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λ-NUMBER OF BANANA TREES

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Abstract. An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at a distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0, 1, \ldots, k\}$, and we are interested to find the minimum k among all possible labelings. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. In this paper, we consider banana trees of type 1, banana trees of type 2 and path-union of k-copies of the star $K_{1,n}$ and find the λ-numbers of them.

1. Introduction

Here, we consider only a simple, finite, connected, undirected graph without loops or multiple edges. For standard terminology and notation, we follow Bondy and Murty [1] or Murugan [2]. Following the standard terminology, we use P_n to denote a path on n vertices, C_n to denote a cycle on n vertices, $K_{1,n}$ to denote a star, T to denote a Tree, Δ to denote the maximum degree of a graph, $\lceil x \rceil$ to denote the least integer greater than or equal to x, and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

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The interest towards $L(2,1)$-labeling comes from the radio channel assignment problem, where each vertex is taken to be a transmitter location, with the label assigned to it determining the channel on which it transmits. We note that the available channels are uniformly spaced in the spectrum justifying integer labeling. The assignment of frequencies to each transmitter (vertex) should avoid interference which depends on the separation each pair of vertices has. In this problem there are two levels of interference which correspond to two constraints in the labeling. For this, we use $L(2,1)$-labeling which is defined as below.

An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0,\ldots,k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. The generalization of this concept is as below.

For positive integers k, d_1, d_2, a k-$L(d_1, d_2)$-labeling of a graph G is a function $f : V(G) \to \{0,1,2,\ldots,k\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v in G, $d_G(u,v) = i$, for $i = 1, 2$. The $L(d_1, d_2)$-number of G, $\lambda_{d_1,d_2}(G)$, is the smallest k such that there exists a k-$L(d_1, d_2)$-labeling of G.

2. Some Existing Results

Distance two labeling or $L(2,1)$-labeling has received the attention of many researchers and here we present some important existing results.

- Yeh [3] have discussed the $L(2,1)$-labeling on various class of graphs like trees, cycles, chordal graphs, Cartesian products of graphs etc.
- Vaidya and Bantava [4] have discussed $L(2,1)$-labeling of cacti.
- Vaidya et.al. [5] have discussed $L(2,1)$-labeling in the context of some graph operations.
- Chang and Kuo [6] provided an algorithm to obtain $\lambda(T)$.
- Murugan [7] has obtained an upper bound of the λ-number for the corona $G_1 \circ G_2$ where G_1 and G_2 are any two graphs and found the condition for the attainment of the bound.

- In [8] Griggs and Yeh have discussed $L(2,1)$-labeling for path, cycle, tree and cube. They have derived results for the graphs of diameter 2. They have shown that the $\lambda(T)$ for trees with maximum degree $\Delta \geq 1$ is either $\Delta + 1$ or $\Delta + 2$.
- Griggs and Yeh [8] proved that if a graph G contain three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$, where Δ is the maximum degree of G.
- Griggs and Yeh [8] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and they proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ at the same time.
- Chang and Kuo [6] proved that $\lambda(G) \leq \Delta^2 + \Delta$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Kral and Skrekovski [9] proved that $\lambda(G) \leq \Delta^2 + \Delta - 1$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Goncalves [10] proved that $\lambda(G) \leq \Delta^2 + \Delta - 2$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.

In spite of all the efforts the conjecture posed by Griggs and Yeh is still open.

3. Banana Trees

Chen, Lu and Yeh [11] defined a banana tree as a graph obtained by connecting a new vertex v to one leaf of each of any number of stars. They have conjectured that banana trees are graceful, which is still open. We define a new family of banana trees by joining a new vertex to the central vertex of any number of stars and call this as banana tree of type 2 and we call the above as banana tree or banana tree of type 1.

Sze-Chin Shee and Yong-Song Ho [12] defined path-union of n copies of a graph as below and they relate the cordiality of the path-union of n copies of a graph to the solution of a system involving an equation and two inequalities, and give some sufficient conditions for that path-union to be cordial. Let $G_1, G_2, \ldots, G_n$ be n (≥ 2) copies of a graph G. A graph obtained by adding an edge to G_i and G_{i+1}, $i = 1, 2, \ldots, n-1$ is called the path-union of n copies of the graph G and it is denoted by $G(n)$.

In this paper, we consider banana tree of type 1, banana tree of type 2 and path-union of k copies of star $K_{1,n}$ and find the λ-number of them.

4. Results

Theorem 4.1. *The λ-number of a banana tree T with k copies of $K_{1,n}$ is*

$$\lambda(T) = \begin{cases} n+1 & \text{if} \quad k \leq n-1 \\ n+2 & \text{if} \quad k = n \\ k+1 & \text{if} \quad k \geq n+1 \end{cases}$$

Proof. Consider k copies of a star $K_{1,n}$. For $i = 1, 2, \ldots, k$, let $v_{i,j}$, $j = 0, 1, 2, \ldots, n$ be the vertices of $K_{1,n}$, where $v_{i,0}$ is the central vertex of the i-th copy of the star $K_{1,n}$ and $v_{i,j}$, $j = 1, 2, \ldots, n$ be the end vertices of the i-th copy of the star $K_{1,n}$.

Case 1: $k \leq n-1$.

Now, we label the vertices $v_{i,j}$, $i = 1, 2, \ldots, n-1$ and $j = 0, 1, 2, \ldots, n$ by the following function f.

$$\begin{aligned} f(v_{i,0}) &= 0, && i = 1, 2, \ldots, n-1 \\ f(v_{i,j}) &= j+1, && i = 1, 2, \ldots, n-1 \text{ and } j = 1, 2, \ldots, n \end{aligned}$$

Introduce a new vertex u and label it with 1. Join this new vertex u to the vertex with label $i+2$ of the i-th copy of $K_{1,n}$, $i = 1, 2, \ldots, n-1$.

Now we prove that this labeling of the banana tree is an $L(2,1)$ labeling. For $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n$,

$$|f(v_{i,0}) - f(v_{i,j})| = j + 1 \geq 2.$$

The label of the edge connecting u with the i-th copy of $K_{1,n} = i + 2 \geq 2$, $i = 1, 2, \ldots, n-1$.

Here, the vertices at distance two have different labelings and so their label difference are always greater than or equal to one. Hence this is an $L(2,1)$ labeling. That is, $\lambda(T) \leq n+1$, if $k \leq n-1$. Since the maximum degree of T is n, $\lambda(T) \geq n+1$. Hence, $\lambda(T) = n+1$, if $k \leq n-1$.

Case 2: $k = n$.

First, we label $v_{i,0}$ by

$$f(v_{i,0}) = \begin{cases} i+3, & i = 1, 2, \ldots, n-1 \\ n-1, & i = n \end{cases}$$

Now, we label the vertices $v_{i,j}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, by the following function f.

Let $i \leq n - 2$.

$$f(v_{i,j}) = \begin{cases} j-1, & j = 1, 2, \ldots, i+2 \\ j+2, & j = i+3, \ldots, n \end{cases}$$

Let $i = n - 1$.

$$f(v_{i,j}) = j, \quad j = 1, 2, \ldots, n.$$

Let $i = n$.

$$f(v_{i,j}) = \begin{cases} j-1, & j = 1, 2, \ldots, n-2 \\ j+2, & j = n-1, n \end{cases}$$

Introduce a new vertex u and label it with 0. Join this new vertex u to the vertex with label $i+1$ of the i-th copy of $K_{1,n}$, $i = 1, 2, \ldots, n$.

Now we prove that this labeling of the banana tree is an $L(2,1)$ labeling.

First we consider adjacent vertices.

When $i \leq n - 2$,

The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= \min\{|(i+3) - (i+2-1)|, |(i+3) - (i+5)|\} \\ &= \min\{2, 2\} = 2. \end{aligned}$$

When $i = n - 1$,

The minimum edge label of the i-th copy of $K_{1,n}$

$$= \ i + 3 - n = i + 3 - (i+1) = 2$$

When $i = n$,

The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= \min\{|(n-1) - (n-3)|, |(n-1) - (n+1)|\} \\ &= \min\{2, 2\} = 2. \end{aligned}$$

The edge label between u and i-th copy of $K_{1,n} = i + 1 \geq 2$, $i = 1, 2, \ldots, n$.

Here, the vertices at distance two have different labelings and so their label difference are always greater than or equal to one. Hence, this is an $L(2,1)$ labeling.

That is, $\lambda(T) \leq n + 2$, if $k = n$.

Suppose $\lambda(T) \leq n+1$, then the labels $0, 1, 2, \ldots, n+1$ are sufficient to give an $L(2,1)$ labeling of T. Among this we give one color to the vertex u, say x.

If $x \neq 0$, then u cannot be adjacent with the vertices with the labels $x-1, x, x+1$. Thus, remaining labels are $n+2-3 = n-1$ only. Therefore, u can be adjacent with these $n-1$ labels only. But there are n copies of $K_{1,n}$ and u must be adjacent with one end vertex of all $K_{1,n}$ with different labels. This is a contradiction.

If $x = 0$, 0 cannot be given to the central vertex of $K_{1,n}$ and some $\alpha \neq 0$ has to be given to the central vertex. Then $\alpha-1, \alpha, \alpha+1$ cannot be given to the end vertices of that $K_{1,n}$. So remaining $n+2-3 = n-1$ labels are available to label n end vertices of that $K_{1,n}$. This is not possible. Therefore, $\lambda(T) = n+2$. That is, $\lambda(T) = n+2$, if $k = n$.

Case 3: $k \geq n+1$.

First, we label $v_{i,0}$ by

$$f(v_{i,0}) = \begin{cases} i+3, & i = 1, 2, \ldots, n-1 \\ n-1, & i = n \\ i-n, & i = n+1, n+2, \ldots, k \end{cases}$$

Now, we label the vertices $v_{i,j}$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$, by the following function f.

Let $i \leq n-2$.

$$f(v_{i,j}) = \begin{cases} j-1, & j = 1, 2, \ldots, i+2 \\ j+2, & j = i+3, \ldots, n \end{cases}$$

Let $i = n-1$.

$$f(v_{i,j}) = j, \quad j = 1, 2, \ldots, n.$$

Let $i = n$.

$$f(v_{i,j}) = \begin{cases} j-1, & j = 1, 2, \ldots, n-2 \\ j+2, & j = n-1, n \end{cases}$$

Let $i \geq n+1$.

$$f(v_{i,j}) = j + (i-n+1), \quad j = 1, 2, \ldots, n.$$

Introduce a new vertex u and label it with 0. Join this new vertex u to the vertex with label $i+1$ of the i-th copy of $K_{1,n}$, $i = 1, 2, \ldots, k$.

Now we prove that this labeling of the banana tree is an $L(2,1)$ labeling.

First we consider adjacent vertices.

When $i \leq n-2$,
The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= \min\{|(i+3)-(i+2-1)|, |(i+3)-(i+5)|\} \\ &= \min\{2,2\} = 2. \end{aligned}$$

When $i = n-1$,
The minimum edge label of the i-th copy of $K_{1,n}$

$$= \; i+3-n = i+3-(i+1) = 2.$$

When $i = n$,
The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= \min\{|(n-1)-(n-3)|, |(n-1)-(n+1)|\} \\ &= \min\{2,2\} = 2. \end{aligned}$$

When $i \geq n+1$,
The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= |(i-n)-\{j+(i-n+1)\}|, \quad j = 1,2,\ldots,n. \\ &\geq 2. \end{aligned}$$

The edge label between u and i-th copy of $K_{1,n} = i+1 \geq 2$, $i = 1,2,\ldots,k$. Here, the vertices at distance two have different labelings and so their label difference are always greater than or equal to one. Hence, this is an $L(2,1)$ labeling.

That is, $\lambda(T) \leq k+1$, if $k \geq n+1$.

Since the maximum degree of T is k, $\lambda(T) \geq k+1$. Hence, $\lambda(T) = k+1$, if $k \geq n+1$. $\square$

Theorem 4.2. *The λ-number of a banana tree of type 2, T with k copies of $K_{1,n}$ is*

$$\lambda(T) = \begin{cases} n+3 & \textit{if} \quad k \leq n \\ k+1 & \textit{if} \quad k \geq n+1 \end{cases}$$

Proof. Consider k copies of a star $K_{1,n}$. For $i = 1,2,\ldots,k$, let $v_{i,j}$, $j = 0,1,2,\ldots,n$ be the vertices of $K_{1,n}$, where $v_{i,0}$ is the central vertex of the i-th copy of the star $K_{1,n}$ and $v_{i,j}$, $j = 1,2,\ldots,n$ be the end vertices of the i-th copy of the star $K_{1,n}$. We introduce a new vertex u and join to $v_{i,0}$, the central vertex of the i-th copy of the star $K_{1,n}$.

Case 1: $k \leq n$.

Now, we label the vertices $v_{i,j}$, $i = 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n$ by the following function f.

$$\begin{array}{rcll} f(v_{i,0}) & = & i+1, & i = 1, 2, \ldots, n \\ f(v_{i,j}) & = & j, & j = 1, 2, \ldots, i-1 \\ & = & j+3, & j = i, i+1, \ldots, n \end{array}$$

Also label the new vertex u with 0.

The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{array}{rl} = & \min\{|(i+1) - (i-1)|, |(i+1) - (n+3)|\} \\ = & \min\{|(i+1) - (i-1)|, |(n+1) - (n+3)|\} \\ = & \min\{2, 2\} = 2. \end{array}$$

The edge label of $uv_{i,0} = i + 1$, $i = 1, 2, \ldots, n$.

Since the vertices at distance 2 are all different, this is an $L(2, 1)$ labeling and $\lambda(T) \leq n + 3$.

Since the maximum degree is $n + 1$, $\lambda(T) \geq n + 2$.

Suppose $\lambda(T) = n + 2$, the labels $0, 1, 2, \ldots, n + 2$ are sufficient to give a $L(2, 1)$-labeling of T. If the label α is given to the vertex u, and if the label x is given to the vertex $v_{i,0}$, then the labels $\alpha, x, x-1$, and $x + 1$ cannot be given to end vertices of the i-th copy of $K_{1,n}$, and so $n + 3 - 4 = n - 1$ labels are available to label the n end vertices of the i-th copy of $K_{1,n}$, which is a contradiction. Hence $\lambda(T) > n + 2$. That is, $\lambda(T) = n + 3$.

Case 2: $k \geq n + 1$.

Now, we label the vertices $v_{i,j}$, $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, n$ by the following function f.

$$f(v_{i,0}) = i + 1, \;\; i = 1, 2, \ldots, k.$$

Let $k \leq n$.

We define

$$f(v_{i,j}) = \begin{cases} j, & j = 1, 2, \ldots, i-1 \\ j+3, & j = i, i+1, \ldots, n \end{cases}$$

Let $k \geq n + 1$.

We define $f(v_{i,j}) = j, \;\; j = 1, 2, \ldots, n$

Also label the new vertex u with 0.

Now we prove that this labeling of the banana tree is an $L(2, 1)$ labeling.

First we consider adjacent vertices.

Let $k \leq n$.

The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= \min\{|(i+1)-(i-1)|, |(i+1)-(n+3)|\} \\ &= \min\{|(i+1)-(i-1)|, |(n+1)-(n+3)|\} \\ &= \min\{2,2\} = 2. \end{aligned}$$

The edge label of $uv_{i,0} = i+1$, $i = 1, 2, \ldots, n$.

Let $k \geq n+1$. The minimum edge label of the i-th copy of $K_{1,n}$

$$\begin{aligned} &= |(i+1)-n| \\ &= |(n+1+1)-n| = 2. \end{aligned}$$

The edge label of $uv_{i,0} = i+1$, $i = 1, 2, \ldots, n$.

Here, the vertices at distance two have different labelings and so their label difference are always greater than or equal to one. Hence, this is an $L(2,1)$ labeling.

That is, $\lambda(T) \leq k+1$, if $k \geq n+1$.

Since the maximum degree of T is k, $\lambda(T) \geq k+1$. Hence,

$$\lambda(T) = k+1,$$

if $k \geq n+1$. □

Theorem 4.3. *The λ-number of T, the path-union of $k(\geq 2)$ copies of $K_{1,n}$, $n \geq 3$, is $n+1$.*

Proof. Let $T_1, T_2, \ldots, T_k$ be $k(\geq 2)$ copies of a star $K_{1,n}$, $n \geq 3$. Let the vertices of T_i be $v_{i,j}$, $i = 1, 2, \ldots, k$ and $j = 0, 1, 2, \ldots, n$ where $v_{i,0}$ is the central vertex of the i^{th} copy of the star $K_{1,n}$ and $v_{i,j}$, $j = 1, 2, \ldots, n$ be the end vertices of the i^{th} copy of the star $K_{1,n}$. Now join $v_{i,n}$ and $v_{i+1,1}$, $i = 1, 2, \ldots, k-1$ by a new edge and the resulting graph is the path-union of the star $K_{1,n}$ and call it as T.

Now, we label the vertices $v_{i,j}$, for all i and $j = 0, 1, \ldots, n$ by the following function f.

$$\begin{aligned} f(v_{i,0}) &= 0, && \text{for all } i \\ f(v_{i,j}) &= j+1, && \text{for all } i \text{ and } j = i, i+1, \ldots, n. \end{aligned}$$

In each copy of $K_{1,n}$, the edge labels are $j+1, j = 1, 2, \ldots, n$ and since all the end vertices of $K_{1,n}$ are distinct, $L(2,1)$ labeling conditions are satisfied. The label of the edge between $v_{i,n}$ and $v_{i+1,1}$, $i = 1, 2, \ldots, n-1$ is $n+1-2 = n-1 \geq 2$.

Therefore, this is an $L(2,1)$ labeling. That is, $\lambda(T) \leq n+1$. Since the maximum degree of T is n, $\lambda(T) \geq n+1$. Hence $\lambda(T) = n+1$. $\square$

5. Conclusion

This work motivates to find $L(2,1)$ labeling on rooted trees which are useful in Decision making, Enumeration, Computer science and so on. Sze-Chin Shee and Yong-Song Ho [12] have proved a sufficient condition for the graph $G^{(n)}$ to be cordial, which is related to the solution of a system involving one equation and two inequalities with their coefficients depending on some binary labeling of G, where G is rooted graph and $G^{(n)}$, the graph obtained from n copies of G by identifying their roots. Chang and Kuo [6] produced a polynomial time algorithm for computing λ-number of a tree. It is based on dynamic programming approach and very recently, a linear time algorithm has been established [13]. The latter algorithm achieves the linear running time by best utilizing an important property of labeling, called label compatibility. Since this property holds for more general labelings, say $L(p,1)$-labeling, the linear time algorithm for $L(2,1)$-labeling of trees can be extended to a linear time algorithm for $L(p,1)$-labeling of trees for a fixed positive integer p.

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