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A COMMON FIXED POINT APPROACH FROM NON-ARCHIMEDEAN MENGER SPACES TO MODULAR METRIC SPACES VIA SIMULATION FUNCTION

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Abstract. In this paper, we prove common fixed point theorems on non-Archimedean Menger spaces by using the concept of simulation function. We also deduce some consequences in modular metric spaces.

1. Introduction and preliminaries

Recently, the notion of simulation function was introduced by [8]. This definition of simulation function was revised in [10] and [12].

Definition 1. *(*[10], [12]*). A mapping* ζ*:* $[0,\infty) \times [0,\infty) \to R$ *is a simulation function if it satisfies the following conditions:*

(ζ_1) $\zeta(t,s) < s - t$ for all $t, s > 0$.
(ζ_2) If $\{t_n\}, \{s_n\}$are sequences in $(0,\infty)$ such that
$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$
and $t_n < s_n$ for all $n \in N$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

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The set of all simulation functions is denoted by Z.

For examples of simulation functions one can refer to [3], [8], [10], [11] [13], [17].

It is clear from (ζ_1) that $\zeta(t,t) < 0$ when $t > 0$.

Last more than half a century saw a tremendous growth in the field of fixed point theory and its applications to study the

existence and uniqueness of common fixed point for different metric structure spaces especially where the probabilistic situations arises such as probabilistic metric spaces. The concept of probabilistic metric space plays a very important role where the distance between the two points are unknown but the probabilities of the possible values of the distance are known.

For terminologies, notations and properties of probabilistic metric spaces, we refer to [5], [6], [9], [15], [16].

Definition 2. *Let R denote the set of reals and R^+ denote the set of the non-negative reals. A mapping $F : R \to R^+$ is called a distribution function if it is non- decreasing and left continuous with $inf_{t\in R}F(t) = 0$ and $sup_{t\in R}\ F(t) = 1$. We will denote by D the set of all distribution functions.*

Definition 3. *Let X be any nonempty set. An ordered pair (X, F) is called a probabilistic metric space(briefly a PM-space) if F is a mapping from $X \times X \to D$ satisfying the following conditions (where we denote $F_{p,\ q}$ the distribution function $F(p, q)$ for $(p, q) \in X \times X$) such that*

$(P_1)F_{p,q}(t) = 1$ *for every* $t > 0$ *if and only if* $p = q$,

$(P_2)F_{p,q}(0) = 0$ *for every* $p, q \in X$,

$(P_3)F_{p,q}(t) = F_{q,p}(t)$ *for every* $p, q \in X$,

$(P_4)F_{p,q}(t_1) = 1$ *an y*

(P_5) $F_{p,q}(t_1) = 1$ *and* $F_{q,r}(t_2) = 1$ *then* $F_{p,r}(\max\{t_1, t_2\}) = 1$ *for all* $p, q, r \in X$ *and* $t_1, t_2 > 0$

Then (X, F) is called a non-Archimedean probabilistic metric space(briefly a N. A. PM-space) .

In metric space (X, d) the metric d induces a mapping $F : X \times X \to D$ such that $F(p, q)(t) = F_{p,q}(t) = H(t - d(p, q))$ for every $p, q \in X$ and $x \in R$, where H is the distributive function defined by

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Definition 4. *A function $\Delta : [0,1] \times [0,1] \to\ [0,1]$ is called a $T-$ norm if it satisfies the following conditions:*

(t_1) $\Delta(a,1) = a$ for every $a \in [0,1]$ and $\Delta(0,0) = 0$,

(t_2) $\Delta(a,b) = \Delta(b,a)$ for every $a,b \in [0,1]$,

(t_3) If $c \geq a$ and $d \geq\ b$ then $\Delta(c,d) \geq\ \Delta(a,b)$,for every $a,b,c \in [0,1]$,

(t_4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$ for every $a,b,c \in [0,1]$.

Definition 5. *A Menger space is a triple (X,F,Δ), where (X,F) is a PM-space and Δ is a T-norm satisfying the following condition:*

$(P_6)F_{p,q}(t_1+t_2) \geq \Delta(F_{p,q}(t_1), F_{q,r}(t_2))$ for every $p,q,r \in X$ and t_1, $t_2 \geq 0$.

If we replace (P_6) by

$(P_7)F_{p,r}(\max\{t_1,t_2\}) \geq\ \Delta(F_{p,q}(t_1), F_{q,r}(t_2))$ for everyp, $q,r \in X$ and t_1, $t_2 \geq 0$

Then (X,F,Δ) is called N. A. Menger space.

Note: We observe that a Menger space (X,F,Δ) is non Archimedean if and only if

$$F_{p,r}(t) \geq \Delta(F_{p,q}(t), F_{q,r}(t)).$$

An important T-norm is the T-norm $\Delta(a,b) = min\{a,b\}$ for all $a,b \in [0,1]$ and this is the unique T-norm such that $\Delta(a,a) \geq\ a$ for every $a \in [0,1]$.Indeed if it satisfies this condition, we have

$\min\{a,b\} \leq \Delta(\min\{a,b\}, \min\{a,b\}\ ) \leq \Delta(a,b)$

$\leq \Delta(\min\{a,b\},1\ ) = \min\ \{a,b\}$

Therefore $\Delta = \min$.

In the sequel, we need the following definitions due to [14].

Definition 6. *Let (X,F,Δ) be a Menger space with continuous T-norm Δ. A sequence $\{x_n\}$ of points in X is said to be convergent to a point $x \in X$ if for every $t > 0$*

$lim_{n\to\ \infty} F_{x_n,x}(t) = 1$.

Definition 7. *Let (X,F,Δ) be a Menger space with continuous T-norm Δ. A sequence $\{x_n\}$ of points in X is said to be Cauchy sequence if for every $t > 0$ and $\lambda > 0$, there exists an integer $N = N(t,\lambda) > 0$ such that $F_{x_n,x_m}(t) > 1-\lambda$ for all $m,n >\ N$.*

Definition 8. *A Menger space (X, F, Δ) with the continuous T-norm Δ is said to be complete if every Cauchy sequence in X converges to a point in X.*

Definition 9. [18]. *Let (X, F, Δ) be a Menger space. The probabilistic metric F is said to be triangular if it satisfies the condition*

$$\frac{1}{F_{p,q}(t)} - 1 \leq (\frac{1}{F_{p,r}(t)} - 1) + (\frac{1}{F_{r,q}(t)} - 1)$$

for every $p, q, r \in X$ and each $t > 0$.

Definition 10. [7].*Two self maps A and B on a set X are said to be weakly compatible if they commute at their coincidence point.*

2. Common fixed point via simulation function on N. A. Menger spaces

Theorem 11. *Let $(X,\ F,\ \Delta)$ be a N. A. Menger space with F triangular and let $A,\ B : X \to X$ be two given mappings. Assume that there exists $\zeta \in Z$ such that*

$$\zeta(\frac{1}{F_{Ax,\ Ay}(t)} - 1, \frac{1}{F_{Bx,\ By}(t)} - 1) \geq 0 \text{ for all } x, y \in X \tag{1}$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Proof. First of all we will prove that if coincidence point of A and B exists then it is unique.

Suppose that v_1 and v_2 are two distinct coincidence points of A and B.Then then there exists two distinct points $u_1, u_2 \in X$ such that

$$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_{2.}$$

It follows by(1) that

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{F_{Au_1,\ Au_2}(t)} - 1, \frac{1}{F_{Bu_1,\ Bu_2}(t)} - 1) \\ &= \zeta(\frac{1}{F_{v_1,v_2}(t)} - 1, \frac{1}{F_{v_1,v_2}(t)} - 1) \\ &< 0, \end{aligned}$$

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. We have $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we construct a sequence $\{x_n\}$ such that

$$Ax_n = Bx_{n+1} \text{ for all } n \in N$$

Let $Ax_n = Bx_{n+1} = y_n$.If $y_n = y_{n+1}$ for some $n \in N$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus x_{n+1} is the unique coincidence point of A and B. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in N$.Hence we have

$$\begin{aligned} 0 &\leq \zeta\Big(\frac{1}{F_{Ax_n,\, Ax_{n+1}}(t)} - 1, \frac{1}{F_{Bx_n,\, Bx_{n+1}}(t)} - 1\Big) \\ &= \zeta\Big(\frac{1}{F_{y_n,\, y_{n+1}}(t)} - 1, \frac{1}{F_{y_{n-1},\, y_n}(t)} - 1\Big) \\ &< S_{y_{n-1},\, y_n}(t) - S_{y_n,\, y_{n+1}}(t) \end{aligned} \tag{2}$$

where $S(y_{n-1}, y_n, t) = \frac{1}{F_{y_{n-1},\, y_n}(t)} - 1$.

Therefore $\{S(y_{n-1}, y_n, t)\}$ is a decreasing sequence of positive real numbers. Thus there exists $z \geq 0$ such that

$$\lim_{n\to\infty} S(y_{n-1}, y_n, t) = z \tag{3}$$

Suppose $z > 0$.Then by (2) and (ζ_2) it follows that

$$0 \leq \lim_{n\to\infty} \sup \zeta(S(y_n, y_{n+1}, t), S(y_{n-1}, y_n, t)) < 0$$

where $t_n = S(y_n, y_{n+1}, t) < S(y_{n-1}, y_n, t) = s_n$ and $t_n, s_n \to z > 0$.
Clearly this is a contradiction and so $z = 0$.By (3) we obtain

$$\lim_{n\to\infty} F_{y_{n-1},\, y_n}(t) = 1 \tag{4}$$

Now we prove that the sequence $\{y_n\}$ is Cauchy. Suppose to the contrary that $\{y_n\}$ is not a Cauchy sequence in X, therefore for some $t_0 > 0$ we do not have $\lim_{m,n\to\infty} \inf F_{y_m,y_n}(t_0) = 1$.

It follows that there exists $0 < \epsilon < 1$and two sub sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$such that n_k is the smallest index for which $n_k >$

$m_k \geq k$ and

(5) $$F_{y_{m_k},\, y_{n_k}}(t_0) \leq 1 - \epsilon$$

and

(6) $$F_{y_{m_k},\, y_{n_{k-1}}}(t_0) > 1 - \epsilon$$

Now we have

$$\begin{aligned} 1 - \epsilon &\geq F_{y_{m_k},\, y_{n_k}}(t_0) \\ &\geq \Delta(F_{y_{m_k},\, y_{n_{k-1}}}(t_0), F_{y_{n_{k-1}},\, y_{n_k}}(t_0)) \\ &> \Delta(1 - \epsilon, F_{y_{n_{k-1}},\, y_{n_k}}(t_0)) \end{aligned}$$

Letting $k \to \infty$ and using (4), we get

(7) $$\lim_{k\to\infty} F_{y_{m_k},\, y_{n_k}}(t_0) = 1 - \epsilon$$

By the same reasoning as above, we obtain

$$\begin{aligned} 1 - \epsilon &\geq F_{y_{m_k},\, y_{n_k}}(t_0) \\ &\geq \Delta(F_{y_{m_k},\, y_{m_{k-1}}}(t_0),\ \Delta(F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0),\ F_{y_{n_{k-1}},\, y_{n_k}}(t_0))) \end{aligned}$$

and

$$F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0) \geq \Delta(F_{y_{m_{k-1}},\, y_{m_k}}(t_0), \Delta(F_{y_{m_k}, y_{n_k}}(t_0),\ F_{y_{n_k}, y_{n_{k-1}}}(t_0)))$$

By letting $k \to \infty$ and using (4) and (7), we obtain

(8) $$\lim_{k\to\infty} F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0) = 1 - \ \in$$

Using (7) and (8), we obtain

$$\begin{aligned} \lim_{k\to\infty} S(y_{m_k}, y_{n_k}, t_0) &= \lim_{k\to\infty} \frac{1}{F_{y_{m_k},\, y_{n_k}}(t_0)} - 1 \\ &= \lim_{k\to\infty} \frac{1 - F_{y_{m_k},\, y_{n_k}}(t_0)}{F_{y_{m_k},\, y_{n_k}}(t_0)} \\ &= \frac{1 - (1 - \epsilon)}{1 - \epsilon} \\ &= \frac{\epsilon}{1 - \epsilon} \end{aligned}$$

and

$$\lim_{k\to\infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = \frac{\epsilon}{1-\epsilon}$$

Let

$$t_k = S(y_{m_k}, y_{n_k}, t_o)$$

$$s_k = S(y_{m_{k-1}}, y_{n_{k-1}}, t_o).$$

Thus by using (1) and (ζ_2), we have

$$0 \leq \lim_{k\to\infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_0), S(y_{m_{k-1}}, y_{n_{k-1}}, t_0)) < 0.$$

The above inequality is not true and hence $\{y_n\}$ is a Cauchy sequence in X.

Now since AX or BX is a complete subset of X therefore there exists $u \in X$ such that $y_n \to Bu$ as $n \to \infty$. If there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Au$ then letting $k \to \infty$ we get $Au = Bu$ and hence the claim. So we suppose that $y_{n_k} \neq Au$ for all $n \in N$.

Since $y_{n-1} \neq y_n$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Bu$ for $k \in N$.Using (1) we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{F_{Ax_{n_{k+1}},\, Au}(t)} - 1, \frac{1}{F_{Bx_{n_{k+1}},\, Bu}(t)} - 1) \\ &= \zeta(S(y_{n_{k+1}}, Au, t), S(y_{n_k}, Bu, t)) \\ &< S(y_{n_k}, Bu, t) - S(y_{n_{k+1},} Au, t). \end{aligned}$$

This shows that $y_{n_{k+1}} \to Au$ and hence $Au = Bu$ is a unique coincidence point of A and B. If A and B are weakly compatible then by using well known result due to [7] we can prove the existence of unique common fixed point of A and B.

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Theorem 12. *Let (X, F, Δ) be a N. A. Menger space with F triangular and $A, B : X \to X$ be two given mappings. Suppose that there exists $\zeta \in Z$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that*

(9) $$\zeta(\frac{1}{F_{Ax,\, Ay}(t)} - 1, \phi(\frac{1}{F_{Bx,\, By}(t)} - 1) \geq 0 \text{ for all } x, y \in X$$

(10) $$0 < \phi(t) \leq t \text{ for all } t \in (0, +\infty) \text{ and } \phi(0) = 0$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X.Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Proof. First of all we will prove that if A and B have at most one coincidence point.

Let v_1 and v_2 be two coincidence points of A and B. Suppose that v_1 and v_2 are distinct. Then there exists two points $u_1, u_2 \in X$ such that

$$Au_1 = Bu_1 = v_1 \neq v_2 = Au_2 = Bu_2$$

then by (9) we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{F_{Au_1,\, Au_2}(t)} - 1, \phi(\frac{1}{F_{Bu_1,\, Bu_2}(t)} - 1)) \\ &< \phi(\frac{1}{F_{v_1,v_2}(t)} - 1) - \frac{1}{F_{v_1,v_2}(t)} - 1) \\ &\leq 0, \end{aligned}$$

but this is a contradiction. Thus we have $v_1 = v_2$.

Let $x_0 \in X$ be arbitrary. Since $AX \subseteq BX$ therefore there exists $x_1 \in X$ such that $Ax_0 = Bx_1$ continuing this process, we obtain

$$Ax_n = Bx_{n+1} \text{ for all } n \in N$$

Let $Ax_n = Bx_{n+1} = y_n$.If $y_n = y_{n+1}$ for some $n \in N$ then $Bx_{n+1} = y_n = y_{n+1} = Ax_{n+1}$.

Thus x_{n+1} is the unique coincidence point of A and B. Therefore let us suppose that $y_n \neq y_{n+1}$ for all $n \in N$.Hence we have

$$\begin{aligned} 0 \leq \zeta(&\frac{1}{F_{Ax_n,\, Ax_{n+1}}(t)} - 1, \phi(\frac{1}{F_{Bx_n,\, Bx_{n+1}}(t)} - 1) \\ &= \zeta(\frac{1}{F_{y_n,\, y_{n+1}}(t)} - 1, \phi(\frac{1}{F_{y_{n-1},y_n}(t)} - 1) \\ &< \phi(\frac{1}{F_{y_{n-1},\, y_n}(t)} - 1), (\frac{1}{F_{y_n,\, y_{n+1}}(t)} - 1) \\ &= S(y_{n-1}, y_{n,}t) - S(y_n, y_{n+1,}t) \text{ for all } n \in N \qquad (11) \end{aligned}$$

where $S(y_{n-1}, y_n, t) = \frac{1}{F_{y_{n-1},\, y_n}(t)} - 1$.

Therefore $\{\ S(y_{n-1}, y_n, t)\}$ is a decreasing sequence of positive real numbers. Thus there exists $z \geq 0$ such that

$$\lim_{n\to\infty} S(y_{n-1}, y_n, t) = z \tag{12}$$

Suppose $z > 0$ then

$$0 \leq \lim_{n\to\infty} \sup \zeta(S(y_n, y_{n+1}, t), \phi(S(y_{n-1}, y_n, t)) < 0$$

where $t_n = S(y_n, y_{n+1}, t), s_n = \phi(S(y_{n-1}, y_n, t)) < S(y_{n-1}, y_n, t)$, and $t_n < s_n,\ t_n, s_n \to z > 0$.

This is a contradiction. Thus we have

$$\lim_{n\to\infty} S(y_{n-1,}\, y_n, t) = 0$$

By (12) we obtain

$$\lim_{n\to\infty} F_{y_n,\, y_{n+1}}(t) = 1 \tag{13}$$

Now we claim that the sequence $\{y_n\}$ is Cauchy sequence in X.Suppose to the contrary that $\{y_n\}$ is not a Cauchy sequence in X, therefore $\lim_{m,n\to\infty} \inf F_{y_m,\, y_n}(t_0) < 1$ for some $t_0 > 0$.

Then there exists $0 < \epsilon < 1$ and two sub sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that n_k is the smallest index for which $n_k > m_k \geq k$ and

$$F_{y_{m_k},\, y_{n_k}}(t_0) \leq 1 - \epsilon \tag{14}$$

and

$$F_{y_{m_k},\, y_{n_{k-1}}}(t_0) > 1 - \epsilon \tag{15}$$

Now we have

$$\begin{aligned} 1 - \epsilon &\geq F_{y_{m_k},\, y_{n_k}}(t_0) \\ &\geq \Delta(F_{y_{m_k},\, y_{n_{k-1}}}(t_0),\ F_{y_{n_{k-1}},\, y_{n_k}}(t_0)) \\ &\geq \Delta(1 - \epsilon,\ F_{y_{n_{k-1}},\, y_{n_k}}(t_0)) \end{aligned}$$

Letting $k \to \infty$ and using (13), we get

$$\lim_{k\to\infty} F_{y_{m_k},\, y_{n_k}}(t_0) = 1 - \epsilon \tag{16}$$

By the same reasoning as above, we obtain

$$\begin{aligned} 1-\epsilon &\geq F_{y_{m_k},\, y_{n_k}}(t_0) \\ &\geq \Delta(F_{y_{m_k},\, y_{m_{k-1}}}(t_0),\ F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0),\ F_{y_{n_{k-1}},\, y_{n_k}}(t_0) \end{aligned}$$

and

$$F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0) \geq \Delta(F_{y_{m_{k-1}},\, y_{m_k}}(t_0),\ \Delta(F_{y_{m_k},\, y_{n_k}}(t_0),\ F_{y_{n_k},\, y_{n_{k-1}}}(t_0)))$$

From the last inequality, by letting $k \to \infty$ and using (13), (16) we get

$$\lim_{k\to\infty} F_{y_{m_{k-1}},\, y_{n_{k-1}}}(t_0) = 1-\epsilon \tag{17}$$

By letting $k \to \infty$ and using (16) and (17) we obtain

$$\begin{aligned} \lim_{k\to\infty} S(y_{m_k}, y_{n_k}, t_0) &= \lim_{k\to\infty} \frac{1}{F_{y_{m_k},\, y_{n_k}}(t_0)} - 1 \\ &= \lim_{k\to\infty} \frac{1 - F_{y_{m_k},\, y_{n_k}}(t_0)}{F_{y_{m_k},\, y_{n_k}}(t_0)} \\ &= \frac{1-(1-\epsilon)}{1-\epsilon} \\ &= \frac{\epsilon}{1-\epsilon} \end{aligned}$$

and

$$\lim_{k\to\infty} S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) = \frac{\epsilon}{1-\epsilon}$$

Let

$$t_k = S(y_{m_k}, y_{n_k}, t_0)$$

$$s_k = \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0)) < S(y_{m_{k-1}}, y_{n_{k-1}} t_0)$$

By (9), we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{F_{Ax_{m_k},\, Ay_{n_k}}(t_0)} - 1, \phi(\frac{1}{F_{Bx_{m_k}, Bx_{n_k}}(t_0)} - 1) \\ &= \zeta(\frac{1}{F_{y_{m_k},\, y_{n_k}}(t_0)} - 1, \phi(\frac{1}{F_{y_{m_{k-1}},\, y_{n_{k-1},}}(t_0)} - 1) \\ &= \zeta(S(y_{m_k}, y_{n_k}, t_0), \phi(S((y_{m_{k-1}}, y_{n_{k-1,}} t_0)) \\ &< \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0) - S(y_{m_k}, y_{n_k}, t_0) \qquad (18) \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

From (18) we deduce that

$$\lim_{k\to\infty} \sup \zeta(S(y_{m_k}, y_{n_k}, t_0), \phi(S(y_{m_{k-1}}, y_{n_{k-1}}, t_0)) = 0.$$

Clearly this is a contradiction to (ζ_2) and hence we conclude that $\{y_n\}$ is a Cauchy sequence in X. Now since AX or BX is a complete subset of X therefore there exists $u \in X$ such that $y_n \to Bu$ as $n \to \infty$. If there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} = Au$ then letting $k \to \infty$ we get $Au = Bu$ and hence the claim. So we suppose that $y_{n_k} \neq Au$ for all $n \in N$.

Since $y_{n-1} \neq y_n$ there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \neq Bu$ for $k \in N$.Using (9) we have

$$\begin{aligned} 0 &\leq \zeta(\frac{1}{F_{Ax_{n_{k+1}},\, Au}(t)} - 1, \phi(\frac{1}{F_{Bx_{n_{k+1}},\, Bu}(t)} - 1) \\ &= \zeta(S(y_{n_{k+1}}, Au, t), \phi(S(y_{n_k}, Bu, t))) \\ &< \phi(S(y_{n_k}, Bu, t) - S(y_{n_{k+1},} Au, t)). \\ &< S(y_{n_k}, Bu, t) - S(y_{n_{k+1}}, Au, t) \text{ for all } n \in N \end{aligned}$$

This shows that $y_{n_{k+1}} \to Au$ and hence $Au = Bu$ is a unique coincidence point of A and B. If A and B are weakly compatible then by using well known result due to [7] we can prove the existence of unique common fixed point of A and B. ∎

Theorem 13. *Let (X, F, Δ) be a N. A. Menger space and $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in Z$ and a function $k \in (0, \frac{1}{2})$ such that for all x, $y \in X$*

(19)

$$\zeta(\frac{1}{F_{Ax,\, Ay}(t)} - 1,\ k \max\{\frac{1}{F_{Bx,\, By}(t)} - 1, \frac{1}{F_{Bx,\, Ax}(t)} - 1, \frac{1}{F_{By,\, Ay}(t)} - 1, \frac{1}{F_{Bx,\, Ay}(t)} - 1\}) \geq$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Corollary 14. *If in (19) we put*

$Bx = x$ for all $x \in X$ then $A : X \to X$ has a unique fixed point in (X, F, Δ).

3. Extended approach to a modular metric space

Definition 15. [1] [2]. *Let* $\omega : (0, \infty) \times X \times X \to [0, \infty)$ *be a function satisfying the following conditions for all* $\lambda, \mu > 0$ *and* $x, , y, z \in X$

(i) $x = y$ *if and only if* $\omega(\lambda, x, y) = 0$ *for all* $\lambda > 0$.

(ii) $\omega(\lambda, x, y) = \omega(\lambda, y, x)$

(iii)$\omega(\lambda + \mu, x, y) \leq \omega(\lambda, x, y) + \omega(\mu, z, y)$.

Then ω *is called a modular metric on* X. *If we replace (i) by*

(iv) $\omega(\lambda, x, x) = 0$ *for all* $\lambda > 0$,

then ω *is called pseudo modular metric on* X. *If we replace (iii) by*

(v) $\omega(\lambda, x, y) \leq \omega(\lambda, x, z) + \omega(\lambda, z, y)$ *for all* $\lambda > 0$ *and* $x, y, z \in X$

Then ω *is called non-Archimedean. Moreover* ω *is called convex if the following inequality is satisfied for all* $\lambda, \mu > 0$ *and* $x, , y, z \in X$

(vi) $\omega(\lambda + \mu, x, z) \leq \frac{\lambda}{\lambda+\mu}\omega(\lambda, x, z) + \frac{\mu}{\lambda+\mu}\omega(\mu, z, y)$.

Remark 16. *(i) A metric on a set* X *is a finite distance between any two points of* X *while a modular on a same set* X *is a way to consider a nonnegative "field of velocities" precisely an average velocity* $\omega(\lambda, x, y)$ *is associated to each* $\lambda > 0, \omega(\lambda, x, y)$ *that is one takes time* λ *to move from* x *to* y

(ii)[4]. *Let* (X, F, Δ) *be a triangular N. A. Menger space. Define a function* $\omega : (0, \infty) \times X \times X \to [0, \infty)$ *as*

$$\omega(\lambda, x, y) = \frac{1}{F_{x,\, y}(\lambda)} - 1 \tag{20}$$

for all $x, y \in X$ and $\lambda > 0$.Then ω_λ is a modular metric on X.

Definition 17. *Let* X_ω *be a modular metric space. Then*

(i) $\{x_n\}$ *in* X_ω *is called* $\omega-$*convergent to* $x \in X_\omega$,*if* $\omega(\lambda, x_n, x) \to 0$ *as* $n \to \infty$ *for all* $\lambda > 0$. *In this case we say* x *is the* $\omega-$*limit of* $\{x_n\}$.

(ii) $\{x_n\}$ *in* X_ω *is called* $\omega-$*Cauchy if* $\omega(\lambda, x_n, x_m) \to 0$ *as* $m, m \to \infty$ *for all* $\lambda > 0$.

(iii) A subset Y *of* X_ω *is called* $\omega-$*complete if any* $\omega-$*Cauchy sequence in* Y *is a* $\omega-$*convergent sequence and it's* $\omega-$*limit is in* Y.

Now we state two existence results for unique fixed point in the setting of modular space. Clearly these results are modular counterparts of Theorem **??** *and Theorem 16.*

Theorem 18. *Let* X_ω *be a non-Archimedean modular metric space and let* $A, B : X \to X$ *be two given mappings. Let there exists* $\zeta \in Z$ *such that*

(21)
$$\zeta(\omega(\lambda, Ax, Ay),\omega(\lambda, Bx, By)) \geq 0 for all x, y \in X \textit{ and for all } \lambda > 0$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

Theorem 19. *Let X_ω be a non-Archimedean modular metric space and let $A, B : X \to X$ be two given mappings. Suppose there exists $\zeta \in Z$ and a function $\phi : [0, \infty) \to [0, \infty)$ such that*

$$\zeta(\omega(\lambda, Ax, Ay),\phi(\omega(\lambda, Bx, By)) \geq 0 for all x, y \in X \textit{ and for all } \lambda > 0$$

$$0 < \phi(t) \leq t \textit{ for all } t \in (0, \infty) \textit{ and } \phi(0) = 0$$

If $AX \subseteq BX$ and AX or BX is a complete subset of X. Then A and B have unique coincidence point in X. Moreover if A and B are weakly compatible then A and B have a unique common fixed point in X.

The proofs of Theorem 18 and Theorem 19 are established by applying Theorem 11 and Theorem 12. We give outline of the proof of Theorem 18.

Proof. Let F be a probabilistic metric induced by ω and defined by (20). It follows that the triple (X, F, Δ) is N. A. Menger space. Then by (21) we have

$$\zeta(\frac{1}{F_{Ax,\, Ay}(\lambda)} - 1 + \frac{1}{F_{Bx,\, By}(\lambda)} - 1) \geq 0$$

for all $x, y \in X_\omega$ and for all $\lambda > 0$.Therefore, we apply Theorem 11 to conclude that A and B have a unique common fixed point in X. ∎

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