\alpha_1}} = A_1 \Rightarrow \log_{q+1} M_g(r) \geq \frac{1}{2} A_1 (\log r)^{\alpha_1}$$

for some $\varepsilon > 0$, for all sufficiently large values of $r$ and also there exists a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$ and such that

$$\limsup_{r\to\infty} \frac{\log_p \varphi(r)}{\left(\log_{q+1} r\right)^{\beta_1}} = B_1 \Rightarrow \log_p \varphi(M_g(r_n)) \geq \frac{1}{2} B_1 \left(\log_{q+1} M_g(r_n)\right)^{\beta_1}.$$

since $M_g(r_n)$ is continuous, increasing and unbounded of $r$.

Therefore, for any $\varepsilon > 0$ ,

$$\begin{aligned}
\lambda_{\log}^{p+q}(f\circ g) &= \liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}} \\
&\geq \liminf_{r\to\infty} \left[\log_p \phi(M_g(r)) \frac{\log_{q+1} M_g(r)}{\log\log r^{1+\varepsilon}}\right] \\
&\geq \frac{\frac{1}{2}B_1.\left(\log_{q+1} M_g(r_n)\right)^{\beta_1} \log_{q+1} M_g(r_n)}{\log\log r^{1+\varepsilon}} \\
&\geq \frac{\frac{1}{2}B_1.\left\{\frac{1}{2}A_1 (\log r_n)^{\alpha_1}\right\}^{\beta_1+1}}{\log\log r^{1+\varepsilon}} \\
&\geq \frac{\left(\frac{1}{2}\right)^{\beta_1+2} B_1 (A_1)^{\beta_1+1} (\log r_n)^{\alpha_1(\beta_1+1)}}{\log\log r^{1+\varepsilon}} = \infty,
\end{aligned}$$

since $\alpha_1(\beta_1+1) > 1$ by our hypothesis.

Next suppose that condition $(II)$ hold, then for any sufficiently small $\varepsilon > 0$,

$$\liminf_{r\to\infty} \frac{\log_{q+1} M_g(r)}{(\log\log r)^{\alpha_2}} = A_2 \Rightarrow \log_{q+1} M_g(r) > (A_2 - \varepsilon)(\log\log r)^{\alpha_2}$$

for all sufficiently large values of $r$ and also there exists a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$ and such that

$$\limsup_{r\to\infty} \frac{\log_{p+1}\varphi(r)}{\left(\log_{q+1} r\right)^{\beta_2}} = B_2 \Rightarrow$$
$$\log_{p+1}\varphi(M_g(r_n)) > (B_2-\varepsilon)\left(\log_{q+1} M_g(r_n)\right)^{\beta_2} \Rightarrow$$
$$\log_p \phi(M_g(r)) > \exp\left\{(B_2-\varepsilon)\left(\log_{q+1} M_g(r_n)\right)^{\beta_2}\right\}.$$

Thus for any sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
\lambda_{\log}^{p+q}(f\circ g) &= \liminf_{r\to\infty} \frac{\log_{p+q+1} M_{f\circ g}(r^{1+\varepsilon})}{\log\log r^{1+\varepsilon}} \\
&\geq \liminf_{r\to\infty} \left[\log_p \phi(M_g(r_n)) \frac{\log_{q+1} M_g(r_n)}{\log\log r_n^{1+\varepsilon}}\right] \\
&\geq \liminf_{r\to\infty} \frac{\exp\left[(B_2-\varepsilon)\left\{\log_{q+1} M_g(r_n)\right\}^{\beta_2}\right]\log_{q+1} M_g(r_n)}{\log\log r_n^{1+\varepsilon}}
\end{aligned}$$

$$\geq \liminf_{r\to\infty} \frac{\exp\left[(B_2-\varepsilon)\{(A_2-\varepsilon)(\log\log r_n)^{\alpha_2}\}^{\beta_2}\right](A_2-\varepsilon)(\log\log r_n)^{\alpha_2}}{\log\log r_n^{1+\varepsilon}}$$
$$\geq \liminf_{r\to\infty} \frac{\exp\left[(B_2-\varepsilon)(A_2-\varepsilon)^{\beta_2}(\log\log r_n)^{\alpha_2\beta_2}\right](A_2-\varepsilon)(\log\log r_n)^{\alpha_2}}{\log\log r_n^{1+\varepsilon}}$$
$$\geq \liminf_{r\to\infty} \frac{\exp\left\{(\log\log r_n)(B_2-\varepsilon)(A_2-\varepsilon)^{\beta_2}(\log\log r_n)^{\alpha_2\beta_2-1}\right\}(A_2-\varepsilon)(\log\log r_n)^{\alpha_2}}{\log\log r_n^{1+\varepsilon}}$$
$$= \liminf_{r\to\infty} \frac{\{\exp(\log\log r_n)\}\exp\left\{(B_2-\varepsilon)(A_2-\varepsilon)^{\beta_2}(\log\log r_n)^{\alpha_2\beta_2-1}\right\}(A_2-\varepsilon)(\log\log r_n)^{\alpha_2}}{\log\log r_n^{1+\varepsilon}}$$
$$\geq \liminf_{r\to\infty} \frac{(\log r_n)\exp\left\{(B_2-\varepsilon)(A_2-\varepsilon)^{\beta_2}(\log\log r_n)^{\alpha_2\beta_2-1}\right\}(A_2-\varepsilon)(\log\log r_n)^{\alpha_2}}{\log\log r_n^{1+\varepsilon}}$$
$$= \infty.$$

Since $\alpha_2\beta_2 - 1 > 0$, $\alpha_2 > 0$ by our hypothesis and as $r \to \infty$, $\dfrac{\log r}{\log\log r^{1+\varepsilon}} \to \infty$ for $\varepsilon > 0$.

Hence the theorem is proved. □

**Theorem 35.** *Let $f(z)$ and $g(z)$ be two entire functions of finite iterated logarithmic order with $i(f) = p$, $i(g) = q$ and $\rho_{\log}^q(g) < \lambda_{\log}^p(f) < \rho_{\log}^p(f)$, then*

$$\limsup_{r\to\infty} \frac{\log_{p+q+1}\mu_{f\circ g}(r)}{\log_p \mu_f(r)} = 0.$$

*Proof.* From the definition of $\rho_{\log}^p(f)$ and $\lambda_{\log}^p(f)$ we get

(65) $$\log_p \mu_f(r) < (\log r)^{\rho_{\log}^p(f)+\varepsilon}$$

for large $r$ and

$$\log_p \mu_f(r) > (\log r)^{\lambda^p_{\log}(f)-\varepsilon} \tag{66}$$

for large $r$.

From (14),

$$\begin{aligned}\log_{p+q+1} \mu_{f\circ g}(r) &\leq \log_{p+q+1}\left[2\mu_f\left(4\mu_g\left(2r\right)\right)\right]\\ &\leq \log_{p+q+1}\left[\mu_f\left(4\mu_g\left(2r\right)\right)\right] + O(1).\end{aligned}$$

Using (65) we have,

$$\begin{aligned}\log_{p+q+1} \mu_{f\circ g}(r) &\leq \log_{q+1}\left[\left\{4\mu_g\left(2r\right)\right\}^{\rho^p_{\log}(f)+\varepsilon}\right] + O\left(1\right)\\ &\leq \log_q\left(\rho^p_{\log}(f)+\varepsilon\right)\log\left\{4\mu_g\left(2r\right)\right\} + O\left(1\right)\\ &\leq \log_q\left(\rho^p_{\log}(f)+\varepsilon\right)\log\left\{\mu_g\left(2r\right)\right\} + O\left(1\right)\\ &\leq \log_q\left(\rho^p_{\log}(f)+\varepsilon\right)\exp_{q-1}\left(\log 2r\right)^{\rho^q_{\log}(g)+\varepsilon}. \end{aligned} \tag{67}$$

From (66) and (67) we get,

$$\frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} \leq \frac{\log_q\left(\rho^p_{\log}(f)+\varepsilon\right)\exp_{q-1}\left(\log 2r\right)^{\rho^q_{\log}(g)+\varepsilon}}{(\log r)^{\lambda^p_{\log}(f)-\varepsilon}}.$$

Since $\rho^q_{\log}(g) < \lambda^p_{\log}(f)$, we choose $\varepsilon > 0$ such that

$$\rho^q_{\log}(g)+\varepsilon < \lambda^p_{\log}(f)-\varepsilon.$$

Therefore we have

$$\limsup_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} = 0.$$

□

**Theorem 36.** *Let $f(z)$ and $g(z)$ be entire functions of finite iterated logarithmic order $p$ and $q$ respectively. If $\rho^q_{\log}(g) < \rho^p_{\log}(f)$ then*

$$\lim_{r\to\infty} \frac{\log_{p+q+1} \mu_{f\circ g}(r)}{\log_p \mu_f(r)} = 0.$$

*Proof.* From Lemma 9 we get for large $r$

$$\begin{aligned}
\log_{p+q+1}\left(\mu_{f\circ g}(r)\right) &\geq \log_{p+q+1}\left[\frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)-|g(0)|\right)\right] \\
&\geq \log_{p+q+1}\left[\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)-|g(0)|\right)\right]+O(1) \\
&\geq \log_{q+1}\left(\log\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right)^{\lambda^p_{\log}(f)-\varepsilon}+O(1) \\
&> \log_q\left(\lambda^p_{\log}(f)-\varepsilon\right)\log\left(\log\frac{1}{8}\mu_g\left(\frac{r}{4}\right)\right)+O(1) \\
&> \log_q\left(\lambda^p_{\log}(f)-\varepsilon\right)\log\left(\log\mu_g\left(\frac{r}{4}\right)\right)+O(1) \\
&> \log_q\left(\lambda^p_{\log}(f)-\varepsilon\right)\exp_{q-1}\left(\log\frac{r}{4}\right)^{\rho^q_{\log}(g)-\varepsilon}+O(1).
\end{aligned}$$

Thus for sufficiently large $r$, there exists a sequence $\{r_n\}$

(68)
$$\log_{p+q+1}\left(\mu_{f\circ g}(r_n)\right)>\log_q\left(\lambda^p_{\log}(f)-\varepsilon\right)\exp_{q-1}\left(\log\frac{r_n}{4}\right)^{\rho^q_{\log}(g)-\varepsilon}+O(1).$$

Also for large $r$,

$$\log_p\mu(r,f)<(\log r)^{\rho^p_{\log}(f)+\varepsilon}.$$

So for the sequence $\{r_n\}$ tending to infinity,

$$\frac{\log_{p+q+1}\left(\mu_{f\circ g}(r_n)\right)}{\log_p\mu(r_n,f)}>\frac{\log_q\left(\lambda^p_{\log}(f)-\varepsilon\right)\exp_{q-1}\left(\log\frac{r_n}{4}\right)^{\rho^q_{\log}(g)-\varepsilon}}{(\log r_n)^{\rho^p_{\log}(f)+\varepsilon}}.$$

Since $\rho^q_{\log}(g)<\rho^p_{\log}(f)$, we choose $\varepsilon>0$ such that

$$\rho^q_{\log}(g)-\varepsilon<\rho^p_{\log}(f)+\varepsilon.$$

So we have

$$\lim_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}(r)}{\log_p\mu_f(r)}=0.$$

□

**Theorem 37.** *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite iterated logarithmic order $p$ and $q$ respectively with $\rho^q_{\log}(g)>0$. Then*

$$\liminf_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}(r)}{\log_{q+1}\mu_g(r)}=0.$$

*Proof.* For a sequence $\{r_n\}$ tending to infinity, from (68),

$$\log_{p+q+1}\left(\mu_{f\circ g}\left(r_n\right)\right) > \log_q\left(\lambda_{\log}^p(f)-\varepsilon\right)\exp_{q-1}\left(\log\frac{r_n}{4}\right)^{\rho_{\log}^q(g)-\varepsilon}+O(1).$$

Also using the definition of $\rho_{\log}^q(g)$ for the entire function $g$, we get

$$\begin{aligned}\log_{q+1}\mu_g(r) &< \log\left(\log r\right)^{\rho_{\log}^q(g)+\varepsilon}\\ &= \left(\rho_{\log}^q(g)+\varepsilon\right)\log\log r\\ &< \left(\rho_{\log}^q(g)+\varepsilon\right)\log r\end{aligned}$$

for large $r$.

Thus for a sequence $\{r_n\}$ tending to infinity, we obtain

$$\frac{\log_{p+q+1}\mu_{f\circ g}\left(r_n\right)}{\log_{q+1}\mu_g(r_n)} > \frac{\log_q\left(\lambda_{\log}^p(f)-\varepsilon\right)\exp_{q-1}\left(\log\frac{r_n}{4}\right)^{\rho_{\log}^q(g)-\varepsilon}}{\left(\rho_{\log}^q(g)+\varepsilon\right)\log r_n}.$$

Since $\rho_{\log}^q(g)>0$ and so we can choose $\varepsilon>0$ such that

$$\rho_{\log}^q(g)-\varepsilon>0.$$

Hence

$$\liminf_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_{q+1}\mu_g\left(r\right)}=0.$$

□

**Remark 38.** *In particular,* $\lambda_{\log}^q(g)>0$, *which implies that* $\rho_{\log}^q(g)>0$, *therefore we have*

$$\limsup_{r\to\infty}\frac{\log_{p+q+1}\mu_{f\circ g}\left(r\right)}{\log_{q+1}\mu_g\left(r\right)}=0.$$

**Theorem 39.** *Let* $f(z)$ *and* $g(z)$ *be transcendental entire function of finite iterated logarithmic order* $p$ *and* $q$ *and let* $\lambda_{\log}^q(g)>0$, *then*

$$\limsup_{r\to\infty}\frac{\log_{p+q+2}\mu_{f\circ g}\left(r\right)}{\log_{q+1}\mu_g\left(r\right)}\le k\frac{\rho_{\log}^q(g)}{\lambda_{\log}^q(g)}$$

*for some positive constant* $k$ *such that* $\log r^2<(\log r)^k$.

*Proof.* From (67) it easily follows that

$$\log_{p+q+1}\mu_{f\circ g}\left(r\right)\le\log_q\left(\rho_{\log}^p(f)+\varepsilon\right)\exp_{q-1}\left(\log 2r\right)^{\rho_{\log}^q(g)+\varepsilon}$$

for large $r$.

So for sufficiently large $r$

$$\begin{aligned}
\log_{p+q+2}\mu_{f\circ g}(r) &\le \log_{q+1}\exp_{q-1}(\log 2r)^{\rho^q_{\log}(g)+\varepsilon}+O(1)\\
&= \log\log(\log 2r)^{\rho^q_{\log}(g)+\varepsilon}+O(1)\\
&\le \log(\log 2r)^{\rho^q_{\log}(g)+\varepsilon}+O(1)\\
&\le \left(\rho^q_{\log}(g)+\varepsilon\right)\log\log 2r+O(1)\\
&\le \left(\rho^q_{\log}(g)+\varepsilon\right)\log\log r^2+O(1)\\
&\le k\left(\rho^q_{\log}(g)+\varepsilon\right)\log\log r+O(1)
\end{aligned}$$

Again we have for sufficiently large $r$

$$\log_{q+1}\mu_g(r) > \left(\lambda^q_{\log}(g)-\varepsilon\right)\log\log r.$$

Hence

$$\limsup_{r\to\infty}\frac{\log_{p+q+2}\mu_{f\circ g}(r)}{\log_{q+1}\mu_g(r)} \le k\frac{\rho^q_{\log}(g)}{\lambda^q_{\log}(g)}.$$

□

**Theorem 40.** *Let $h(z)$ and $f(z)$ be two entire functions of finite iterated logarithmic order such that $\rho^s_{\log}(h) < \lambda^p_{\log}(f)$ then*

$$\lim_{r\to\infty}\frac{\log_{q+s}\mu_{h\circ g}(r)}{\log_{p+q}\mu_{f\circ g}(r)} = 0$$

*for any nonconstant entire function $g(z)$ of finite iterated order $\rho^q_{\log}(g)$.*

*Proof.* We have from Niino [7],

$$\begin{aligned}
\mu_{f\circ g}(r) &\ge \frac{r-r'}{r}M_{f\circ g}(r')\\
&= \frac{r-r+r^{-\beta}}{r}M_{f\circ g}\left(r-r^{-\beta}\right)\\
&= \frac{1}{r^{\beta+1}}M_{f\circ g}\left(r-r^{-\beta}\right).
\end{aligned}$$

Now, from the definition of $\lambda^p_{\log}(f)$ we get,

$$(\log r)^{\lambda^p_{\log}(f)-\varepsilon} < \log_p M_f(r)$$

for large $r$ and also from the definition of $\rho^s_{\log}(h)$ we have,

$$(\log r)^{\rho^s_{\log}(h)+\varepsilon} \ge \log_s M_h(r)$$

for large $r$.

Hence

$$\begin{aligned}
\mu_{f\circ g}(r) &\geq \frac{1}{r^{\beta+1}}\exp_p \log_p M_{f\circ g}\left(r-r^{-\beta}\right)\\
&\geq \frac{1}{r^{\beta+1}}\exp_p \log_p M_f\left(\frac{M_g\left(r-r^{-\alpha}\right)-|g(0)|}{5r^{2(\alpha+1)}}-|g(0)|\right)\\
&> \frac{1}{r^{\beta+1}}\exp_p\left(\log\left\{\frac{M_g\left(r-r^{-\alpha}\right)-|g(0)|}{5r^{2(\alpha+1)}}-|g(0)|\right\}\right)^{\lambda^p_{\log}(f)-\varepsilon}\\
&> \frac{1}{r^{\beta+1}}\exp_p\left(\log\left\{\frac{M_g\left(r-r^{-\alpha}\right)}{6r^{2(\alpha+1)}}\right\}\right)^{\lambda^p_{\log}(f)-\varepsilon}
\end{aligned}$$

$$(69) \qquad > \exp_p\left(\log\left(M_g(r,g)\right)\right)^{\lambda^p_{\log}(f)-\varepsilon}.$$

On the other hand, for large $r$,

$$\begin{aligned}
\mu_{h\circ g}(r) &\leq M_{h\circ g}(r)\\
&\leq M_h\left(M_g(r)\right)\\
&= \exp_s \log_s M_h\left(M_g(r)\right)\\
&= \exp_s\left(\log\left(M_g(r)\right)\right)^{\rho^s_{\log}(h)+\varepsilon}. \qquad (70)
\end{aligned}$$

Choose $\varepsilon > 0$ such that

$$\rho^s_{\log}(h)+\varepsilon < \lambda^p_{\log}(f)-\varepsilon.$$

Thus from (69) and (70) we get,

$$\frac{\mu_{h\circ g}(r)}{\mu_{f\circ g}(r)} < \frac{\exp_s\left(\log\left(M_g(r)\right)\right)^{\rho^s_{\log}(h)+\varepsilon}}{\exp_p\left(\log\left(M_g(r)\right)\right)^{\lambda^p_{\log}(f)-\varepsilon}}.$$

Therefore

$$\lim_{r\to\infty}\frac{\mu_{h\circ g}(r)}{\mu_{f\circ g}(r)} = 0.$$

□

**Theorem 41.** *Let $f$ and $g$ be entire functions of finite iterated logarithmic order such that $0 < \lambda^p_{\log}(f) < \rho^q_{\log}(g) < \infty$ and $\rho^s_{\log}(h) = \rho^t_{\log}(k)$. Then the entire functions $h$ and $k$ with finite iterated logarithmic order $s$ and $t$ respectively satisfy*

$$\liminf_{r\to\infty}\frac{\log_{p+s}\mu_{f\circ h}(r)}{\log_{q+t}\mu_{g\circ k}(r)} = 0.$$

*Proof.* Now for sufficiently large $r$, using (15) we obtain,

$$\begin{aligned}
\log_p\left[\mu_{f\circ h}(r)\right] &\geq \log_p\left[\frac{1}{2}\mu_f\left(\frac{1}{8}\mu_h\left(\frac{r}{4}\right)+O(1)\right)\right] \\
&\geq \left(\log\left(\frac{1}{8}\mu_h\left(\frac{r}{4}\right)+O(1)\right)\right)^{\lambda^p_{\log}(f)-\varepsilon} \\
&\geq \left[\log\left(\frac{1}{9}\mu_h\left(\frac{r}{4}\right)\right)\right]^{\lambda^p_{\log}(f)-\varepsilon} \\
&> \left[\log\left(\mu_h\left(\frac{r}{4}\right)\right)\right]^{\lambda^p_{\log}(f)-\varepsilon}.
\end{aligned}$$

And so for a sequence $\{r_n\}$ with $r_n \geq r_0$

$$\begin{aligned}
\log_{p+s}\left[\mu_{f\circ h}(r_n)\right] &\geq \log_s\left[\log\left(\mu_h\left(\frac{r}{4}\right)\right)\right]^{\lambda^p_{\log}(f)-\varepsilon} \\
&= \log_{s-1}\left(\lambda^p_{\log}(f)-\varepsilon\right)\log\log\mu_h\left(\frac{r}{4}\right) \\
&= \log_{s-1}\left(\lambda^p_{\log}(f)-\varepsilon\right)\exp_{s-2}\log_s\mu_h\left(\frac{r}{4}\right) \\
&= \log_{s-1}\left(\lambda^p_{\log}(f)-\varepsilon\right)\exp_{s-2}\left(\frac{r_n}{4}\right)^{\rho^s_{\log}(h)-\varepsilon}. \qquad (71)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\log_{q+t}\left[\mu_{g\circ k}(r)\right] &\leq \log_{q+t} M_{g\circ k}(r) \\
&\leq \log_t\log_q M_g\left(M_k(r)\right) \\
&\leq \log_t\left[\log M_k(r)\right]^{\rho^q_{\log}(g)+\varepsilon} \\
&\leq \log_{t-1}\left(\rho^q_{\log}(g)+\varepsilon\right)\exp_{t-2}\log_t M_k(r) \\
&= \log_{t-1}\left(\rho^q_{\log}(g)+\varepsilon\right)\exp_{t-2}\left(\log r\right)^{\rho^t_{\log}(k)+\varepsilon}. \qquad (72)
\end{aligned}$$

Since $0 < \lambda^p_{\log}(f) < \rho^q_{\log}(g) < \infty$ and $\rho^s_{\log}(h) = \rho^t_{\log}(k)$, choose $\varepsilon$ such that $\lambda^p_{\log}(f)-\varepsilon < \rho^q_{\log}(g)+\varepsilon$ and $\rho^s_{\log}(h)-\varepsilon = \rho^t_{\log}(k)+\varepsilon$ , then from (71) and (72) it follows that as $r\to\infty$

$$\frac{\log_{p+s}\mu_{f\circ h}(r)}{\log_{q+t}\mu_{g\circ k}(r)} = 0.$$

Hence proves the theorem. □

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