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ON δ–β–GENERALIZED CLOSED SETS IN TOPOLOGICAL SPACES

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Abstract. The concepts of δ–β–open sets and δ–β–continuous functions have been introduced by Hatir and Noiri [12] and the ideas were further investigated and their properties have been explored in [13]. In the present paper we introduce a new notion of generalized closed sets called $\delta\beta g$–closed sets in topological spaces which is the more general form of generalized δ–closed, δ–generalized–semi–closed, δ generalized preclosed and δ–β–closed sets. Extending this idea to define and study $\delta\beta g$–quotient maps and $\delta\beta g$-regular and $\delta\beta g$–normal spaces, the authors have explored further characterizations of the new concept.

1. Introduction

Levine [17], Monsef et al. [21] and Velićko [29] introduced semi–open sets, β-open sets, δ-open and δ-closed sets, respectively. The initiation of study of g–closed sets was done by Levine [18] in 1970. A subset A of a topological space X is said to be generalized closed (resp. g-closed) if $Cl(A) \subset U$ whenever $A \subset U$ and U is open.

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This notion has been studied extensively in recent years by many topologist because generalized closed sets are not only natural generalizations of closed sets but also suggestions of several new properties of topological spaces.

The spaces in which the concepts of closed sets and g–closed sets coincide are called $T_{1/2}$–spaces. The concept of g–closed sets has been generalized and investigated in the last twenty years by weaker forms of open sets such as, α–open [23], preopen [20], semi-open [17] and β–open [21]. By combining the concepts of δ–closedness and g–closedness, Dontchev and Ganster [4] proposed a new class of generalised closed sets called δ–generalized–closed sets and also introduced the notation of $T_{3/4}$–spaces as the spaces where every δ–generalized–closed set is δ–closed. Dontchev et al. [5] also introduced and studied generalized δ–closed sets in topological spaces. Park et al. [26] propounded and investigated the concept of δ–generalized–semi–closed sets. Benchalli [3] introduced the notion of δ generalized preclosed (briefly, δgp–closed) sets.

The purpose of the present paper is to define a new class of generalized–closed sets called *δ–β–generalized–closed* (briefly $\delta\beta g$–closed) sets. In the section 2, we put some basic definitions and results which are used to carry out our work. In section 3, we define $\delta\beta g$–closed sets and study some basic properties and its relation with some already existing closed sets in topological spaces. Subsequently we define and investigate $\delta\beta g$–open sets. In section 4, we introduce and study $\delta\beta g$–continuous functions. In section 5, we initiate and explore the notions of $\delta\beta$–quotient maps and $\delta\beta g$–quotient maps by using $\delta\beta g$–closed sets and $\delta\beta g$–open sets. In the last section, we introduce and study the notions of $\delta\beta g$–regular and $\delta\beta g$–normal spaces.

1.1. **Preliminaries.** Throughout this paper, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. The function $f : (X, \tau) \to (Y, \sigma)$ denotes a single valued function of a space (X, τ) into a space (Y, σ). Let A be a subset of a space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. $O(X)$ and $C(X)$ denote collection of open subsets of X and collection of closed subsets of X, respectively.

Here we recall the following known definitions and properties.

The δ–interior [29] of a subset A of a space X is the union of all regular open subsets of X contained in A and is denoted by $Int_{\delta}(A)$. The subset A is said to be δ–open [29] if $A = Int_{\delta}(A)$. i.e., a set is δ–open if it is the union of regular open sets. The complement of a δ–open set is said to be δ–closed. Alternatively, a subset A of (X, τ) is said to be δ–closed [29] if $A = Cl_{\delta}(A)$, where $Cl_{\delta}(A) = \{x \in X : Int(Cl(U)) \cap A \neq \phi,\ U \in \tau \text{ and } x \in U\ \}$. The family of δ–open sets forms a topology on X and it is denoted by τ_{δ}.

Definition 1. *Let (X, τ) be a topological space. A subset A of X is said to be*

(1) *preopen* [20] *if* $A \subseteq Int(Cl(A))$.
(2) *semi–open*[17] *if* $A \subseteq Cl(Int(A))$.
(3) *α–open* [23] *if* $A \subseteq Int(Cl(Int(A))$.
(4) *β–open* [21] *if* $A \subseteq Cl(Int(Cl(A)))$.
(5) *δ–semiopen* [26] *if* $A \subseteq Cl(Int_{\delta}(A))$.
(6) *δ–β–open* [12] *if* $A \subseteq Cl(Int(Cl_{\delta}(A)))$.

The complement of semi–open sets (resp. α–open sets, preopen sets, β–open sets, δ–semiopen sets and δ–β–open) are called semi–closed sets [17](resp. α–closed sets [23], preclosed sets [20], β–closed sets [21], δ–β–closed [12] and δ–semiclosed [26]).

Definition 2. *Let (X, τ) be a topological space and A a subset of X.*

(1) *The union of all δ–β–open (resp. δ–semiopen, preopen* [20]*) sets contained in A is called the δ–β–interior (resp. δ–semiinterior, preinterior* [20]*)of A and is denoted by* $_{\beta}Int_{\delta}(A)$ [12] *(resp.* $_{s}Int_{\delta}(A)$ [25], $pInt(A)$ [20]*).*
(2) *The intersection of all δ–β–closed (resp. δ–semiclosed, preclosed* [20]*) sets containing A is called the δ–β–closure* [12] *(resp. δ–semiclosure* [26]*, preclosure* [20] *) of A and is denoted by* $_{\beta}Cl_{\delta}(A)$ *(resp.* $_{s}Cl_{\delta}(A)$, $pCl(A)$ [20]*).*

Lemma 3. [12] *For a subset A of a topological space (X, τ), the following properties hold:*

(1) $_{s}Int_{\delta}(A) = A \cap Cl(Int_{\delta}(A))$; $_{s}Cl_{\delta}(A) = A \cup Int(Cl_{\delta}(A))$,
(2) $_{\beta}Int_{\delta}(A) = A \cap Cl(Int(Cl_{\delta}(A)))$; $_{\beta}Cl_{\delta}(A) = A \cup Int(Cl(Int_{\delta}(A)))$.

Definition 4. *Let (X, τ) be a topological space. A subset A of X is said to be*

(1) *g–closed* [18] *if $Cl(A) \subset U$ whenever $A \subset U$ and U is open in X.*
(2) *δ–generalized closed (briefly δg–closed)* [4] *if $Cl_\delta(A) \subset U$ whenever $A \subset U$ and U is open in X.*
(3) *generalized δ–closed (briefly $g\delta$–closed)* [5] *if $Cl(A) \subset U$ whenever $A \subset U$ and U is δ–open in X.*
(4) *δ–generalized–semi–closed (briefly δgs–closed)* [27] *if ${}_sCl_\delta(A) \subset U$ whenever $A \subset U$ and U is δ–open in X.*
(5) *δ generalized preclosed (briefly, δgp–closed)*[3] *if $pCl(A) \subseteq U$ whenever $A \subset U$ and U is δ–open in X.*

The complement of a g–closed [18] (resp. δg-closed [4], $g\delta$–closed [5], δgs–closed [27], δgp–closed [3]) set is said to be g–open (resp. δ–generalized–open, generalized δ–open, δ–generalized–semiopen, δgp–open).

Definition 5. *A function $f : (X, \tau) \to (Y, \sigma)$ is to said to be*

(1) *g–continuous* [2] *if the preimage of every open set in (Y, σ) is g–open in (X, τ).*
(2) *δ–continuous function* [24] *if the preimage of every regular open set in Y is δ–open in X. δ–generalized–continuous* [4] *if the preimage of every open set of (Y, σ) is δ–generalized–open in (X, τ).*
(3) *generalized δ–continuous (briefly $g\delta$–continuous)* [5] *if the preimage of every open set of (Y, σ) is generalized δ–open (briefly $g\delta$–open) in (X, τ).*
(4) *δ–generalized–semi–continuous (briefly δgs–continuous)* [27] *if the preimage of every open set of (Y, σ) is δ–generalized–semi–open (briefly δgs–open) in (X, τ).*
(5) *δ–generalized–semi–irresolute* [27] *if the preimage of every δ–generalized–semi–open (briefly δgs–open) set of (Y, σ) is δ–generalized–semi–open (briefly δgs–open) in (X, τ).*
(6) *δ–β–continuous* [12] *if the preimage of every open set of (Y, σ) is δ–β–open in (X, τ).*
(7) *δ–β–irresolute* [1] *if the preimage of every δ–β–open set of (Y, σ) is δ–β–open in (X, τ).*

2. Main Results

2.1. δ–β–Generalized Closed Sets.

Definition 6. *A subset A of a space (X, τ) is said to be δ–β–generalized–closed ($\delta\beta g$–closed) if ${}_{\beta}Cl_{\delta}(A) \subset U$ whenever $A \subset U$ and U is δ–open in X. The family of all $\delta\beta g$–closed sets in a topological space X is denoted by $\delta\beta GC(X)$.*

Theorem 7. *Every generalized δ–closed set is $\delta\beta g$–closed but not conversely.*

Proof. Let A be any subset of a space X. Suppose A is a generalized δ–closed set and U is any δ–open set containing A in X. Then $Cl(A) \subset U$. Since ${}_{\beta}Cl_{\delta}(A) \subset Cl(A)$, thus we have ${}_{\beta}Cl_{\delta}(A) \subset U$ and hence A is $\delta\beta g$–closed. □

Example 8. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{d\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, c, d\}\}$. Then $(X,\ \tau)$ is a topological space and $A = \{a, d\}$ is $\delta\beta g$–closed but it is not generalized δ–closed.*

Theorem 9. *Every δ–generalized semiclosed set is $\delta\beta g$–closed but the converse is not true.*

Proof. Let A be any δ–*generalized*–semiclosed subset of the space X and U be any δ–open set containing A. Since A is δ–generalized–semiclosed, $sCl_{\delta}(A) \subset U$. Since ${}_{\beta}Cl_{\delta}(A) \subset sCl_{\delta}(A)$, ${}_{\beta}Cl_{\delta}(A) \subset U$ and hence A is $\delta\beta g$–closed. □

Example 10. *Let $X = \{a, b, c, d\}$ and $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a, b\}, \{b, d\}, \{a, b, d\}\}$. Then $(X,\ \tau)$ is a topological space and $A = \{a, d\}$ is $\delta\beta g$–closed but it is not δ–generalized semiclosed.*

Theorem 11. *Every δ–generalized preclosed set is $\delta\beta g$–closed but the converse is not true.*

Proof. Let A be any δ–*generalized*–preclosed subset of the space X and U be any δ–open set containing A. Since A is δ–generalized–preclosed, $pCl(A) \subset U$. Since ${}_{\beta}Cl_{\delta}(A) \subset pCl(A)$, ${}_{\beta}Cl_{\delta}(A) \subset U$ and hence A is $\delta\beta g$–closed. □

Example 12. *Consider the Example 10, let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, d\}, \{a, b, d\}\}$. Then $(X,\ \tau)$be a topological space. Let $A = \{a, d\}$ which is δ–β–generalized–closed but it is not δ–generalized preclosed.*

Theorem 13. *Every δ–β–closed set is $\delta\beta g$–closed but not conversely.*

Proof. Let A be a δ–β–closed subset of a space X and U be any δ–open set containing A. Since A is δ–β–closed, ${}_{\beta}Cl_{\delta}(A) = A$ and hence ${}_{\beta}Cl_{\delta}(A) \subset U$. Therefore A is $\delta\beta g$–closed. □

Example 14. *Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then (X, τ) is a topological space and $A = \{a, c, d\}$ is $\delta\beta g$–closed but it is not δ–β–closed.*

Theorem 15. *Let A be a $\delta\beta g$–closed subset of a topological space (X, τ). Then ${}_{\beta}Cl_{\delta}(A) \setminus A$ does not contain any non–empty δ–closed set.*

Proof. Let A be any $\delta\beta g$–closed set in (X, τ) and suppose G is any δ–closed set contained in ${}_{\beta}Cl_{\delta}(A) \setminus A$. Then G^c is a δ–open set in (X, τ) such that $A \subset G^c$. Since A is $\delta\beta g$–closed, ${}_{\beta}Cl_{\delta}(A) \subset G^c$. This implies that $G \subset ({}_{\beta}Cl_{\delta}(A))^c$. We already have $G \subset ({}_{\beta}Cl_{\delta}(A) \setminus A)$. It follows that $G \subset ({}_{\beta}Cl_{\delta}(A))^c \cap ({}_{\beta}Cl_{\delta}(A) \setminus A) = \phi$ □

Theorem 16. *If A is a δ–open and $\delta\beta g$–closed subset of a topological space (X, τ), then A is δ–β–closed.*

Proof. Since A is δ–open and $\delta\beta g$–closed, we have ${}_{\beta}Cl_{\delta}(A) \subset A$ and therefore ${}_{\beta}Cl_{\delta}(A) = A$. Hence A is δ–β–closed. □

Theorem 17. *Let A be a $\delta\beta g$–closed subset of a space (X, τ). Then A is δ–β–closed if and only if ${}_{\beta}Cl_{\delta}(A) \setminus A$ is δ–closed.*

Proof. Let A be any δ–β–closed set, then we have ${}_{\beta}Cl_{\delta}(A) = A$. Therefore ${}_{\beta}Cl_{\delta}(A) \setminus A = \phi$, which is δ–closed.
Conversely, assume that ${}_{\beta}Cl_{\delta}(A) \setminus A$ is δ–closed. Now A is $\delta\beta g$–closed and ${}_{\beta}Cl_{\delta}(A) \setminus A$ is δ–closed, then by Theorem 15 ${}_{\beta}Cl_{\delta}(A) \setminus A = \phi$ i.e. ${}_{\beta}Cl_{\delta}(A) = A$. This shows that A is δ–β–closed. □

Theorem 18. *Let A be a $\delta\beta g$–closed subset of a topological space (X, τ) and suppose $A \subseteq B \subseteq {}_{\beta}Cl_{\delta}(A)$, then B is $\delta\beta g$–closed.*

Proof. Let U be any δ–open set in X such that $B \subseteq U$, then $A \subseteq B \subseteq U$. Since A is $\delta\beta g$–closed, ${}_{\beta}Cl_{\delta}(A) \subseteq U$. Since $B \subseteq {}_{\beta}Cl_{\delta}(A)$, ${}_{\beta}Cl_{\delta}(B) \subseteq {}_{\beta}Cl_{\delta}({}_{\beta}Cl_{\delta}(A)) = {}_{\beta}Cl_{\delta}(A) \subseteq U$ and ${}_{\beta}Cl_{\delta}(B) \subseteq U$. This implies that B is $\delta\beta g$–closed in X. □

Remark 19. (1) *The union of two $\delta\beta g$–closed sets need not be $\delta\beta g$–closed.*

(2) *The intersection of two $\delta\beta g$–closed sets need not be $\delta\beta g$–closed.*

Example 20. *Let (X, τ) be a topological spaces in the Example 8.*

(1) *Let $A = \{c\}$ and $B = \{d\}$. Then A and B are $\delta\beta g$–closed sets but their union $A \cup B$ is not $\delta\beta g$–closed in X.*
(2) *Let $A = \{b, c, d\}$ and $B = \{a, c, d\}$. Then A and B are $\delta\beta g$–closed sets but their intersection $A \cap B$ is not $\delta\beta g$–closed in X.*

Theorem 21. *The intersection of a $\delta\beta g$–closed set and a δ–closed set is δ–β–closed.*

Proof. Let A be any $\delta\beta g$–closed set and G be any δ–closed set. Suppose V is a δ–open set such that $A \cap G \subseteq V$, then $A \subseteq (V \cup (X \setminus G))$. Since A is $\delta\beta g$–closed, ${}_{\beta}Cl_{\delta}(A) \subseteq V\cup\ (X \setminus G)$. Since every δ–closed set is δ–β–closed, ${}_{\beta}Cl_{\delta}(A \cap G) \subseteq {}_{\beta}Cl_{\delta}(A) \cap {}_{\beta}Cl_{\delta}(G) = {}_{\beta}Cl_{\delta}(A) \cap G \subseteq (V \cup (X \setminus G)) \cap G = (V \cap G) \cup \phi \subset V$. This proves the result. □

Definition 22. *Let A be a subset of a topological space (X, τ) . The δ–β–generalized–closure of A (briefly $\delta\beta g$–closure) is defined as the intersection of all $\delta\beta g$–closed sets containing A and is denoted by $\delta\beta g$–$Cl(A)$.*

Theorem 23. *For the $\delta\beta g$–closure of a set A in a topological space (X, τ), the following properties hold:*

(1) *If A is $\delta\beta g$–closed, then $A =\delta\beta g$–$Cl(A)$.*
(2) *If $A \subset B \subset X$, then $\delta\beta g$–$Cl(A) \subset \delta\beta g$–$Cl(B)$.*
(3) *$\delta\beta g$–$Cl(A) \cup \delta\beta g$–$Cl(B) \subset \delta\beta g$–$Cl(A \cup B)$.*
(4) *$x \in \delta\beta g$–$Cl(A)$ if and only if $A\cap U \neq \phi$ for every $U \in\delta\beta GO(X)$ containing x.*
(5) *$\delta\beta g$–$Cl(\delta\beta g$–$Cl(A))= \delta\beta g$–$Cl(A)$.*

Proof. The proof is obvious. □

Remark 24. (1) *The union of two $\delta\beta g$–closed sets need not be $\delta\beta g$–closed.*
(2) *The intersection of two $\delta\beta g$–closed sets need not be $\delta\beta g$–closed.*

Example 25. *Let (X, τ) be a topological spaces in the Example 8.*

(1) *Let $A = \{c\}$ and $B = \{d\}$. Then A and B are $\delta\beta g$–closed sets but their union $A \cup B$ is not $\delta\beta g$–closed in X.*

(2) *Let $A = \{b, c, d\}$ and $B = \{a, c, d\}$. Then A and B are $\delta\beta g$–closed sets but their intersection $A \cap B$ is not $\delta\beta g$–closed in X.*

Theorem 26. *The intersection of a $\delta\beta g$–closed set and a δ–closed set is δ–β–closed.*

Proof. Let A be any $\delta\beta g$–closed set and G be any δ–closed set. Suppose V is a δ–open set such that $A \cap G \subseteq V$, then $A \subseteq (V \cup (X \setminus G))$. Since A is $\delta\beta g$–closed, ${}_{\beta}Cl_{\delta}(A) \subseteq V\cup$ $(X \setminus G)$. Since every δ–closed set is δ–β–closed, ${}_{\beta}Cl_{\delta}(A \cap G) \subseteq {}_{\beta}Cl_{\delta}(A) \cap {}_{\beta}Cl_{\delta}(G) = {}_{\beta}Cl_{\delta}(A) \cap G \subseteq (V \cup (X \setminus G)) \cap G = (V \cap G) \cup \phi \subset V$. This proves the result. □

Definition 27. *Let A be a subset of a topological space (X, τ) . The δ–β–generalized–closure of A (briefly $\delta\beta g$–closure) is defined as the intersection of all $\delta\beta g$–closed sets containing A and is denoted by $\delta\beta g$–$Cl(A)$.*

Theorem 28. *For the $\delta\beta g$–closure of a set A in a topological space (X, τ), the following properties hold:*

(1) *If A is $\delta\beta g$–closed, then $A =\delta\beta g$–$Cl(A)$.*
(2) *If $A \subset B \subset X$, then $\delta\beta g$–$Cl(A) \subset \delta\beta g$–$Cl(B)$.*
(3) *$\delta\beta g$–$Cl(A) \cup \delta\beta g$–$Cl(B) \subset \delta\beta g$–$Cl(A \cup B)$.*
(4) *$x \in \delta\beta g$–$Cl(A)$ if and only if $A\cap U \neq \phi$ for every $U \in\delta\beta GO(X)$ containing x.*
(5) *$\delta\beta g$–Cl($\delta\beta g$–$Cl(A)$)= $\delta\beta g$–$Cl(A)$.*

Proof. The proof is obvious. □

Remark 29. *Since every δ–open set is δ–β–open, ${}_{\beta}Cl_{\delta}(A) \subseteq \delta$–$Cl(A)$, for any subset A of X.*

Theorem 30. *If A and B are $\delta\beta g$–closed sets such that δ–$Cl(A) \subseteq {}_{\beta}Cl_{\delta}(A)$ and δ–$Cl(B) \subseteq {}_{\beta}Cl_{\delta}(B)$. Then $A \cup B$ is $\delta\beta g$–closed.*

Proof. Let V be any δ–open set such that $A \cup B \subseteq V$. Then $A \subseteq V$ and $B \subseteq V$. Since A and B are both $\delta\beta g$–closed, ${}_{\beta}Cl_{\delta}(A) \subseteq V$ and ${}_{\beta}Cl_{\delta}(B) \subseteq V$. By assumption δ–$Cl(A) \subseteq {}_{\beta}Cl_{\delta}(A)$ and δ–$Cl(B) \subseteq {}_{\beta}Cl_{\delta}(B)$. Since every δ–open set is δ–β–open. Therefore δ–$Cl(A \cup B)$ $= \delta$–$Cl(A) \cup \delta$–$Cl(B) = {}_{\beta}Cl_{\delta}(A) \cup {}_{\beta}Cl_{\delta}(B) \subseteq V \cup V = V$. Since ${}_{\beta}Cl_{\delta}(A \cup B) \subset \delta$–$Cl(A \cup B) \subseteq V$, $A \cup B$ is $\delta\beta g$–closed. □

Definition 31. *A space (X, τ) is called a $T\delta\beta_{3/4}$–space if every $\delta\beta g$–closed set is δ–β–closed.*

Theorem 32. *For a topological space (X, τ), the following properties are equivalent:*

(1) *(X, τ) is a $T\delta\beta_{3/4}$–space.*
(2) *Every singleton $\{x\}$ is either δ–β–open or δ–closed.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and suppose $\{x\}$ is not δ–closed. Then $X \setminus \{x\}$ is not δ–open. Thus $X \setminus \{x\}$ is $\delta\beta g$–closed. Since (X, τ) is a $T\delta\beta_{3/4}$–space, $X \setminus \{x\}$ is δ–β–closed i.e. $\{x\}$ is δ–β–open in (X, τ).

(2) $\Rightarrow$ (1). Let A be a $\delta\beta g$–closed set in (X, τ) and suppose $x \in_\beta Cl_\delta(A)$. Since $\{x\}$ is either δ–β–open or δ–closed, we have following two cases:
Case(i): Let $\{x\}$ be δ–β–open and suppose $x \notin A$. Then $A \subset X \setminus \{x\}$ and $X \setminus \{x\}$ is δ–β–closed. Therefore $x \notin_\beta Cl_\delta(A)$. Therefore ${}_\beta Cl_\delta(A) \subset A$. Since $A \subset_\beta Cl_\delta(A)$, ${}_\beta Cl_\delta(A) = A$ and hence A is δ–β–closed.
Case(ii): Let $\{x\}$ be δ–closed and suppose that $x \notin A$. Then $A \subset X\setminus\{x\}$ and $X\setminus\{x\}$ is δ-open. Since A is $\delta\beta g$–closed, ${}_\beta Cl_\delta(A) \subset X\setminus\{x\}$ and hence $x \notin_\beta Cl_\delta(A)$. Therefore, ${}_\beta Cl_\delta(A) \subset A$. Since $A \subset_\beta Cl_\delta(A)$, ${}_\beta Cl_\delta(A) = A$ and hence A is δ-β-closed.

From above two cases, it follows that every $\delta\beta g$–closed set is δ–β–closed. Therefore the space (X, τ) is a $T\delta\beta_{3/4}$–space. □

2.2. δ–β–**Generalized Open Sets.**

Definition 33. *A subset A of X is said to be $\delta\beta g$–open if its complement is $\delta\beta g$–closed.*

Theorem 34. *Let A be any subset of a space (X, τ). Then A is $\delta\beta g$–open if and only if $G \subset_\beta Int_\delta(A)$ whenever G is δ–closed and $G \subset A$.*

Proof. Necessity. Let A be a $\delta\beta g$–open set with $G \subset A$, where G is a δ–closed set. Then $X \setminus A$ is a $\delta\beta g$–closed set contained in $X \setminus G$, where $X \setminus G$ is a δ–open set. Hence ${}_\beta Cl_\delta(X \setminus A) \subset (X \setminus G)$ and hence $X \setminus_\beta Int_\delta(A) \subset (X \setminus G)$. Thus $G \subset_\beta Int_\delta(A)$.
Sufficiency. Let $X \setminus A \subset V$ and V be δ–open in X. Then $X \setminus V \subset A$ and $X\setminus V$ is δ–closed. Thus $X\setminus V \subset_\beta Int_\delta(A)$ and hence ${}_\beta Cl_\delta(X\setminus A) = X \setminus_\beta Int_\delta(A) \subset V$. Therefore $X \setminus A$ is a $\delta\beta g$–closed set and hence A is $\delta\beta g$–open. □

Theorem 35. *If A is $\delta\beta g$–open and B is any set in X such that ${}_\beta Int_\delta(A) \subseteq B \subset A$, then B is $\delta\beta g$–open in X.*

Proof. Its proof follows from the Definition 33 and from Theorem 18. $\square$

Theorem 36. *If A is $\delta\beta g$–open and B is any set in X such that ${}_{\beta}Int_{\delta}(A) \subseteq B \subset A$, then $A \cap B$ is $\delta\beta g$–open in X.*

Proof. Suppose A is $\delta\beta g$–open and ${}_{\beta}Int_{\delta}(A) \subseteq B$, then $A\cap {}_{\beta}Int_{\delta}(A) \subseteq A \cap B \subseteq A$. Since ${}_{\beta}Int_{\delta}(A) \subseteq A$, we have ${}_{\beta}Int_{\delta}(A) \subseteq A \cap B \subseteq A$. Therefore by Theorem 35 $A \cap B$ is $\delta\beta g$–open. $\square$

Definition 37. *Let A be a subset of a topological space (X, τ) . The δ–β–generalized–interior of A is defined as the union of all $\delta\beta g$–open sets contained in A and is denoted by $\delta\beta g$–$Int(A)$.*

Theorem 38. *For the $\delta\beta g$–interior of a set A in a topological space (X, τ), the following properties hold:*

(1) *If A is $\delta\beta g$–open, then $A =\delta\beta g$–$Int(A)$.*
(2) *If $A \subset B \subset X$, then $\delta\beta g$–$Int(A) \subset \delta\beta g$–$Int(B)$.*
(3) *$\delta\beta g$–Int($\delta\beta g$–$Int(A)$) $= \delta\beta g$–$Int(A)$.*
(4) *$\delta\beta g$–$Int(A) \cup \delta\beta g$–$Int(B) \subseteq \delta\beta g$–$Int(A \cup B)$.*

Proof. Proof above is obvious. $\square$

Theorem 39. *Let A be any subset of a space (X, τ). If A is $\delta\beta g$–open, then $V = X$, whenever V is δ–open and ${}_{\beta}Int_{\delta}(A) \cup (X \setminus A) \subset V$.*

Proof. Let V be a δ–open set in X and ${}_{\beta}Int_{\delta}(A) \cup (X \setminus A) \subset V$. Then, we have

$$X \setminus V \subset X\setminus\ [{}_{\beta}Int_{\delta}(A)) \cup (X \setminus A] = (X \setminus_{\beta} Int_{\delta}(A)) \cap (X \setminus (X \setminus A))$$
$$= {}_{\beta}Cl_{\delta}(X \setminus A) \setminus (X \setminus A).$$

Since $X \setminus V$ is δ–closed and $X \setminus A$ is $\delta\beta g$–closed, by Theorem 15 $X \setminus V = \phi$ and hence $V = X$. $\square$

2.3. δ–β–Generalized–Continuous and δ–β–Generalized–Irresolute Functions.

Definition 40. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be δ–β–generalized–continuous(briefly $\delta\beta g$–continuous) if the inverse image of each open set in Y is $\delta\beta g$–open in X.*

Theorem 41. (1) *Every δgs–continuous function is $\delta\beta g$–continuous.*
(2) *Every $g\delta$–continuous function is $\delta\beta g$–continuous.*

(3) *Every δ–β–continuous function is $\delta\beta g$–continuous.*

Proof. The proofs of (1),(2) and (3) are obvious by Theorems 7, 9 and 13, respectively. □

Remark 42. *The converse of (1) and (2) of the above Theorem 41 need not be true in general as shown by the following examples 43 and 44, respectively.*

Example 43. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}\}$, then (X, τ) is a topological space. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{2, 3\}, \{2\}, \{2, 4\}, \{2, 3, 4\}\}$, then (Y, σ) is a topological space.*
Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as $f(a) = 3$, $f(c) = 1$, $f(b) = 2$, $f(d) = 4$, then f is $\delta\beta g$–continuous, since the preimage of every open set in Y is $\delta\beta g$–open in X. But f is not δgs–continuous, since the preimage of an open set $A = \{2, 4\}$ in Y is $\{b, d\}$, which is $\delta\beta g$–open, but this is not δgs–open in X.

Example 44. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, b\}, \{d\}, \{a, b, d\}\}$, then (X, τ) is a topological space. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{1\}, \{2\}, \{1, 2\}, \{2, 4\}, \{1, 2, 4\}\}$ then (Y, σ) is a topological space.*
Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as $f(a) = 2$, $f(c) = 4$, $f(b) = 1$, $f(d) = 3$, then f is $\delta\beta g$–continuous, since the preimage of every open set in Y is $\delta\beta g$–open in X. But f is not $g\delta$–continuous, since the preimage of an open set $A = \{2, 4\}$ in Y is $\{a, c\}$, which is $\delta\beta g$–open in X. But $\{a, c\}$ is not $g\delta$–open in X.

Remark 45. *The converse of (3) of the above Theorem 41 need not be true in general as shown by the following example.*

Example 46. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{b\}, \{a, b, c\}\}$, then (X, τ) is a topological space. Let $Y = \{1, 2, 3, 4\}$, $\sigma = \{Y, \phi, \{2\}, \{4\}, \{2, 4\}, \{2, 3, 4\}\}$ then (Y, σ) is a topological space. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined as $f(a) = 1$, $f(b) = 2$, $f(c) = 4$, $f(d) = 3$, then f is $\delta\beta g$–continuous, since the preimage of every open set in Y is $\delta\beta g$–open in X. But f is not δ–β–continuous, since the preimage of an open set $A = \{4\}$ in Y is $\{c\}$, which is $\delta\beta g$–open in X. But $\{c\}$ is not δ–β–open in X.*

Theorem 47. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function from a topological space X into a topological space Y.*

(1) *The following statements are equivalent*
   (*i*) *f is $\delta\beta g$–continuous.*
   (*ii*) *The inverse image of each closed set in Y is $\delta\beta g$–closed in X.*

(2) *If $f : (X, \tau) \to (Y, \sigma)$ is $\delta\beta g$–continuous, then the following properties hold:*
   (*i*) *For each $p \in X$ and each open set O in Y with $f(p) \in O$, there exist a $\delta\beta g$–open set B in X such that $p \in B$ and $f(B) \subseteq O$.*
   (*ii*) *For every subset A of X, $f(\delta\beta g\text{–}Cl(A)) \subseteq Cl(f(A))$.*

Proof. (1) (*i*) $\Leftrightarrow$ (*ii*). This is direct from the Definition 33 and Definition 40 .

(2) (*i*) Since f is $\delta\beta g$–continuous, for each $p \in X$ and each open set O in Y with $f(p) \in O$, $p \in f^{-1}(O) \in \delta\beta GO(X)$. Let $B = f^{-1}(O)$, then we have $p \in B$ and $f(B) \subseteq O$.

(*ii*) Let A be any subset of X, then $Cl(f(A))$ is closed in Y. Since f is $\delta\beta g$–continuous, $f^{-1}(Cl(f(A))$ is $\delta\beta g$–*closed* in X. Since $A \subset f^{-1}(f(A)) \subseteq f^{-1}(Cl(f(A))$, it follows that $\delta\beta g\text{–}Cl(A) \subseteq \delta\beta g\text{–}Cl(f^{-1}(Cl(f(A)))) = f^{-1}(Cl(f(A))$. Hence f($\delta\beta g$–$Cl(A)) \subseteq Cl(f(A))$.

□

Theorem 48. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *For every subset A of X, $f(\delta\beta g\text{–}Cl(A)) \subseteq Cl(f(A))$.*
(2) *For each subset B of Y, $\delta\beta g\text{–}Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.*
(3) *For each subset A of Y, $f^{-1}(Int(B)) \subseteq \delta\beta g\text{–}Int(f^{-1}(B)$.*

Proof. (1) $\Rightarrow$ (2). Let B be any subset of Y. By (1), $f(\delta\beta g\text{–}Cl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) \subseteq Cl(B)$ and hence $\delta\beta g\text{–}Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.

(2) $\Rightarrow$ (3). Let B be any subset of Y. By (2), $\delta\beta g\text{–}Cl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B))$.

This implies that $\delta\beta g\text{–}Cl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B))$ and hence $X\setminus$ $(\delta\beta g\text{–}Int(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B))$. Therefore, $f^{-1}(Int(B)) \subseteq (\delta\beta g\text{–}Int(f^{-1}(B))$.

(3) $\Rightarrow$ (1). Let A be any subset of X. By (3), we have

$$f^{-1}(Int(Y \setminus f(A))) \subseteq \delta\beta g\text{–}Int(f^{-1}(Y \setminus f(A)),$$
$$f^{-1}(Y \setminus Cl(f(A)) \subseteq \delta\beta g\text{–}Int(X \setminus f^{-1}(f(A)) \text{ and}$$
$$X \setminus f^{-1}(Cl(f(A))) \subseteq \delta\beta g\text{–}Int(X \setminus A) = X \setminus \delta\beta g\text{–}Cl(A).$$

Therefore, $\delta\beta g$–$Cl(A) \subseteq f^{-1}(Cl(f(A)))$ and hence $f(\delta\beta g\text{–}Cl(A)) \subseteq Cl(f(A))$. □

Definition 49. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(1) *$\delta\beta g$–irresolute if $f^{-1}(V) \in \delta\beta GO(X)$ for each $V \in \delta\beta GO(Y)$.*
(2) *$\delta\beta g$–preserving if $f(U) \in \delta\beta GO(Y)$ for each $U \in \delta\beta GO(X)$.*

Theorem 50. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two functions. Then the following properties hold:*

(1) *If $f : (X, \tau) \to (Y, \sigma)$ is $\delta\beta g$–continuous and $g : (Y, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is $\delta\beta g$–continuous.*
(2) *If $f : (X, \tau) \to (Y, \sigma)$ is $\delta\beta g$–irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is $\delta\beta g$–continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is $\delta\beta g$–continuous.*
(3) *If $f : (X, \tau) \to (Y, \sigma)$ is $\delta\beta g$–irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is $\delta\beta g$–irresolute, then $g \circ f : (X, \tau) \to (Z, \eta)$ is $\delta\beta g$–irresolute.*

Proof. (1) The proof is obvious.

(2) Let V be any open set in (Z, η). Since g is $\delta\beta g$–continuous, $f^{-1}(V)$ is $\delta\beta g$–open in (Y, σ). Since f is $\delta\beta g$–irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\delta\beta g$–open in (X, τ). It follows that $g \circ f : (X, \tau) \to (Z, \eta)$ is $\delta\beta g$–continuous.
(3) The proof is obvious.

□

2.4. δ–β–Quotient and $\delta\beta g$–Quotient Maps.

Definition 51. *A surjective map $f : (X, \tau) \to (Y, \sigma)$ is called a δ–β–quotient map provided a subset V of Y is open (resp. closed) in (Y, σ) if and only if $f^{-1}(V)$ is δ–β–open (resp. δ–β–closed) in (X, τ).*

Definition 52. *A surjective map $f : (X, \tau) \to (Y, \sigma)$ is called a $\delta\beta g$–quotient map provided a subset V of Y is open (resp. closed) in (Y, σ) if and only if $f^{-1}(V)$ is $\delta\beta g$–open (resp. $\delta\beta g$–closed) in (X, τ).*

Remark 53. *Every δ–β–quotient map is δ–β–continuous and every $\delta\beta g$–quotient map is $\delta\beta g$–continuous. But the converse of these need not be true in general.*

Example 54. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{b\}, \{d\}, \{c\}, \{b, d\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$, then (X, τ) is a topological space. Let $Y = \{w, x, y, z\}$ and $\sigma = \{Y, \phi, \{w\}, \{x, y\}, \{w, x, y\}\}$ then (Y, σ) is a topological space.*
Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = y$, $f(b) = z$, $f(c) = w$, $f(d) = x$. Then f is δ–β–continuous and hence $\delta\beta g$–continuous, but it is neither δ–β–quotient nor $\delta\beta g$–quotient. Because $f^{-1}(z) = b$ is δ–β–open and therefore $\delta\beta g$–open in (X, τ), but z is not open in (Y, σ).

Theorem 55. *If $f : (X, \tau) \to (Y, \sigma)$ is a δ–β–quotient map and $g : (Y, \sigma) \to (Z, \eta)$ is a quotient map, then $g \circ f : (X, \tau) \to (Z, \eta)$ is δ–β–quotient.*

Proof. Since g is quotient map, U is an open set of Z if and only if $g^{-1}(U)$ is open in Y. Since $g^{-1}(U)$ is open in Y and f is δ–β–quotient, we have $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is δ–β–open in X if and only if $g^{-1}(U)$ is open in Y. Therefore U is open in Z if and only $(g \circ f)^{-1}(U)$ is δ–β–open in X. This shows that $g \circ f$ is δ–β–quotient. □

Theorem 56. *Let $p : (X, \tau) \to (Y, \sigma)$ be a δ–β–quotient map. Let (Z, η) be a space and $g : (X, \tau) \to (Z, \eta)$ be a map that is constant on the set $p^{-1}(\{y\})$ for each $y \in Y$. Then g induces a map $f : (Y, \sigma) \to (Z, \eta)$ such that $f \circ p = g$. For the induced map f, the following properties hold:*

(1) f is continuous if and only if g is δ–β–continuous,
(2) f is a quotient map if and only if g is a δ–β–quotient map.

Proof. Suppose for each element $y \in Y$, $p^{-1}(\{y\}) \subseteq X$. Since g is a constant map on the set $p^{-1}(\{y\})$ for $y \in Y$, $g(p^{-1}(\{y\}))$ is one point set in Z and let $g(p^{-1}(\{y\})) = f(y)$(say). Then a map $f : (Y, \sigma) \to (Z, \eta)$ is defined such that for each $x \in X$, $f(p(x)) = g(x)$.

(1) Let f be continuous. Then, since p is a δ-β-quotient map, the map $g = f \circ p$ is δ–β–continuous. Conversely, suppose that g is δ–β–continuous. Then, for any open set U in Z, $g^{-1}(U)$ is δ–β–open in X. Therefore $g^{-1}(U) = p^{-1}(f^{-1}(U))$ is again δ–β–open in X. Since p is a δ–β–quotient map, $f^{-1}(U)$ is open in Y and hence f is continuous.

(2) Suppose f is a quotient map, then by Theorem 55, $g = f \circ p$ is δ–β–quotient. Conversely, suppose g is δ–β–quotient and therefore it is surjective. Then it follows that f is also surjective. Let U be any open set of Z. Since p is a δ–β–quotient map, then the set $p^{-1}(f^{-1}(U))$ is δ–β–open in X if and only if $f^{-1}(U)$ is open in Y. Since $p^{-1}(f^{-1}(U)) = g^{-1}(U)$, therefore $g^{-1}(U)$ is δ–β–open in X. Since g is a δ–β–quotient map, U is open in Z. Thus U is open in Z if and only if $f^{-1}(U)$ is open in Y. This shows that f is a quotient map. □

Corollary 57. *Let $p : (X, \tau) \to (Y, \sigma)$ be a $\delta\beta g$–quotient map. Under the same assumption with Theorem 56, for the induced map f, the following properties hold:*

(1) f is continuous if and only if g is $\delta\beta g$–continuous,

(2) f is a quotient map if and only if g is a $\delta\beta g$–quotient map.

Proof. It is obvious. □

Theorem 58. *Let X, Y and Z is a topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is $\delta\beta g$–preserving $\delta\beta g$–irresolute surjection and $g : (Y, \sigma) \to (Z, \eta)$ is a $\delta\beta g$–quotient map, then their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is $\delta\beta g$–quotient.*

Proof. Let U is an open set in (Z, η). Then $g^{-1}(U)$ is $\delta\beta g$–open in (Y, σ), since g is $\delta\beta g$–quotient. Since f is $\delta\beta g$–irresolute, $f^{-1}(g^{-1}(U))$ $= (g \circ f)^{-1}(U)$ is $\delta\beta g$–open in (X, τ). It implies that $g \circ f$ is $\delta\beta g$–continuous. Assume that $f^{-1}(g^{-1}(U))$ is $\delta\beta g$–open in (X, τ) for a set U in (Z, η). Since f is $\delta\beta g$–preserving, $f(f^{-1}(g^{-1}(U)))$ is $\delta\beta g$–open in (Y, σ). Since f is surjective, $f(f^{-1}(g^{-1}(U)))= g^{-1}(U)$ is $\delta\beta g$–open in (Y, σ). Since g is $\delta\beta g$–quotient, $g(g^{-1}(U))$ i.e. U is open in (Z, η). Hence $g \circ f$ is $\delta\beta g$–quotient. □

2.5. $\delta\beta g$–Regular and $\delta\beta g$–Normal spaces.

Definition 59. *A topological space (X, τ) is said to be $\delta\beta g$–regular if for every $\delta\beta g$–closed set G and every point $x \notin G$, there exists disjoint δ–β–open sets U and V such that $G \subseteq U$ and $x \in V$.*

Theorem 60. *For a space (X, τ) the following are equivalent:*

(1) *(X, τ) is $\delta\beta g$–regular.*

(2) *For every $x \in X$ and every $\delta\beta g$–open set P containing x, there exists a δ–β–open set Q such that $x \in Q \subset {}_{\beta}Cl_{\delta}(Q) \subset P$.*

(3) *Every $\delta\beta g$–closed set G and every point $x \notin G$, there exists a δ–β–open set P containing 'x' such that ${}_{\beta}Cl_{\delta}(P) \cap G = \phi$.*

Proof. (1) ⇒ (2). Assume that (X, τ) be a $\delta\beta g$–regular space. Let P be a $\delta\beta g$–open set containing a point $x \in X$. Since $X \setminus P$ is $\delta\beta g$–closed and $x \notin X \setminus P$, by the hypothesis, there exist δ–β–open sets R and S such that $X \setminus P \subseteq R$, $x \in Q$ and $R \cap Q = \phi$. Now, $R \cap Q = \phi$ implies that $x \in Q \subset X \setminus R \subset P$ and hence we obtain $x \in Q \subset {}_{\beta}Cl_{\delta}(Q) \subset X \setminus R \subset P$.
(2) ⇒ (3). Let $x \in X$ and G be a $\delta\beta g$–closed set such that $x \notin G$. Then $x \in (X \setminus G)$ and $(X \setminus G)$ is $\delta\beta g$–open in X. By hypothesis, there exists a δ–β–open subset P of X such that $x \in P \subset {}_{\beta}Cl_{\delta}(P) \subset (X \setminus G)$, which implies that ${}_{\beta}Cl_{\delta}(P) \cap G = \phi$.
(3) ⇒ (1). Let $x \in X$ and P be a $\delta\beta g$–open set not containing x. By (iii), $(X \setminus P)$ is δ–closed set and $x \notin (X \setminus P)$. Therefore, there exists a δ–β–open set R with $x \in R$ such that ${}_{\beta}Cl_{\delta}(R) \cap P = \phi$. Let $V = X \setminus {}_{\beta}Cl_{\delta}(R)$. Then R and V are δ–β–open sets such that $x \in R$, $P \subset V$ and $R \cap V \subset R \cap (X \setminus R) = \phi$. It follows that (X, τ) is $\delta\beta g$–regular. □

Theorem 61. *If $f : (X, \tau) \to (Y, \sigma)$ is a δ–β–continuous injection of a topological space X into a regular space Y. If the image of $\delta\beta g$–closed set in X is closed, then X is $\delta\beta g$–regular.*

Proof. Let $x \in X$ and suppose A be any $\delta\beta g$–closed set not containing x in X. Then by assumption, A is closed in X. Since f is closed map, $f(A)$ is a closed set in Y not containing $f(x)$. Since Y is regular, there exist disjoint open sets P and Q in Y such that $f(x) \in P$ and $f(A) \subseteq Q$. Since f is δ–β–continuous, $f^{-1}(P)$ and $f^{-1}(Q)$ are disjoint δ–β–open sets in X containing x and A respectively. Hence X is $\delta\beta g$–regular. □

Definition 62. *A topological space (X, τ) is said to be $\delta\beta g$–normal if for every pair of disjoint $\delta\beta g$–closed subsets F and G of X, there exist disjoint δ–β–open subsets U and V of X such that $F \subseteq U$ and $G \subseteq V$.*

Theorem 63. *For a topological space (X, τ), the following are equivalent:*

(1) *X is $\delta\beta g$–normal.*
(2) *For every pair of $\delta\beta g$–open subsets U and V of X with $U \cup V = X$, there exist δ–β–closed subsets F and G of X such that $F \subseteq U$, $G \subseteq V$ and $F \cup G = X$.*

(3) *For every $\delta\beta g$–closed set M and every $\delta\beta g$–open set N in X such that $M \subseteq N$, there exists a δ–β–open subset V of X such that $M \subseteq V \subseteq {}_{\beta}Cl_{\delta}(V) \subseteq N$.*

(4) *For every pair of disjoint $\delta\beta g$–closed subsets M and N of X, there exists a δ–β–open subset V of X such that $M \subseteq V$ and ${}_{\beta}Cl_{\delta}(V) \cap N = \phi$.*

(5) *For every pair of disjoint $\delta\beta g$–closed subsets M and N of X, there exist δ–β–open subsets U and V of X such that $M \subseteq U$, $N \subseteq V$ and ${}_{\beta}Cl_{\delta}(U) \cap {}_{\beta}Cl_{\delta}(V) = \phi$.*

Proof. (1) $\Rightarrow$ (2). Let U and V be any pair of $\delta\beta g$–open subsets of a $\delta\beta g$–normal space X with $U \cup V = X$. Then $X \setminus U$ and $X \setminus V$ are disjoint $\delta\beta g$–closed subsets of X. According to the assumption, there exist disjoint δ–β–open subsets P and Q of X such that $X \setminus U \subseteq P$ and $X \setminus V \subseteq Q$. Let $F = X \setminus P$ and $G = X \setminus Q$. Then, F and G are δ–β–closed subsets in X such that $F \subset U$, $G \subset V$ and $F \cup G = X$.

(2) $\Rightarrow$ (3). Let M be a $\delta\beta g$–closed set and N be a $\delta\beta g$–open set in X such that $M \subset N$. Then $X \setminus M$ and N are $\delta\beta g$–open subsets of X such that $(X \setminus M) \cup N = X$. By assumption, there exist δ–β–closed subsets F and G of X such that $F \subseteq (X \setminus M)$, $G \subseteq N$ with $F \cup G = X$. Therefore, we have $M \subseteq (X \setminus F) \subseteq G \subseteq N$. Let $V = X \setminus F$. Then V is a δ–β–open subset of X. Since G is δ–β–closed in X, ${}_{\beta}Cl_{\delta}(V) \subseteq G$. It follows that $M \subseteq V \subseteq {}_{\beta}Cl_{\delta}(V) \subseteq N$.

(3) $\Rightarrow$ (4). Let M and N be disjoint $\delta\beta g$–closed subsets of X. Then $M \subseteq X \setminus N$, where $X \setminus N$ is $\delta\beta g$–open. By assumption, there exists a δ–β–open subset V of X such that $M \subseteq V \subseteq {}_{\beta}Cl_{\delta}(V) \subseteq X \setminus N$. Hence ${}_{\beta}Cl_{\delta}(V) \cap N = \phi$.

(4) $\Rightarrow$ (5). Let M and N be any disjoint $\delta\beta g$–closed subsets of X. By assumption there exists a δ–β–open set U containing M with ${}_{\beta}Cl_{\delta}(U) \cap N = \phi$. Since ${}_{\beta}Cl_{\delta}(U)$ is δ–β–closed, then it is $\delta\beta g$–closed. Therefore ${}_{\beta}Cl_{\delta}(U)$ and N are disjoint $\delta\beta g$–closed subsets of X. Again by assumption, there exists a δ–β–open set V in X such that $N \subseteq V$ and ${}_{\beta}Cl_{\delta}(U) \cap {}_{\beta}Cl_{\delta}(V) = \phi$.

(5) $\Rightarrow$ (1). Let M and N be any disjoint $\delta\beta g$–closed subsets of X. By hypothesis, there exist δ–β–open sets U and V such that $M \subseteq U$, $N \subseteq V$ and ${}_{\beta}Cl_{\delta}(U) \cap {}_{\beta}Cl_{\delta}(V) = \phi$. Thus, we have $U \cap V = \phi$ and hence X is $\delta\beta g$–normal. □

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