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A CHARACTERIZATION OF 0-COMPLETENESS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we introduce the concept of p-point in a partial metric space and extend Weston's characterization of metric completeness to partial metric spaces in terms of p-point. As a consequence of this study, we obtain the celebrated Banach Contraction Principle in the framework of 0-complete partial metric space.

1. Introduction

In 1994, Matthews [10] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved the well known Banach Contraction Principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [2, 3, 4, 5, 6, 7, 8, 17] are viable and have opened new avenues for application in different fields of mathematics and applied sciences. It is interesting to note that in partial metric spaces, self-distance of an arbitrary point need not be equal to zero. Matthews [10] introduced a class of open p-balls in partial metric spaces which generates a T_0 topology on X.

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This will facilitate the initiation of open and closed sets, neighbourhoods and other related notions in partial metric spaces. In this work, we shall discuss some topological aspects of partial metric spaces and prove Cantor's intersection theorem, Urysohn's lemma in this setting. We also prove that every partial metric space is first countable and hence continuity is equivalent to sequential continuity.

In 1977, Weston [20] has proved a completeness criterion of metric spaces which has got some relation with the family of real valued semicontinuous functions carried over the space. In fact, he had proved a necessary and sufficient condition for the metric space (X, d) to be complete in terms of the notion of d-point for lower semicontinuous functions. Later on, several authors successfully characterized metric completeness in terms of fixed point theory(see [11, 12, 13, 14, 15, 18, 19]). In this study, our main purpose is to introduce the concept of p-point in partial metric spaces and extend Weston's characterization [20] of metric completeness to partial metric spaces in terms of p-point. As a consequence of this study, we obtain the celebrated Banach Contraction Principle in the framework of 0-complete partial metric space.

2. Some Basic Concepts

In this section, we begin with some basic facts and properties of partial metric spaces.

Definition 2.1. [10] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

(p_1) $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2), it follows that $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Moreover, it is valuable to note that the axiom (p_4) is stronger than triangle inequality.

Example 2.2. [10] *Let $X = [0, \infty)$ and let $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a partial metric space.*

Example 2.3. [10] *Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and let $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, p) is a partial metric space.*

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X,\, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

We now visualise the open balls in a particular case.

Example 2.4. *Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$, then p is a partial metric on X. In this case, an open p-ball $B_p(x,\epsilon)$ is given by*

$$\begin{aligned} B_p(x,\epsilon) &= \{ y \in X : p(x,y) < p(x,x) + \epsilon \} \\ &= \{ y \in X : \max\{x,y\} < x + \epsilon \} \\ &= \{ y \in X : y < x + \epsilon \} \\ &= [0, x+\epsilon). \end{aligned}$$

Theorem 2.5. *If $U \in \tau_p$ and $x \in U$, then there exists $r > 0$ such that $B_p(x,r) \subseteq U$.*

Proof. Since U is an open set containing x, there exists an open p-ball, say $B_p(y,\epsilon)$ such that $x \in B_p(y,\epsilon) \subseteq U$. Then $p(x,y) < p(y,y) + \epsilon$. Let us choose $0 < r < p(y,y) - p(x,y) + \epsilon$ and consider the open p-ball $B_p(x,r)$. Then it is easy to verify that $B_p(x,r) \subseteq B_p(y,\epsilon) \subseteq U$. □

Remark 2.6. *Let (X,p) be a partial metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_p if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.*

Let $x_n \to x$ w.r.t. τ_p and $\epsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B_p(x,\epsilon)$ for all $n \geq n_0$. This gives that $p(x_n,x) - p(x,x) < \epsilon$ for all $n \geq n_0$. Since $p(x_n,x) - p(x,x) \geq 0$, it follows that $\mid p(x_n,x) - p(x,x) \mid < \epsilon$ for all $n \geq n_0$. This proves that $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Conversely, suppose that $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. We shall show that $x_n \to x$ w.r.t. τ_p. Let $U \in \tau_p$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B_p(x,\epsilon) \subseteq U$. By hypotheses, it follows that

$$\lim_{n\to\infty} (p(x_n,x) - p(x,x)) = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that $p(x_n,x) - p(x,x) < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_p(x,\epsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_p on X.

Definition 2.7. [10] *Let (X, p) be a partial metric space and let (x_n) be a sequence in X. Then*

(i) *(x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.*

(ii) *(x_n) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.*

(iii) *(X, p) is said to be complete if every Cauchy sequence (x_n) in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

Definition 2.8. [16] *A sequence (x_n) in (X, p) is called 0-Cauchy if*

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 2.9. *Let (X, p) be a partial metric space.*

(a) *(see [1, 9]) If $p(x_n, z) \to p(z, z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$.*

(b) *(see [16]) If (X, p) is complete, then it is 0-complete.*

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.10. [16] *The space $X = [0, \infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = \max\{x, y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p), but it is not a 0-Cauchy sequence.*

3. Some Topological Aspects

Theorem 3.1. *Let (X, p) be a partial metric space with the topology τ_p defined above and A be any nonempty subset of X. Then,*

(i) *A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;*

(ii) *for any $x \in \overline{A}$ and for any $\epsilon > 0$, we have $B_p(x, \epsilon) \cap A \neq \emptyset$.*

Proof. (i) Suppose that A is a closed subset of X. Let (x_n) be a sequence in A such that $x_n \to x$ as $n \to \infty$. We shall show that $x \in A$. If possible, suppose that $x \notin A$. So $x \in X \setminus A$ and $X \setminus A$ is open. Then there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subseteq X \setminus A$.

Therefore, $B_p(x,\epsilon) \cap A = \emptyset$. Since $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. So for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x) - p(x, x) < \epsilon$, for all $n \geq n_0$. This gives that $x_n \in B_p(x, \epsilon)$, for all $n \geq n_0$. Hence $x_n \in B_p(x, \epsilon) \cap A$, for all $n \geq n_0$, which leads to a contradiction with $B_p(x, \epsilon) \cap A = \emptyset$. So, $x \in A$.

Conversely, assume that the condition holds i.e., for any sequence (x_n) in A which converges to x, we have $x \in A$. Let us prove that A is closed. In fact, we have to show that $X \setminus A$ is open. So for any $x \in X \setminus A$, we need to prove that there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subseteq X \setminus A$ i.e., $B_p(x, \epsilon) \cap A = \emptyset$. If possible, suppose that for any $\epsilon > 0$, we have $B_p(x, \epsilon) \cap A \neq \emptyset$. So for any $n \geq 1$, choose $x_n \in B_p(x, \frac{1}{n}) \cap A$. Then $x_n \in A$ for all $n \geq 1$ and $p(x_n, x) - p(x, x) < \frac{1}{n}$ for all $n \geq 1$. Therefore, $\lim_{n\to\infty}(p(x_n, x) - p(x, x)) = 0$. That is, $x_n \to x$ as $n \to \infty$ in (X, p). Hence, by assumption $x \in A$, which is a contradiction. So for any $x \in X \setminus A$, there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subseteq X \setminus A$ i.e., $X \setminus A$ is open and hence A is closed in X.

(ii) It follows from definition that $\overline{A}$ is the smallest closed subset which contains A. Set $A^* = \{x \in X : \textit{for any } \epsilon > 0, \exists\, a \in A \textit{ such that } p(x, a) < p(x, x) + \epsilon\}$. Obviously, $A \subseteq A^*$. Next we prove that A^* is closed. Let (x_n) be a sequence in A^* such that $x_n \to x$ as $n \to \infty$. We have to prove that $x \in A^*$. Since $x_n \to x$ as $n \to \infty$, we have $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x) - p(x, x) < \frac{\epsilon}{2}$, for all $n \geq n_0$. As $x_n \in A^*$, there exists $a_n \in A$ such that $p(x_n, a_n) < p(x_n, x_n) + \frac{\epsilon}{2}$. Hence,

$$\begin{aligned} p(x, a_n) &\leq p(x, x_n) + p(x_n, a_n) - p(x_n, x_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} + p(x, x) \\ &= \epsilon + p(x, x), \ \textit{for all } n \geq n_0. \end{aligned}$$

In particular, $p(x, a_{n_0}) < \epsilon + p(x, x)$, which implies that $x \in A^*$. Therefore, by part (i), it follows that A^* is closed and contains A. The definition of $\overline{A}$ assures that $\overline{A} \subseteq A^*$, which implies the conclusion of (ii). □

Theorem 3.2. *Every closed subset of a complete partial metric space is complete.*

Proof. Let (X, p) be a complete partial metric space and Y be a closed subset of X. Let (y_n) be a Cauchy sequence in (Y, p_Y), where $p_Y : Y \times Y \to \mathbb{R}^+$ is defined by $p_Y(u, v) = p(u, v)$ for all $u, v \in Y$. Then (y_n) is also a Cauchy sequence in (X, p). As (X, p) is complete, there exists $x \in X$ such that $y_n \to x$ as $n \to \infty$. By applying Theorem 3.1, it follows that $x \in Y$. Thus (y_n) converges in (Y, p_Y). So, (Y, p_Y) becomes a complete partial metric space. □

Remark 3.3. *$x \in \overline{A}$ if and only if every open set U containing x intersects A.*

Definition 3.4. *Let (X, p) be a partial metric space, $A \subseteq X$ and $x \in X$. Then $p(x, A)$ is defined as follows:*

$$p(x, A) = \inf \{p(x, a) - p_{xa} : a \in A\},$$

where $p_{xa} = \min \{p(x, x), p(a, a)\}$. Obviously, $p(x, A) \geq 0$ and $p(x, A) = 0$ if $x \in A$.

We now prove the following theorem.

Theorem 3.5. *Let (X, p) be a partial metric space, $A \subseteq X$ and $x \in X$. If $p(x, A) = 0$ then $x \in \overline{A}$.*

Proof. Let $p(x, A) = 0$ and $U \in \tau_p$, $x \in U$. Then there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subseteq U$. Since $p(x, A) = 0$, there exists $x_\epsilon \in A$ such that $p(x, x_\epsilon) - p_{xx_\epsilon} < \epsilon$. This implies that $p(x, x_\epsilon) - p(x, x) \leq p(x, x_\epsilon) - p_{xx_\epsilon} < \epsilon$. Therefore, $x_\epsilon \in B_p(x, \epsilon) \subseteq U$ and $x_\epsilon \in A$. Hence, $U \cap A \neq \emptyset$. The last theorem ensures that $x \in \overline{A}$.

□

Next we prove the property of first countability of partial metric spaces.

Theorem 3.6. *Let (X, p) be a partial metric space and $x \in X$ be arbitrary. Then there exists a countable collection $\{B_n\}_{n=1}^{\infty}$ of open neighbourhoods of x such that for any neighbourhood U of x, there exists $m \in \mathbb{N}$ with $B_m \subseteq U$.*

Proof. For each $n \in \mathbb{N}$, we consider $B_n = B_p(x, \frac{1}{n})$. Clearly, $\{B_n : n \in \mathbb{N}\}$ is a countable family of open p-balls centered at x. Let U be any neighbourhood of x. Then there exists $r > 0$ such that $B_p(x, r) \subseteq U$. We choose $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Then, $B_m = B_p(x, \frac{1}{m}) \subseteq B_p(x, r) \subseteq U$. □

Definition 3.7. *Let (X, p) be a partial metric space and $A \subseteq X$. The diameter of A, denoted by $diam(A)$, is defined by*

$$diam(A) = \sup\{p(x, y) : x,\ y \in A\}.$$

Clearly, $0 \leq diam(A) \leq \infty$. The subset A is said to be bounded if $diam(A)$ is finite. Otherwise, A is said to be unbounded.

It follows from the above definition that if $A \subseteq B$, then $diam(A) \leq diam(B)$. Hence, it is worth mentioning that $diam(A) \leq diam(\overline{A})$.

We now prove Cantor's intersection theorem in partial metric spaces.

Theorem 3.8. *If a partial metric space (X, p) is 0-complete, then every descending sequence (A_n) of nonempty closed sets with $diam(A_n) \to 0$ as $n \to \infty$, the intersection $A = \bigcap_{n=1}^{\infty} A_n$ consists of exactly one point.*

Proof. Let (X, p) be a 0-complete partial metric space and let (A_n) be a descending sequence of nonempty closed sets with $diam(A_n) \to 0$ as $n \to \infty$. As each A_n is nonempty, we choose a point $x_n \in A_n$, for each $n \in \mathbb{N}$. We shall show that (x_n) is 0-Cauchy in (X, p). For $m,\ n \in \mathbb{N}$ with $m > n$, we have $A_m \subseteq A_n$ which gives that $x_m,\ x_n \in A_n$. Therefore,

$$p(x_n, x_m) \leq diam(A_n) \to 0 \ as \ n \to \infty.$$

i.e., $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.
This shows that (x_n) is a 0-Cauchy sequence in (X, p). Then by hypothesis, there exists $x \in X$ such that $x_n \to x$ with $p(x, x) = 0$ i.e., $p(x_n, x) \to p(x, x) = 0$ as $n \to \infty$. We prove that $x \in \bigcap_{n=1}^{\infty} A_n$.
Let $U \in \tau_p$ and $x \in U$. Then there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subseteq U$. As $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x) < \epsilon = p(x, x) + \epsilon$, for all $n \geq n_0$. Therefore, $x_n \in B_p(x, \epsilon) \subseteq U$, for all $n \geq n_0$. Again, $x_m \in A_n$, for all $m \geq n$ as $x_m \in A_m \subseteq A_n$, for all $m \geq n$. So, $U \cap A_n \neq \emptyset$, for all $n \in \mathbb{N}$. This proves that $x \in \overline{A_n} = A_n$, for all n, A_n being closed. Hence $x \in \bigcap_{n=1}^{\infty} A_n$.

Now, let $y \in \bigcap_{n=1}^{\infty} A_n$ with $y \neq x$. Then for each $n \in \mathbb{N}$, we have x, $y \in A_n$. Therefore,

$$0 \leq p(x,y) \leq diam(A_n) \to 0 \; as \; n \to \infty$$

which gives that $p(x,y) = 0$ and hence $x = y$, a contradiction. This proves that A contains exactly one point. □

Definition 3.9. *Let (X, p_1) and (Y, p_2) be two partial metric spaces. A function $f : (X, p_1) \to (Y, p_2)$ is said to be continuous at a point $a \in X$, if corresponding to every $\epsilon > 0$, $\exists\, \delta > 0$ such that*

$$x \in B_{p_1}(a, \delta) \; implies \; f(x) \in B_{p_2}(f(a), \epsilon).$$

f is said to be continuous on X if it is continuous at each point of X.

Obviously, the concept of continuity of a real valued function on a partial metric space turns out to be a special case of the above definition by considering $Y = \mathbb{R}$ and $p_2(y, z) =\mid y - z \mid$ for all y, $z \in \mathbb{R}$. For such real valued functions on a partial metric space, we can prove the following theorem, as exact duplicates of the corresponding proofs for real valued continuous functions on a metric space.

Theorem 3.10. *Let f and g be real valued functions on a partial metric space (X, p). If f and g are continuous at a point $a \in X$ and $g(x) \neq 0$ for all $x \in X$, then so are $f \pm g$, fg, αf $(for\ any\ \alpha \in \mathbb{R})$ and $\frac{f}{g}$.*

Theorem 3.11. *Let (X, p_1) and (Y, p_2) be two partial metric spaces. Then a function $f : (X, p_1) \to (Y, p_2)$ is continuous at a point $a \in X$ if and only if for each sequence (x_n) in X converging to a in (X, p_1), the sequence $(f(x_n))$ in Y converges to $f(a)$ in (Y, p_2).*

Proof. Suppose that f is continuous at $a \in X$. Then for a given $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$x \in B_{p_1}(a, \delta) \; implies \; f(x) \in B_{p_2}(f(a), \epsilon).$$

Since (x_n) converges to a in (X, p_1), we have $\lim_{n\to\infty} p_1(x_n, a) = p_1(a, a)$. So, there exists $n_0 \in \mathbb{N}$ such that $p_1(x_n, a) < p_1(a, a) + \delta$, for all $n \geq n_0$. This shows that $x_n \in B_{p_1}(a, \delta)$, for all $n \geq n_0$. By hypothesis, it follows that $f(x_n) \in B_{p_2}(f(a), \epsilon)$, for all $n \geq n_0$. Then $p_2(f(x_n), f(a)) < p_2(f(a), f(a)) + \epsilon$, for all $n \geq n_0$. Consequently, it follows that $\lim_{n\to\infty} p_2(f(x_n), f(a)) = p_2(f(a), f(a))$. Therefore, $(f(x_n))$

converges to $f(a)$ in (Y, p_2).

Conversely, suppose the condition holds but f is not continuous at $a \in X$. Then there exists $\epsilon > 0$ such that for each $\delta > 0$, $\exists\, x_\delta \in X$ with $x_\delta \in B_{p_1}(a, \delta)$ but $f(x_\delta) \notin B_{p_2}(f(a), \epsilon)$. In particular, for each $n \in \mathbb{N}$, $\exists\, x_n \in X$ with $x_n \in B_{p_1}(a, \frac{1}{n})$ but $f(x_n) \notin B_{p_2}(f(a), \epsilon)$. It then follows that $x_n \to a$ but $p_2(f(x_n), f(a)) \geq p_2(f(a), f(a)) + \epsilon$, for all $n \in \mathbb{N}$ i.e., $(f(x_n))$ does not converge to $f(a)$ in (Y, p_2). This contradicts the assumed hypothesis. □

We now present analogue of Urysohn's lemma in partial metric spaces.

Theorem 3.12. *For any two nonempty disjoint closed subsets U, V of a partial metric space (X, p), there exists a function $f : X \to \mathbb{R}$ such that $f(U) = \{0\}$, $f(V) = \{1\}$ and $0 \leq f(x) \leq 1$ for all $x \in X$.*

Proof. For $A \subseteq X$ and $x \in X$, we use the notation $p_A(x)$ for the function $p(x, A)$. We now show that the function $f(x) = \frac{p_U(x)}{p_U(x)+p_V(x)}$ is the desired function. If $p_U(x) + p_V(x) = 0$ for some $x \in X$, then $p_U(x) = p_V(x) = 0$ and hence $x \in \overline{U} = U$ and $x \in \overline{V} = V$, which contradicts the fact that $U \cap V = \emptyset$. Therefore, f is well defined. Obviously, $0 \leq f(x) \leq 1$ for all $x \in X$. Now, $x \in U$ *implies* $p_U(x) = 0$ *implies* $f(x) = 0$ and $x \in V$ *implies* $p_V(x) = 0$ *implies* $f(x) = 1$. □

4. A Characterization of 0-Completeness

Inspired by Weston [20], we now characterize 0-completeness via lower semicontinuity, introduce a strict order " $\ll$ " and establish Banach Contraction Principle in the setting of 0-complete partial metric spaces.

Definition 4.1. *Let (X, p) be a partial metric space. A function $\varphi : X \to \mathbb{R}$ is called lower semicontinuous if, for each sequence $(x_n) \subseteq X$ converges to a point $x \in X$ with $p(x, x) = 0$, we have*

$$\varphi(x) \leq \liminf_{n\to\infty} \varphi(x_n).$$

Definition 4.2. *Let (X, p) be a partial metric space. A function $f : X \to \mathbb{R}$ is called uniformly continuous on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$p(x, y) < \delta \text{ implies } \mid f(x) - f(y) \mid < \epsilon.$$

Definition 4.3. *Let (X, p) be a partial metric space and $h : X \to \mathbb{R}$ be a function. A point $x_0 \in X$ is called a p-point for h if for every point $x \in X$ other than x_0,*

$$h(x_0) - h(x) < p(x_0, x).$$

Example 4.4. *Let $X = [0, \infty)$ and let $p(x, y) = max\{x, y\}$ be a partial metric on X. Let $h : X \to \mathbb{R}$ be defined by $h(x) = 2x$ for all $x \in X$. Then, $h(0) - h(x) = -2x < p(0, x)$ for every $x \in X$ with $x \neq 0$. Therefore, 0 is a p-point for h. But if $g : X \to \mathbb{R}$ is defined by $g(x) = \frac{x}{2}$ for all $x \in X$. Then every point of X is a p-point for g.*

Example 4.5. *Let $X = [0, 1]$ be equipped with the partial metric given as*

$$p(x, y) = \begin{cases} \mid x - y \mid, & if \;\; x, y \in [0, 1), \\ \\ 1, & if \; x = 1, \text{ or } y = 1. \end{cases}$$

Let $h : X \to \mathbb{R}$ be defined by $h(x) = 1 - x$ for all $x \in X$. Then, $h(1) - h(x) = -h(x) < p(1, x)$ for every $x \in X$ with $x \neq 1$. Therefore, 1 is a p-point for h.

Theorem 4.6. *If the partial metric space (X, p) is 0-complete then any lower semicontinuous function $h : X \to \mathbb{R}$ which is bounded below has a p-point. If (X, p) is not 0-complete, then there is a uniformly continuous function $g : X \to \mathbb{R}$ which is bounded below but has no p-point.*

Proof. To prove the first part, we assume that (X, p) is 0-complete and $h : X \to \mathbb{R}$ is a lower semicontinuous function which is bounded below. For any point $x_1 \in X$, we construct a sequence (x_n) in the following way:

For each $n \in \mathbb{N}$, let

$$c_n = \inf\{h(x) : h(x_n) - h(x) \geq p(x_n, x) > 0\}$$

and let x_{n+1} be a point such that

$$h(x_n) - h(x_{n+1}) \geq p(x_n, x_{n+1}) \tag{4.1}$$

and

$$h(x_{n+1}) < c_n + n^{-1}. \tag{4.2}$$

In above construction, we assume that none of x_n is a p-point for h. In case, x_n is a p-point for h, then we have nothing to prove. It follows from condition (4.1) that the sequence $(h(x_n))$ is nonincreasing in $\mathbb{R}$.

Also, it is bounded below. So, the sequence $(h(x_n))$ is convergent.

For $m \geq n$, we have

$$\begin{aligned} h(x_n) - h(x_m) &= h(x_n) - h(x_{n+1}) + h(x_{n+1}) - h(x_{n+2}) \\ &\quad + \cdots + h(x_{m-1}) - h(x_m) \\ &\geq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\geq p(x_n, x_m). \end{aligned} \tag{4.3}$$

Hence,

$$p(x_n, x_m) \leq h(x_n) - h(x_m) \to 0 \ as \ m, \ n \to \infty.$$

This yields that the sequence (x_n) is 0-Cauchy in (X, p). By 0-completeness of X, it follows that the sequence (x_n) converges to a point $x_0 \in X$ such that $p(x_0, x_0) = 0$. From condition (4.3), it follows that

$$h(x_m) \leq h(x_n) - p(x_n, x_m) \tag{4.4}$$

for all $m \geq n$. By using condition (4.4), Lemma 2.9 and lower semi-continuity of the function h, one can obtain that

$$\begin{aligned} h(x_0) &\leq \liminf_{m\to\infty} h(x_m) \\ &\leq \liminf_{m\to\infty} [h(x_n) - p(x_n, x_m)] \\ &= h(x_n) - p(x_n, x_0) \end{aligned}$$

for all $n \geq 1$.

Thus,

$$h(x_n) - h(x_0) \geq p(x_n, x_0) \tag{4.5}$$

for all $n \geq 1$.

If x_0 is not a p-point for h, then for some $x(\neq x_0) \in X$, we have

$$h(x_0) - h(x) \geq p(x_0, x) > 0. \tag{4.6}$$

Using conditions (4.5) and (4.2), we obtain

$$h(x) \leq h(x_{n+1}) + h(x) - h(x_0) < c_n + n^{-1} + h(x) - h(x_0). \tag{4.7}$$

In view of condition (4.6), we can choose n so that condition (4.7) gives that $h(x) < c_n$.

From conditions (4.5) and (4.6), it follows that

$$\begin{aligned} h(x_n) - h(x) &= h(x_n) - h(x_0) + h(x_0) - h(x) \\ &\geq p(x_n, x_0) + p(x_0, x) \\ &> 0, \end{aligned}$$

which implies that $h(x_n) > h(x)$. So, $x_n \neq x$ and therefore $p(x_n, x) > 0$.

Moreover,

$$h(x_n) - h(x) \geq p(x_n, x_0) + p(x_0, x) > p(x_n, x) > 0.$$

It now follows from the definition of c_n that $h(x) \geq c_n$, which contradicts the fact that $h(x) < c_n$. Thus, x_0 is a p-point for h.

Now suppose that (X, p) is not 0-complete. So there exists a 0-Cauchy sequence (x_n) in X which does not converge to a point $x \in X$ such that $p(x, x) = 0$. We show that for any $x \in X$, the sequence $(2p(x, x_n))$ is Cauchy in $\mathbb{R}$.

For $x \in X$, we have

$$p(x, x_n) \leq p(x, x_m) + p(x_m, x_n) - p(x_m, x_m) \leq p(x, x_m) + p(x_m, x_n)$$

and so

$$p(x, x_n) - p(x, x_m) \leq p(x_m, x_n).$$

Interchanging n and m, we obtain

$$p(x, x_m) - p(x, x_n) \leq p(x_m, x_n).$$

Therefore,

$$| \, p(x, x_n) - p(x, x_m) \, | \leq p(x_m, x_n). \tag{4.8}$$

As (x_n) is 0-Cauchy, it follows from condition (4.8) that the sequence $(2p(x, x_n))$ is Cauchy in $\mathbb{R}$. Let $g(x)$ be its limit. Clearly, $g(x) > 0$ for every $x \in X$. Because $g(x) = 0$ for some $x \in X$ implies that $\lim_{n\to\infty} p(x, x_n) = 0$. Then,

$$\begin{aligned} p(x, x) &\leq p(x, x_n) + p(x_n, x) - p(x_n, x_n) \\ &\leq p(x, x_n) + p(x_n, x) \\ &\to 0, \ \ as \ n \to \infty, \end{aligned}$$

which gives that $p(x,x)=0$. Therefore, $\lim_{n\to\infty} p(x,x_n)=p(x,x)=0$, a contradiction.

Thus, the function g is bounded below. If $x_0\in X$, then

$$\begin{aligned} g(x_0)-g(x) &= \lim_{n\to\infty} 2p(x_0,x_n)-\lim_{n\to\infty} 2p(x,x_n)\\ &= \lim_{n\to\infty}\left[2p(x_0,x_n)-2p(x,x_n)\right]\\ &\le \lim_{n\to\infty} 2p(x_0,x)\\ &= 2p(x_0,x). \end{aligned}$$

Interchanging x_0 and x, we obtain

$$g(x)-g(x_0)\le 2p(x_0,x).$$

Thus,

$$\mid g(x_0)-g(x)\mid\le 2p(x_0,x).$$

Let $\epsilon>0$ be a given real number. We choose $\delta=\frac{\epsilon}{2}$ such that

$$p(x_0,x)<\delta \textit{ implies } \mid g(x_0)-g(x)\mid<\epsilon.$$

So, g is uniformly continuous. Also,

$$\frac{1}{2}\left[g(x_0)+g(x)\right]=\frac{1}{2}\left[\lim_{n\to\infty} 2p(x_0,x_n)+\lim_{n\to\infty} 2p(x,x_n)\right]\ge p(x_0,x).$$

Now,

$$\begin{aligned} g(x_0)-g(x) &= \frac{1}{2}\left[g(x_0)+g(x)\right]+\frac{1}{2}\left[g(x_0)-3g(x)\right]\\ &\ge p(x_0,x)+\frac{1}{2}\left[g(x_0)-3g(x)\right]. \end{aligned} \tag{4.9}$$

The definition of g implies that $g(x_m)\to 0$ as $m\to\infty$. Therefore, $3g(x)<g(x_0)$ if $x=x_m$ and m is large. It now follows from condition (4.9) that $g(x_0)-g(x)>p(x_0,x)$ if $x=x_m$ and m is large. So, x_0 is not a p-point for g.

□

The following corollary is the result of Weston [20].

Corollary 4.7. *If the metric space (X,d) is complete then any lower semicontinuous function $X\to\mathbb{R}$ which is bounded below has a d-point. If (X,d) is not complete there is a uniformly continuous function $X\to\mathbb{R}$ which is bounded below but has no d-point.*

Proof. The proof can be obtained from Theorem 4.6 by taking $p=d$. □

Remark 4.8. *When p and h are given, a relation can be defined on X as follows:*

$$x \ll y \; if \text{ and only } if \; h(y) - h(x) \geq p(x, y).$$

This relation " $\ll$ " is a strict order on X. In fact, " $\ll$ " is antisymmetric, transitive and irreflexive.

Definition 4.9. *A point x_0 in (X, p) is said to be a minimal point w.r.t. $\ll$ if and only if $x \ll x_0$ implies $x = x_0$.*

Remark 4.10. *A point of X is a p-point for h if and only if it is a minimal point w.r.t. $\ll$.*

Proof. Let $x_0 \in X$ be a p-point for h. Then,

$$h(x_0) - h(x) < p(x, x_0), \; for \text{ all } x \in X \text{ and } x \neq x_0. \tag{4.10}$$

Now $x \ll x_0$ implies that $h(x_0) - h(x) \geq p(x, x_0)$. This gives that $x = x_0$. Because if $x \neq x_0$, then by condition (4.10) it follows that $h(x_0) - h(x) < p(x, x_0)$, a contradiction. Therefore, x_0 is a minimal point w.r.t. $\ll$.

Conversely, let x_0 be a minimal point w.r.t. $\ll$. Then $x \ll x_0$ implies that $x = x_0$. That is, $x \ll x_0$ does not hold for all $x \in X$ with $x \neq x_0$. Therefore, $h(x_0) - h(x) < p(x, x_0)$ for all $x \in X$ with $x \neq x_0$. This gives that x_0 is a p-point for h. □

Remark 4.11. *If a function $f : X \to X$ is such that it may be possible to choose p and h so that the relation $\ll$ has the property that $fx \neq x$ implies $fx \ll x$, then any p-point for h is a fixed point for f.*

Proof. Let $x_0 \in X$ be a p-point for h. Then,

$$h(x_0) - h(x) < p(x, x_0), \; for \text{ all } x \in X \text{ and } x \neq x_0. \tag{4.11}$$

If $fx_0 \neq x_0$, then by hypothesis $fx_0 \ll x_0$ which implies that

$$h(x_0) - h(fx_0) \geq p(fx_0, x_0),$$

which contradicts the condition (4.11). So, it must be the case that $fx_0 = x_0$. This shows that x_0 is a fixed point of f. □

We now apply Theorem 4.6 and Remark 4.11 to prove Banach Contraction Principle in 0-complete partial metric spaces .

Theorem 4.12. *Let (X, p) be a 0-complete partial metric space and let $f : X \to X$ be a mapping satisfying the following condition:*

$$p(fx, fy) \leq \alpha \, p(x, y) \tag{4.12}$$

for all x, $y \in X$, where $0 \le \alpha < 1$ is a constant. Then f has a unique fixed point u in X and $p(u,u) = 0$.

Proof. Let $h(x) = \beta\, p(fx, x)$, where $\beta = \frac{1}{1-\alpha} > 0$ and $x \in X$. We first show that $h : X \to \mathbb{R}$ is a lower semicontinuous function. Let $y_n \to y$ in (X,p) with $p(y,y) = 0$. Then, $\lim_{n\to\infty} p(y, y_n) = p(y,y) = 0$. We have to show that

$$h(y) \le \liminf_{n\to\infty} h(y_n).$$

By using condition (4.12), we have

$$\begin{aligned} h(y) = \beta\, p(fy, y) &\le \beta\,[p(fy, y_n) + p(y_n, y) - p(y_n, y_n)] \\ &\le \beta\,[p(fy, y_n) + p(y_n, y)] \\ &\le \beta\,[p(fy, fy_n) + p(fy_n, y_n) - p(fy_n, fy_n) + p(y_n, y)] \\ &\le \beta\,[p(fy, fy_n) + p(fy_n, y_n) + p(y_n, y)] \\ &\le \beta\,[\alpha\, p(y, y_n) + p(fy_n, y_n) + p(y_n, y)] \\ &= \beta(\alpha + 1)\, p(y, y_n) + h(y_n). \end{aligned}$$

This gives that,

$$h(y) \le \liminf_{n\to\infty} h(y_n).$$

Thus, h is a lower semicontinuous function on a 0-complete partial metric space (X,p) which is also bounded below. Therefore, Theorem 4.6 ensures the existence of a p-point u(say) for h.

We now show that $fx \ne x$ implies $fx \ll x$.

Let $fx \ne x$. By using condition (4.12), we obtain

$$\begin{aligned} h(x) - h(fx) &= \beta\,[p(fx, x) - p(f^2x, fx)] \\ &\ge \beta\,[p(fx, x) - \alpha\, p(fx, x)] \\ &= \beta\,(1-\alpha)\, p(fx, x)] \\ &= p(fx, x). \end{aligned}$$

Thus f satisfies the condition that $fx \ne x$ implies $fx \ll x$. By applying Remark 4.11, it follows that the p-point u for h is a fixed point for f in X.

For uniqueness, let $v \in X$ be another fixed point of f. Then, by condition (4.12), we get

$$p(u,v) = p(fu, fv) \le \alpha\, p(u,v).$$

Since $0 \leq \alpha < 1$, it follows that $p(u, v) = 0$ and hence $u = v$.

Moreover, $p(u, u) = p(fu, fu) \leq \alpha\, p(u, u)$ gives that $p(u, u) = 0$. □

Remark 4.13. *In view of Lemma 2.9 and Example 2.10, it follows that every complete partial metric space is* 0*-complete but the converse may not hold, in general. Thus the results of this section are obtained under the weaker assumption that the given partial metric space is* 0*-complete.*

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