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WEAK FORMS OF OPEN FUNCTIONS BETWEEN MINIMAL STRUCTURE SPACES AND BOUNDARY PRESERVATION

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Abstract. In this note we continue the study of almost $M-$open functions between spaces with minimal structure, also taking into account the unified theory of weakly $M-$open functions developed by Noiri and Popa. Our main result is a characterization of almost $M-$open functions via preservation of boundary under inverse image, generalizing a classical characterization of open functions in topological spaces. We partially extend this result to the setting of generalized closure spaces, which allows us to obtain, as a special case, a new characterization of weakly $M-$open functions in terms of $m-\theta-$boundary preservation.

1. Introduction

Two decades ago, Popa and Noiri initiated the study of functions between spaces endowed with minimal structures [31]. A family $m_X \subset \mathcal{P}(X)$ is called a *minimal structure* (shortly, *$m-$structure)* on X if $\emptyset \in m_X$ and $X \in m_X$. The couple (X, m_X) is called a space with minimal structure or an $m-$space, for short.

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The members of m_X are called $m-$open sets and their complements form the class of $m-$closed sets. Arbitrary unions of $m-$open sets play an important role and will be called almost $m-$open sets, while complements of almost $m-$open sets will be called almost $m-$closed sets. A minimal structure m_X is said *to have property* $(\mathcal{B})$ if it closed under arbitrary unions and in this case it is called a *generalized topology* [5]. In topological spaces, each of the following classes of sets forms a minimal structure which plays an important role in the study of various forms of generalized continuity: semi-open sets [8], preopen sets [11], semi-preopen sets [2], $\alpha-$open sets [19], $\beta-$open sets [1], $\delta-$open sets [39], $\theta-$open sets [39].

Using the framework of minimal structure spaces, Popa and Noiri obtained unified theories for generalized forms of continuous functions [31], [32], [33], [34], [35], [36], [24], [25], contra-continuous functions [23], open functions [28], [29], [30], closed functions [26], [27]. This unifying approach, that encompasses a broad range of generalizations of continuous functions, open functions and closed functions, opened new perspectives for research on these concepts.

In this note we continue the study of a class of generalized open functions between minimal structure spaces introduced in our earlier paper [13], also taking into account the unified theory of weakly open functions developed by Noiri and Popa in [28]. A function between $m-$spaces is $M-$open if it maps $m-$open sets to $m-$open sets, respectively is almost $M-$open if it maps almost $m-$open sets to almost $m-$open sets. Let us recall the following special cases of $M-$open functions: semi-open functions [22], almost open functions (in the sense of Singal) [20] and [21], preopen functions [11], $\alpha-$open functions [12], $\beta-$open functions [1], semi-preopen functions [18].

Our main result is a characterization of almost $M-$open functions via preservation of boundary under inverse image. We prove that a function between minimal structure spaces $f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open if and only if $f^{-1}(m_Y Fr(B)) \subset m_X Fr(f^{-1}(B))$ for all $B \subset Y$. Finally, we partially extend to generalized closure spaces this result on boundary preservation under the inverse image of an almost open function, obtaining a new characterization of weakly $M-$open functions as a special case. The topological study of generalized closure spaces [7], [37] has many applications to mathematical models used in various fields [3], [38].

2. Preliminaries

A function u from the power set $\mathcal{P}(X)$ of a non-empty set X into itself is called a *generalized closure operator on* X (GCO, for short) and the pair (X, u) is said to be a *generalized closure space* (GCS, for short).

Definition 1. [37] *A GCO* $u : \mathcal{P}(X) \to \mathcal{P}(X)$ *is called grounded if* $u(\emptyset) = \emptyset$*, isotone if* $A \subset B \subset X$ *implies* $u(A) \subset u(B)$*, expansive if* $A \subset u(A)$ *for every* $A \subset X$*, contractive if* $A \subset u(A)$ *for every* $A \subset X$*, idempotent if* $u(u(A)) = u(A)$ *for every* $A \subset X$*, sublinear if* $u(A \cup B) \subset u(A) \cup u(B)$ *for every* $A, B \subset X$*.*

A *Čech closure operator* is a GCO which is grounded, expansive, isotone and sublinear. A *Kuratowski closure operator* is a GCO which is grounded, expansive, isotone, idempotent and sublinear.

The GCS (X, u) is said to be *isotonic* if u is isotone, a *closure space* if u is expansive, isotone and idempotent, a *neighborhood space* or *monotone space* if u is grounded, expansive and isotone.

The $u-$*interior operator*, $u - Int : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by

$$u - Int(A) = X \setminus u(X \setminus A)$$

and is called the dual of the GCO u. Note that u is the dual of the GCO $u - Int$.

Remark 2. *Let* $u : \mathcal{P}(X) \to \mathcal{P}(X)$*. Then* u *is isotone if and only if* $u - Int$ *is isotone,* u *is idempotent if and only if* $u - Int$ *is idempotent, and* u *is grounded if and only if* $u - Int(X) = X$*.*

For every family of sets $\mathcal{F}$ we will denote by $\mathcal{U}(\mathcal{F})$ the family of all unions of sets that belong to $\mathcal{F}$. Note that a minimal structure m_X has property $(\mathcal{B})$ if and only if $\mathcal{U}(m_X) = m_X$.

Spaces with minimal structure are generalized closure spaces [14], [15]. The fundamental generalized closure operators associated to a minimal structure have been introduced by Maki in [10].

Definition 3. *Let* $m_X \subset \mathcal{P}(X)$ *be a minimal structure. For each subset* $A \subset X$ *the* m_X-*closure of* A *and the* m_X-*interior of* A *are defined as follows:*

$$m_X Cl(A) := \cap\{F : A \subset F \text{ and } X \setminus F \in m_X\},$$

$$m_X Int(A) := \cup\{U : U \subset A \text{ and } U \in m_X\}.$$

Remark 4. *The above definitions are equivalent to:* $x \in m_X - Cl(A)$ *if and only if* $D \cap A \neq \emptyset$ *for every* $D \in m_X$ *containing* x*, respectively* $x \in m_X - Int(A)$ *if and only if there exists* $D \in m_X$ *containing* x *such that* $D \subset A$*.*

Lemma 5. [10] *Let* $m_X \subset \mathcal{P}(X)$ *be a minimal structure. For all subset* A *and* B *of* X *the following properties hold:*

(i) $m_X Cl(X \setminus A) = X \setminus m_X Int(A)$ *and* $m_X Int(X \setminus A) = X \setminus m_X Cl(A)$*;*

(ii) $m_X Cl(A) = A$ *if* $X \setminus A \in m_X$ *and* $m_X Int(B) = B$ *if* $B \in m_X$*;*

(iii) $m_X Cl(\emptyset) = \emptyset$, $m_X Cl(X) = X$ *and* $m_X Int(\emptyset) = \emptyset$, $m_X Int(X) = X$*;*

(iv) If $A \subset B$*, then* $m_X Cl(A) \subset m_X Cl(B)$ *and* $m_X Int(A) \subset m_X Int(B)$*;*

(v) $m_X Int(A) \subset A \subset m_X Cl(A)$*;*

(vi) $m_X Cl(m_X Cl(A)) = m_X Cl(A)$ *and* $m_X Int(m_X Int(A)) = m_X Int(A)$.

The above lemma shows that the GCO's $m_X Cl$ and $m_X Int$ are dual to each other and are both grounded, isotone and idempotent. In addition, $m_X Cl$ is expansive, while $m_X Int$ is contractive.

Definition 6. *Let* (X, m_X) *be an* $m-$*space. A set* $A \subset X$ *is said to be almost* $m-$*open if* $m_X Int(A) = A$*. A set* $B \subset X$ *is said to be almost* $m-$*closed if* $m_X Cl(B) = B$*.*

Since $m_X Cl$ and $m_X Int$ are dual to each other, a set is almost $m-$closed if and only if its complement is almost $m-$open. Clearly, every $m-$closed set is almost $m-$closed, while the converse holds if the minimal structure has property $(\mathcal{B})$. Note that $A \subset X$ is almost $m-$open if and only if $A \in \mathcal{U}(m_X)$. Therefore, a set is almost $m-$closed if and only if this set is an arbitrary intersection of $m-$closed sets. Since $m_X Cl$ and $m_X Int$ are idempotent, for all subsets A and B of X, the set $m_X Cl(A)$ is almost $m-$closed and the set $m_X Int(B)$ is almost $m-$open.

Unlike the standard closure operator Cl of a topological space, the GCO $m_X Cl$ need not to be sublinear, even if m_X has property $(\mathcal{B})$, as the following example shows.

Example 7. *Let* m_X *be a minimal structure such that there exist* A, $B \in m_X$ *with* $A \cap B \notin \mathcal{U}(m_X)$*. Then* $(A \cap B) \setminus m_X - Int(A \cap B)$ *is* *non-empty. But* $(A \cap B) \setminus m_X - Int(A \cap B) = m_X - Cl\,(C \cup D) \setminus (m_X - Cl(C) \cup m_X - Cl(D))$*, where* $C := X \setminus A$ *and* $D := X \setminus B$*.*

The following GCOs $m_X Cl_\theta$ and $m_X Int_\theta$ have been introduced and studied in [24].

Definition 8. *Let* (X, m_X) *be a space with* $m-$*structure and* $S \subset X$*. A point* $x \in X$ *is called:*

(a) an $m - \theta-$*adherent point of* S *if* $m_X Cl(U) \cap S \neq \emptyset$ *for every* $U \in m_X$ *containing* x*;*

(b) an $m-\theta-$*interior point of* S *if* $m_X Cl(V) \subset S$ *for some* $V \in m_X$ *containing* x*.*

Definition 9. *The set* $m_X Cl_\theta(S)$ *containing all the* $m - \theta-$*adherent points of* S *is called the* $m - \theta-$*closure of* S*. The set* $m_X Int_\theta(S)$ *containing all the* $m - \theta-$*interior points of* S *is called the* $m - \theta-$*interior of* S*.*

A set $A \subset X$ is said to be $m - \theta-$closed in (X, m_X) if $m_X Cl_\theta(A) = A$. The complements of $m - \theta-$closed sets are called $m - \theta-$open sets.

The following properties have been proved in [24]. The GCOs $m_X Cl_\theta$ and $m_X Int_\theta$ are dual to each other, grounded and isotone. A set $A \subset X$ is $m - \theta-$open in (X, m_X) if and only if $m_X Int_\theta(A) = A$. For every $A \subset X$,

$$m_X Int_\theta(A) \subset m_X Int(A) \subset A \subset m_X Cl(A) \subset m_X Cl_\theta(A).$$

In general, $m_X Cl_\theta$ and $m_X Int_\theta$ are not idempotent [17].

Lemma 10. *Let* (X, m_X) *be an* $m-$ *space,* A *and* B *subsets of* X*, and* $x \in X$*. The following properties hold:*

(i) $m_X Int_\theta\,(A) \in \mathcal{U}\,(m_X)$*, in particular every* $\theta - m-$*open set is almost* $m-$*open;*

(ii) $m_X Cl_\theta\,(B)$ *is almost* $m-$*closed, in particular every* $\theta - m-$*closed set is almost* $m-$*closed;*

(iii) If $x \in m_X Cl_\theta\,(A)$*, then for every* $\theta - m-$*open set* $D \subset X$ *containing* x *we have* $D \cap A \neq \emptyset$*.*

Proof. (i) We may assume that $m_X Int_\theta\,(A)$ is non-empty. Denote $B = m_X Int_\theta\,(A)$. For every $x \in B$, there exists $U_x \in m_X$ such that $x \in U_x \subset m_X Cl(U_x) \subset B$. Then $B = \bigcup\limits_{x \in B} U_x$, hence $B \in \mathcal{U}\,(m_X)$.

(ii) Write $m_X Cl_\theta\,(B) = X \setminus m_X Int_\theta\,(X \setminus B)$ and use (i) with $A = X \setminus B$.

(iii) Assume that there exists some $\theta-m-$open set $D \subset X$ containing x such that $D \cap A = \emptyset$. Then $x \in D \subset X \setminus A$ and $D = m_X Int_\theta(D)$, hence $x \in m_X Int_\theta(D) \subset m_X Int_\theta(X \setminus A)$. $\square$

Corollary 11. [24, Lemma 3.6 (6)]*If m_X has property $(\mathcal{B})$, then $m_X Int_\theta(A)$ is $m-$open and $m_X Cl_\theta(A)$ is $m-$closed, for every $A \subset X$.*

Moreover, by [24, Lemma 3.6 (5)] it is known that $m_X Cl(A) = m_X Cl_\theta(A)$ whenever $A \in m_X$. A more general property holds.

Lemma 12. *Let (X, m_X) be an $m-$ space. If $A \in \mathcal{U}(m_X)$, then $m_X Cl_\theta(A) = m_X Cl(A)$.*

Proof. The inclusion $m_X Cl(A) \subset m_X Cl_\theta(A)$ holds for every $A \subset X$.

Assume that $A \in \mathcal{U}(m_X)$. Write $A = \cup\{A_i : i \in I\}$, where $A_i \in m_X$ for every $i \in I$. We prove that $X \setminus m_X Cl(A) \subset X \setminus m_X Cl_\theta(A)$, hence the reverse inclusion of the above also holds.

Let $x \in X \setminus m_X Cl(A)$. Since $x \in m_X Int(X \setminus A)$, there exists $U \in m_X$ containing x such that $U \cap A = \emptyset$, i.e. $U \cap A_i = \emptyset$ for every $i \in I$. But $A_i \in m_X$ and $U \cap A_i = \emptyset$ implies the stronger relation $A_i \cap m_X Cl(U) = \emptyset$; indeed, if there exists some $y \in A_i \cap m_X Cl(U)$, then $y \in m_X Cl(U)$ and $A_i \in m_X$ contains y, therefore $A_i \cap U \neq \emptyset$. Consequently, we have $A_i \cap U = \emptyset$ for every $i \in I$, hence $A \cap m_X Cl(U) = \emptyset$. Since $x \in U \subset m_X Cl(U) \subset X \setminus A$ and $U \in m_X$, it follows that $x \in m_X Int_\theta(X \setminus A) = X \setminus m_X Cl_\theta(A)$. $\square$

2.1. **Weak forms of open functions in spaces with minimal structures.** In [13] we introduced the notions of $M-$open function and almost $M-$open function as counterparts of open functions in the setting of $m-$spaces. The notion of almost $M-$open function is more general than that of $M-$open function and is a natural dual of the notion of $M-$continuous function, since a bijective function $f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open if and only if its inverse is $M-$continuous.

Definition 13. [13, Definitions 3.1 and 3.2]*A function $f : (X, m_X) \to (Y, m_Y)$ is said to be:*

(i) $M-$open at $x \in X$ if for each $U \in m_X$ containing $x \in X$ we have $f(U) \in m_Y$;

(ii) $M-$open if it is open at each point in X;

(iii) almost $M-$open at $x \in X$ if for each $U \in m_X$ containing $x \in X$ there exists $V \in m_Y$ such that $f(x) \in V \subset f(U)$.

(iv) almost $M-$open if it is almost $M-$open at each point in X.

It is easy to see that $f:(X,m_X)\to(Y,m_Y)$ is $M-$open if and only if it maps $m-$open sets to $m-$open sets. Similarly, it was shown that $f:(X,m_X)\to(Y,m_Y)$ is almost $M-$open if and only if it maps almost $m-$open sets to almost $m-$open sets [13, Corollary 3.1.]. Obviously, every $M-$open function is almost $M-$open. The converse is false in general [13, Example 3.1.], but holds if m_Y has property $(\mathcal{B})$. See [28, Lemma 4.2, Remark 4.2] for more connections to previously studied notions of generalized open functions.

We summarize below several characterizations of (global) almost $M-$open functions, proved in [13, Lemma 3.1, Lemma 3.2, Corollary 3.1, Theorem 5.1].

Lemma 14. *The following are equivalent for a function $f:(X,m_X)\to(Y,m_Y)$:*
(i) f is almost $M-$open;
(ii) $f(m_XInt(A))\subset m_YInt(f(A))$ for all $A\subset X$;
(iii) If $U\in m_X$, then $f(U)\in\mathcal{U}(m_X)$;
(iv) $m_XInt\left(f^{-1}(B)\right)\subset f^{-1}\left(m_YInt(B)\right)$ for all $B\subset Y$;
(v) $f^{-1}\left(m_YCl(B)\right)\subset m_XCl\left(f^{-1}(B)\right)$ for all $B\subset Y$.
(vi) f maps almost $m-$open sets to $m-$almost $m-$open sets.

In [28] Noiri and Popa introduced and investigated the notion of weakly $M-$open function, that is more general than that of almost $M-$open function.

Definition 15. *A function $f:(X,m_X)\to(Y,m_Y)$ is said to be* weakly $M-$open *if for each $U\in m_X$, $f(U)\subset m_YInt\left(f(m_XCl(U))\right)$.*

Lemma 16. *A function $f:(X,m_X)\to(Y,m_Y)$ is weakly $M-$open if and only if for each $U\in\mathcal{U}(m_X)$, $f(U)\subset m_YInt\left(f(m_XCl(U))\right)$.*

Proof. The sufficiency is obvious. The necessity follows writing $U\in\mathcal{U}(m_X)$ as $U=\bigcup\limits_{i\in I}U_i$, whence $f(U)=\bigcup\limits_{i\in I}f(U_i)\subset\bigcup\limits_{i\in I}m_YInt\left(f(m_XCl(U_i))\right)\subset m_YInt\left(f(m_XCl(U))\right)$. □

The following characterizations for weakly $M-$open functions have been proved in [28].

Theorem 17. [28, Theorem 3.1.] *For a function $f:(X,m_X)\to(Y,m_Y)$ the following properties are equivalent:*

(i) f is weakly $M-$open;
(ii) $f(m_X Int_\theta(A)) \subset m_Y Int(f(A))$ for every subset A of X;
(iii) $m_X Int_\theta(f^{-1}(B)) \subset f^{-1}(m_Y Int(B))$ for every subset B of Y;
(iv) $f^{-1}(m_Y Cl(B)) \subset m_X Cl_\theta(f^{-1}(B))$ for every subset B of Y;
(v) For each $x \in X$ and each $U \in m_X$ containing x, there exists $V \in m_Y$ containing $f(x)$, such that $V \subset f(m_X - Cl(U))$.

Every almost $M-$open function is weakly $M-$open, by Theorem 17 and Lemma 14, due to the inclusion $m_X Int_\theta(A) \subset m_X Int(A)$ for every subset A of X. The converse is false in general, as shown in [28, Remark 4.2]. However, Noiri and Popa provided several types of additional assumptions under which weakly $M-$open functions are necessarily almost $M-$open.

Namely, Noiri and Popa proved that every weakly $M-$open function $f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open if at least one of the following conditions is satisfied:

(a) f is *strongly $M-$continuous*, i.e. $f(m_X - Cl(A)) \subset f(A)$ for every $A \subset X$ [28, Theorem 4.1];

(b) The space (X, m_X) is *$m-$regular*, i.e. for each $m-$closed set $F \subset X$ and each $x \in X \setminus F$ there exist disjoint $m-$open subsets U and V of X such that $x \in U$ and $F \subset V$ [28, Theorem 4.2];

(c) f satisfies the *weakly $M-$open interiority condition*, that is, for every $U \in m_X$ the following inclusion holds: $m_Y Int(f(m_X Cl(U))) \subset f(U)$ [28, Theorem 4.3];

(d) f is complementary *weakly $M-$open*, i.e. $f(m_X Fr(U))$ is $m-$closed in (Y, m_Y) for each $U \in m_X$, and f is a bijection, while m_X has property $(\mathcal{B})$ and m_Y is closed under finite intersection [28, Theorem 4.4].

3. Boundary preservation under the inverse image of an weak open function

Open functions between topological spaces can be characterized by the property of boundary preservation under inverse image, i.e. a function $f : (X, \tau_X) \to (Y, \tau_Y)$ is open if and only if $f^{-1}(\partial B) \subset \partial f^{-1}(B)$ for every $B \subset Y$.

We extend this characterization to almost $M-$open functions between minimal structure spaces.

We recall the definition of the boundary (frontier) of a set in an $m-$space.

Definition 18. [35] $m_X Fr(A) = m_X Cl(A) \cap m_X Cl(X \setminus A)$.

As a direct consequence of the above definition and of Lemma 5, we get the alternative formulas

$$\begin{aligned} m_X Fr(A) &= (X \setminus m_X Int(X \setminus A)) \cap (X \setminus m_X Int(A)) \\ &= m_X Cl(A) \setminus m_X Int(X \setminus A) \\ &= m_X Cl(X \setminus A) \setminus m_X Int(A) \end{aligned}$$

Remark 19. *(i)* $X \setminus m_X Fr(A) = m_X Int(A) \cup m_X Int(X \setminus A)$.

(ii) $x \in m_X Fr(A)$ *if and only if every set* $U \in m_X$ *containing* x *intersects both* A *and* $(X \setminus A)$.

Lemma 20. *Let* (X, m_X) *be a space with minimal structure and* $A \subset X$*. The intersection* $m_X Fr(A) \cap A$ *is empty if and only if* A *is almost* $m-$*open.*

Proof. By Remark 19, $m_X Fr(A) \cap A$ is empty if and only if $A \subset m_X Int(A) \cup m_X Int(X \setminus A)$. But the intersection $A \cap m_X Int(X \setminus A)$ is always empty, therefore $m_X Fr(A) \cap A$ is empty if and only if $A \subset m_X Int(A)$, which is equivalent to $A \in \mathcal{U}(m_X)$. $\square$

Our main result is the following characterization of almost $M-$open functions.

Theorem 21. *A function* $f : (X, m_X) \to (Y, m_Y)$ *is almost* $M-$*open if and only if*

$$f^{-1}(m_Y Fr(B)) \subset m_X Fr\left(f^{-1}(B)\right) \text{ for each } B \subset Y.$$

Proof. Necessity. Let $B \subset Y$. By the definition of the boundary,

$$f^{-1}(m_Y Fr(B)) = f^{-1}\left(m_Y Cl(B)\right) \cap f^{-1}(m_Y Cl(Y \setminus B)).$$

Assume that $f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open. Lemma 14 shows that

$$\begin{aligned} f^{-1}(m_Y Cl(B)) &\subset m_X Cl\left(f^{-1}(B)\right) \text{ and} \\ f^{-1}(m_Y Cl(Y \setminus B)) &\subset m_X Cl\left(f^{-1}(Y \setminus B)\right) = m_X Cl\left(X \setminus f^{-1}(B)\right), \end{aligned}$$

therefore $f^{-1}(m_Y Fr(B)) \subset m_X Cl(f^{-1}(B)) \cap m_X Cl(X \setminus f^{-1}(B)) = m_X Fr(f^{-1}(B))$.

Sufficiency. Let $B \subset Y$. Assume that $f^{-1}(m_Y Fr(B)) \subset m_X Fr(f^{-1}(B))$. This is equivalent to $X \setminus m_X Fr(f^{-1}(B)) \subset X \setminus f^{-1}(m_Y Fr(B))$.

We will prove that $m_X Int\left(f^{-1}(B)\right) \subset f^{-1}\left(m_Y Int(B)\right)$, then we will apply Lemma 14.

Using Remark 19 we see that $X \setminus m_X Fr\left(f^{-1}(B)\right) = m_X Int(f^{-1}(B)) \cup m_X Int(f^{-1}(Y \setminus B))$ and

$$\begin{aligned} X \setminus f^{-1}\left(m_Y Fr(B)\right) &= f^{-1}\left(Y \setminus m_Y Fr(B)\right) \\ &= f^{-1}\left(m_Y Int(B)\right) \cup f^{-1}\left(m_Y Int(Y \setminus B)\right). \end{aligned}$$

Our assumption is equivalent to

$$\begin{aligned} & m_X Int(f^{-1}(B)) \cup m_X Int(f^{-1}(Y \setminus B)) \\ \subset\ & f^{-1}\left(m_Y Int(B)\right) \cup f^{-1}\left(m_Y Int(Y \setminus B)\right). \end{aligned} \tag{1}$$

But the intersection $m_X Int(f^{-1}(B)) \cap f^{-1}\left(m_Y Int(Y \setminus B)\right)$ is empty, since $m_X Int(f^{-1}(B)) \subset X \setminus f^{-1}(Y \setminus B)$ and $f^{-1}\left(m_Y Int(Y \setminus B)\right) \subset f^{-1}(Y \setminus B)$.

Then inclusion (1) implies $m_X Int\left(f^{-1}(B)\right) \subset f^{-1}\left(m_Y Int(B)\right)$, q.e.d. $\square$

We also give a pointwise version of the previous theorem, that is stronger than that.

Theorem 22. *$f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open at $x \in X$ if and only if for every $B \subset Y$ such that $x \in f^{-1}(m_Y Fr\,(B))$ it follows that $x \in m_X Fr\left(f^{-1}(B)\right)$.*

Proof. Necessity. Let $f : (X, m_X) \to (Y, m_Y)$ be almost $M-$open at $x \in X$. Assume by contradiction that $x \in f^{-1}(m_Y Fr\,(B)) \setminus m_X Fr\left(f^{-1}(B)\right)$ for some $B \subset Y$.

Denote $A := f^{-1}(B)$. Since $x \in X \setminus m_X Fr\,(A) = m_X Int(A) \cup m_X Int(X \setminus A)$, there exists $U \in m_X$ containing x such that either $U \subset A$ or $U \subset X \setminus A$.

Case 1. Assume that $U \subset A$. Then $f(U) \subset f(A) = f\left(f^{-1}(B)\right) \subset B$. As f is almost $M-$open at x, there exists $V \in m_Y$ such that $f(x) \in V \subset f(U) \subset B$. It follows that $f(x) \in m_Y Int(B)$, a contradiction with $f\,(x) \in m_Y Fr\,(B)$.

Case 2. Assume that $U \subset X \setminus A$. But $X \setminus A = X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$, hence $f(U) \subset Y \setminus B$. As f is almost $M-$open at x, there exists $W \in m_Y$ such that $f(x) \in W \subset f(U) \subset Y \setminus B$. It follows that $f(x) \in m_Y Int(Y \setminus B)$, a contradiction with $f\,(x) \in m_Y Fr\,(B)$.

Sufficiency. Assume by contradiction that there exists some $x \in X$ such that $x \in f^{-1}(m_Y Fr\,(B))$ implies $x \in m_X Fr\,(f^{-1}(B))$ for every $B \subset Y$, but f is not almost $M-$open at x.

By the last assertion, there exists $U \in m_X$ containing x such that for every $V \in m_Y$ containing $f\,(x)$ the set V is not included in $f(U)$. Since each $V \in m_Y$ containing $f(x)$ intersects $Y \setminus f(U)$ and since $f(x) \in V \cap f(U)$, Remark 19 (ii) shows that $f(x) \in m_Y Fr\,(f(U))$. Therefore, $x \in f^{-1}(m_Y Fr\,(f(U)))$.

For $B := f\,(U)$ our assumption shows that $x \in f^{-1}(m_Y Fr\,(f(U)))$ implies $x \in m_X Fr\,(f^{-1}(f(U)))$.

On the other hand, $U \in m_X$ contains x and $U \subset f^{-1}(f(U))$, therefore $x \in m_X Int\,(f^{-1}(f(U)))$.

We obtained $x \in m_X Fr\,(f^{-1}(f(U)))$ and $x \in m_X Int\,(f^{-1}(f(U)))$, a contradiction. $\square$

4. Weak forms of open functions between generalized closure spaces

In this section we generalize the notions of almost $M-$open functions and weakly $M-$open functions to the setting of generalized closure spaces. Our aim is to extend the result on the boundary preservation under the inverse image of an almost open function.

Definition 23. *Let (X, u) be a generalized closure space. A set $A \subset X$ is said to be:*
(i) $u-$closed if $u(A) = A$;
(ii) almost $u-$closed if $u(A) \subset A$;
(iii) $u-$open if $A = u - Int(A)$;
(iv) almost $u-$open if $A \subset u - Int(A)$.

Using the terminology from [6], a subset A in an almost $u-$open in a generalized closure space (X, u) if and only if A is a neighborhood of itself.

Given a GCS (X, u), we denote by $C_u(X)$, $a - C_u(X)$, $O_u(X)$ and $a - O_u(X)$ the family of all $u-$closed sets, almost $u-$closed subsets, $u-$open subsets and almost $u-$open subsets, respectively.

Clearly, $C_u(X) \subset a - C_u(X)$ and $O_u(X) \subset a - O_u(X)$, while the reverse inclusions hold if u is expansive. Note that $U \in O_u(X)$ if and only if $X \setminus U \in C_u(X)$, and $U \in a - O_u(X)$ if and only if $X \setminus U \in a - C_u(X)$.

By the Knaster-Tarski theorem, $C_u(X) \neq \emptyset$ whenever u is isotone.

If u is idempotent, then $u(A) \in C_u(X)$ and $u - Int(A) \in O_u(X)$ for every $A \subset X$.

Example 24. *Let $m_X \subset \mathcal{P}(X)$ be a minimal structure. Denote $u := m_X Cl$. Then $O_u(X) = a - O_u(X) = \mathcal{U}(m_X)$ and $C_u(X) = a - C_u(X) = \{X \setminus A : A \in \mathcal{U}(m_X)\}$.*

The boundary of a set in a GCS is a concept which naturally generalizes the boundary of a set in a topological space, useful in some applications in Theoretical Computer Science [4].

Definition 25. *Let (X, u) be a GCS. The boundary of $A \subset X$ in (X, u) is*

$$u - Fr(A) = u(A) \cap u(X \setminus A).$$

Note that $u - Fr(A) = u(A) \setminus u - Int\,(A)$ and $X \setminus u - Fr(A) = u - Int(A) \cup u - Int(X \setminus A)$.

We recall that the notion of $m-\theta-$boundary (called $m-\theta-$frontier) $m_X Fr_\theta\,(A)$ of a set A in a space with minimal structure (X, m_X) was introduced in [24, Definition 5.4].

Definition 26. *The $m - \theta-$boundary of a set $A \subset X$ in (X, m_X) is $m_X Fr_\theta\,(A) = m_X Cl_\theta(A) \cap m_X Cl_\theta(X \setminus A)$.*

Note that $m_X Fr_\theta = u - Fr$ for $u = m_X Cl_\theta$.

Lemma 27. *Let (X, u) be a GCS, $A \subset X$ and $x \in X$. Consider the following assertions:*

(i) $x \in u(A)$;

(ii) $D \cap A \neq \emptyset$ for every almost $u-$open set $D \subset X$ containing x.

a) If u is isotone, then (i) implies (ii).

b) If u is expansive and idempotent, then (ii) implies (i).

Proof. a) Let u be isotone. Assume by contradiction that $x \in u(A)$ and that $D \cap A = \emptyset$ for some almost $u-$open set $D \subset X$ containing x. The assumptions that $A \subset X \setminus D$, u is isotone and $X \setminus D$ is almost $u-$closed imply $x \in u(A) \subset u\,(X \setminus D) \subset X \setminus D$. We get $u \in X \setminus D$, a contradiction.

b) Let u be expansive and idempotent. Assume by contradiction that $D \cap A \neq \emptyset$ for every almost $u-$open set $D \subset X$ containing x and that $x \in X \setminus u(A) = u - Int(X \setminus A)$. The set $D_A = u - Int(X \setminus A)$ is $u-$open, since u is idempotent, therefore $D_A \cap A \neq \emptyset$. On the other hand, since u is expansive it follows $D_A \subset X \setminus A$, which contradicts $D_A \cap A \neq \emptyset$. □

Corollary 28. *Let (X,u) be a GCS, $A \subset X$ and $x \in X$. Consider the following assertions:*
(i) $x \in u - Fr(A)$;
(ii) $D \cap A \neq \emptyset$ and $D \cap (X \setminus A) \neq \emptyset$ for every almost $u-$open set $D \subset X$ containing x.
a) If u is isotone, then (i) implies (ii).
b) If u is expansive and idempotent, then (ii) implies (i).

We recall some weak forms of open functions between generalized closure spaces, as defined in [16].

Definition 29. *Let (X,u) and (Y,v) be generalized closure spaces. A function $f:(X,u) \to (Y,v)$ is said to be:*
a) open if $f(A)$ is $v-$open whenever $A \subset X$ is $u-$open;
b) almost open if $f\,(u - Int(A)) \subset v - Int(f(A))$ for every $A \subset X$;
c) weakly open if $f\,(A) \subset v - Int(f(u(A))$ whenever $A \subset X$ is almost $u-$open.

Example 30. *Let (X, m_X) and (Y, m_Y) be spaces with minimal structure.*
a) For $u = m_X Cl$ and $v = m_Y Cl$, a function $f:(X,u) \to (Y,v)$ is almost open if and only if $f:(X,m_X) \to (Y,m_Y)$ is almost $M-$open, by Lemma 14. Note that in this setting the notions of open function and almost open functions are the same, since $u = m_X Cl$ and $v = m_Y Cl$ are expansive.
b) For $u = m_X Cl$ and $v = m_Y Cl$, a function $f:(X,u) \to (Y,v)$ is weakly open if and only if $f:(X,m_X) \to (Y,m_Y)$ is weakly $M-$open, by Lemma 16.
c) For $u = m_X Cl_\theta$ and $v = m_Y Cl$, a function $f:(X,u) \to (Y,v)$ is almost open if and only if $f:(X,m_X) \to (Y,m_Y)$ is weakly $M-$open, by Theorem 17 [28, Theorem 3.1].

We compare the above forms of weak open functions.
Lemma *Let $f:(X,u) \to (Y,v)$ be almost open. Then:*
(i) The set $f(A)$ is almost $v-$open whenever $A \subset X$ is almost $u-$open.
(ii) If u is expansive and v is isotone, then f is weakly open.

Proof. Let $A \subset X$ be almost $u-$open, i.e. $A \subset u - Int(A)$. Then:

(i) $f(A) \subset f(u - Int(A)) \subset v - Int(f((A))$, hence $f(A)$ is almost $v-$open.

(ii) Assuming that u is expansive and v is isotone and using $v - Int(f(A)) \subset v - Int(f(u(A))$, it follows that $f(A) \subset v - Int(f(u(A))$. □

Remark 31. *If u is expansive and idempotent and v is isotone, then every function $f : (X, u) \to (Y, v)$ that maps almost $u-$open sets to almost $v-$open sets is an almost open function, in particular every open function is almost open.*

We are also interested in pointwise versions for the above types of weak open functions between generalized closure spaces.

Definition 32. *Let (X, u) and (Y, v) be generalized closure spaces. A function $f : (X, u) \to (Y, v)$ is said to be:*

a) open at $x \in X$ if for each $u-$open set $U \subset X$ containing x the set $f(U)$ is $v-$open,

b) almost open at $x \in X$ if for each almost $u-$open set U containing x there exists an almost $u-$open set V containing $f(x)$ such that $V \subset f(U)$;

c) weakly open at $x \in X$ if for each almost $u-$open set U containing x there exists an almost $u-$open set V containing $f(x)$ such that $V \subset f(u(U))$.

Obviously, every function that is open at a point is almost open at that point. If u is expansive, then every function that is almost open at a point is also weakly open at that point.

We compare the pointwise version and the global version for the notions of almost open function and of weakly open function in generalized closure spaces, respectively.

Proposition 33. *Let (X, u) and (Y, v) be generalized closure spaces and $f : (X, u) \to (Y, v)$. If f is almost open, then f is almost open at every point $x \in X$. The converse holds if u is an expansive, idempotent operator and v is isotone.*

Proof. Assume that $f : (X, u) \to (Y, v)$ is almost open. Fix an arbitrary $x \in X$. Let $U \subset X$ be an almost $u-$open set containing x. Then $f(x) \in f(U) \subset f(u - Int(U)) \subset v - Int(f(U))$, therefore $V := f(U)$ is a almost $v-$open set containing $f(x)$ that is included in $f(U)$. We proved that f is almost open at x.

Let $f : (X, u) \to (Y, v)$ be almost open at every point $x \in X$. Assume that u is an expansive, idempotent operator and v is isotone. We prove that $f\,(u - Int(A)) \subset v - Int\,(f(A))$ for every $A \subset X$. We may suppose that $u - Int(A)$ is nonempty.

Fix an arbitrary $x \in u - Int(A)$. Since u is idempotent, the set $u - Int(A)$ is $u-$open. As f is almost open at x, there exists an almost $v-$open set $V \subset Y$ such that $f(x) \in V \subset f\,(u - Int(A))$. But u is expansive, therefore $f\,(u - Int(A)) \subset f(A)$. Since v is isotone,

$$V \subset f(A) \text{ implies } v - Int(V) \subset v - Int(f(A)),$$

but $f(x) \in V \subset v - Int(V)$, hence $f(x) \in v - Int(f(A))$, q.e.d. $\square$

Proposition 34. *Let (X, u) and (Y, v) be generalized closure spaces and $f : (X, u) \to (Y, v)$.*

(i) Assume that u is expansive. If f is weakly open, then f is weakly open at every point $x \in X$.

(ii) Assume that v is an expansive, isotone and idempotent operator. If f is weakly open at every point $x \in X$, then f is weakly open.

Proof. (i) Assume u is expansive. Let $f : (X, u) \to (Y, v)$ be weakly open. Fix an arbitrary $x \in X$. Let $U \subset X$ be an almost $u-$open set containing x. Then $f(x) \in f(U) = f(u - Int(U)) \subset v - Int\,(f(u\,(U)))$. Then $V := f(U)$ is an almost $v-$open set containing $f(x)$ that is included in $f(U)$. We proved that f is almost open at x.

(ii) Assume that v is an expansive, isotone and idempotent operator. Let $f : (X, u) \to (Y, v)$ be weakly open at every point $x \in X$. We prove that $f\,(A) \subset v - Int\,(f(u(A)))$ for every almost $u-$open set $A \subset X$. We may suppose that A is nonempty.

Fix an arbitrary $x \in A$. As f is weakly open at x, there exists an almost $v-$open set $V \subset Y$ such that $f(x) \in V \subset f\,(u(A))$. But v is expansive, therefore $V = v - Int(V)$. Moreover, since v is idempotent and isotone, we have $v - Int(V) = v - Int(v - Int(V)) \subset v - Int\,(f(A))$.

We get $f\,(x) \in v - Int\,(f(A))$, as desired. $\square$

We give for a counterpart for [13, Theorem 5.1], providing necessary conditions, respectively sufficient conditions for a function $f : (X, u) \to (Y, v)$ to be almost open, under minimal assumptions on the closure operators.

Lemma 35. *Let (X,u) and (Y,v) be generalized closure spaces. For $f:(X,u)\to(Y,v)$ consider the following assertions:*
(i) f is almost open;
(ii) $u-Int\left(f^{-1}(B)\right)\subset f^{-1}\left(v-Int(B)\right)$ for every $B\subset Y$;
(iii) $f^{-1}\left(v(C)\right)\subset u\left(f^{-1}(C)\right)$ for every $C\subset Y$.
Then (ii) and (iii) are equivalent and
a) For v isotone, (i) implies (ii);
b) For u isotone, (ii) implies (i).

Proof. The equivalence between (ii) and (iii) is clear, since $f^{-1}\left(v(Y\setminus B)\right)=X\setminus f^{-1}\left(v-Int(B)\right)$ and $u\left(f^{-1}(Y\setminus B)\right)=X\setminus u-Int\left(f^{-1}(B)\right)$, for every $B\subset Y$.

a) Let v be isotone. Assume that f is almost open. Let $B\subset Y$. Then $f\left(u-Int\left(f^{-1}(B)\right)\right)\subset v-Int\left(f(f^{-1}(B)\right)$. Since $f(f^{-1}(B)\subset B$ and v is isotone, $v-Int\left(f(f^{-1}(B)\right)\subset v-Int(B)$. It follows that $f\left(u-Int\left(f^{-1}(B)\right)\right)\subset v-Int(B)$, hence $u-Int\left(f^{-1}(B)\right)\subset f^{-1}\left(v-Int(B)\right)$.

b) Let u be isotone. Assume that (ii) holds. Let $A\subset X$. We prove that $f\left(u-Int\left(A\right)\right)\subset v-Int\left(f(A)\right)$. Since $A\subset f^{-1}(f(A))$ and u is isotone, $u-Int\left(A\right)\subset u-Int\left(f^{-1}(f(A))\right)$, hence $f\left(u-Int\left(A\right)\right)\subset f\left(u-Int\left(f^{-1}(f(A))\right)\right)$. Using (ii) for $B=f(A)$, it follows that $f\left(u-Int\left(f^{-1}(f(A))\right)\right)\subset v-Int\left(f(A)\right)$. Then $f\left(u-Int\left(A\right)\right)\subset v-Int\left(f(A)\right)$, q.e.d. □

Now we generalize the result on the boundary preservation under the inverse image of an almost open function, from the setting of minimal structure spaces to that of generalized closure spaces.

Theorem 36. *Let $(X,u),(Y,v)$ be generalized closure spaces and $f:(X,u)\to(Y,v)$. Consider the following assertions:*
(i) f is almost open.
(ii) $f^{-1}\left(v-Fr(B)\right)\subset u-Fr(f^{-1}(B))$ for every $B\subset Y$.
a) For v isotone, (i) implies (ii);
b) For u isotone expansive and v expansive, (ii) implies (i).

Proof. a) Let v be isotone. Assume that f is almost open. Let $B\subset Y$. Then $f^{-1}\left(v-Fr(B)\right)=f^{-1}\left(v(B)\cap v(Y\setminus B)\right)$. By Lemma 35 (a),

$$\begin{aligned} f^{-1}\left(v-Fr(B)\right) &= f^{-1}\left(v(B)\right)\cap f^{-1}\left(v(Y\setminus B)\right) \\ &\subset u\left(f^{-1}(B)\right)\cap u\left(X\setminus f^{-1}(B)\right)=u-Fr(B). \end{aligned}$$

b) Let u be isotone expansive and v be expansive. Assume that (ii) holds. Let $B \subset Y$. We will prove that $u - Int\left(f^{-1}(B)\right) \subset f^{-1}\left(v - Int(B)\right)$, then we may apply Lemma 35 (b), since u is isotone.

Taking complements of both members in (ii), we get

$$(2) \qquad \begin{aligned} & u - Int\left(f^{-1}(B)\right) \cup u - Int\left(f^{-1}(Y \setminus B)\right) \\ \subset \;& f^{-1}\left(v - Int(B)\right) \cup f^{-1}\left(v - Int(Y \setminus B)\right). \end{aligned}$$

Having u expansive, $u - Int(f^{-1}(B)) \subset f^{-1}(B) = X \setminus f^{-1}(Y \setminus B)$. Since v is expansive, we get $f^{-1}\left(v - Int(Y \setminus B)\right) \subset f^{-1}(Y \setminus B)$. It follows that $u - Int\left(f^{-1}(B)\right) \cap f^{-1}\left(v - Int(Y \setminus B)\right) = \emptyset$.

Then (2) implies $u - Int\left(f^{-1}(B)\right) \subset f^{-1}\left(v - Int(B)\right)$, q.e.d. □

Corollary 37. *Let (X, m_X) and (Y, m_Y) be spaces with minimal structure. A function $f : (X, m_X) \to (Y, m_Y)$ is weakly $M-$open if and only if*

$$f^{-1}\left(m_Y Fr(B)\right) \subset m_X Fr_\theta\left(f^{-1}(B)\right) \text{ for all } B \subset Y.$$

Proof. It suffices to note that $f : (X, m_X) \to (Y, m_Y)$ is weakly $M-$open if and only if the function between GCS's $f : (X, m_X Cl_\theta) \to (Y, m_Y Cl)$ is almost open, as well as that the generalized closure operators $m_X Cl_\theta$ and $m_Y Cl$ are both isotone and expansive. □

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