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A UNIFIED FORM OF SEPARATION AXIOMS IN IDEAL TOPOLOGICAL SPACES

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Abstract. We introduce the notion of $mIO(X)$-structures determined by operators Int, Cl and Cl* on an ideal topological space $(X, \tau, \mathcal{I})$. By using $mIO(X)$-structures, we obtain a unified form of some separation axioms containing semi-I-T_i, β-I-T_i, b-I-T_i ($i = 0$, 1, 2) and other.

1. Introduction

In 1975, Maheshwari and Prasad [15] introduced new separation axioms semi-T_0, semi-T_1 and semi-T_2. In 1990, Kar and Bhattacharyya [13] introduced new separation axioms pre-T_0, pre-T_1 and pre-T_2. Tong [27] introduced the notion of D-sets and used these sets to introduce a separation axioms D_1 which is strictly between T_0 and T_1. Borsan [2] and Caldas [3] introduced the notions of s-D-sets and a separation axiom s-D_1 which is strictly between semi-T_0 and semi-T_1. Caldas [4] and Jafari [11] introduced independently the notions of p-D-sets and separation axioms p-D_1.

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In [23] and [24], the present authors introduced the notions of minimal structures, m-spaces, m-continuous functions and M-continuous functions. In [19], the authors introduced and studied the notions of m-T_i spaces and m-D_i spaces ($i = 0, 1, 2$) generalizing the notions of T_i, sT_i, p-T_i, D_i-spaces ($i = 0, 1, 2$).

The notion of ideal topological spaces were introduced in [14] and [28]. By using b-I-open sets, β-I-open sets and semi-I-open sets, some separation axioms in ideal topological spaces are introduced in [1], [7] and [26], respectively, and other papers.

In this paper, by using some results in [19], [20] and [21], we unify and generalize several results obtained in [1], [7], [26] and [17].

2. Preliminaries

Definition 2.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [23], [24] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (briefly m-closed).

Definition 2.2. Let (X, m_X) be an m-space. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [16] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Lemma 2.1. [16] *Let (X, m_X) be an m-space. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$*,*

(2) If $(X \setminus A) \in m_X$, then $\mathrm{mCl}(A) = A$ *and if $A \in m_X$, then* $\mathrm{mInt}(A) = A$*,*

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$*,*

(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$*,*

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$*.*

Definition 2.3. An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [16] if the union of any family of subsets belonging to m_X belongs to m_X.

Lemma 2.2. [25] *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,
(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 2.4. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* at $x \in X$ [23] if for each m_Y-open set V containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be *M-continuous* if it has this property at each $x \in X$.

Theorem 2.1. [23] *For a function $f : (X, m_X) \to (Y, m_Y)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*
(1) f is M-continuous;
(2) $f^{-1}(V)$ is m_X-open for every m_Y-open set V of Y;
(3) $f^{-1}(F)$ is m_X-closed for every m_Y-closed set F of Y;

Definition 2.5. An m-space (X, m_X) is said to be
(1) m-T_0 [19] if for any pair of distinct points x, y of X, there exists an m_X-open set containing x but not y or an m_X-open set containing y but not x,
(2) m-T_1 [19] if for any pair of distinct points x, y of X, there exist an m_X-open set containing x but not y and an m_X-open set containing y but not x,
(3) m-T_2 [23] if for any pair of distinct points x, y of X, there exist m_X-open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 2.6. A subset A of an m-space (X, m_X) is called an *m-D-set* [19] if there exist two m_X-open sets U and V such that $U \neq X$ and $A = U - V$.

Remark 2.1. Every m_X-open set different from X is an m-D-set since we can take as follows $A = U$ and $V = \emptyset$.

Definition 2.7. An m-space (X, m_X) is said to be
(1) m-D_0 [19] if for any distinct points $x, y \in X$, there exists an m-D-set of X containing x but not y or an m-D-set of X containing y but not x,
(2) m-D_1 [19] if for any distinct points $x, y \in X$, there exist an m-D-set of X containing x but not y and an m-D-set of X containing y but not x,
(3) m-D_2 [19] if for any distinct points $x, y \in X$, there exist m-D-sets U, V of X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Remark 2.2. By Definitions 2.5 and 2.7, we have the following diagram [19]:

$$\begin{array}{ccccc} m\text{-}T_2 & \Rightarrow & m\text{-}T_1 & \Rightarrow & m\text{-}T_0 \\ \Downarrow & & \Downarrow & & \Downarrow \\ m\text{-}D_2 & \Rightarrow & m\text{-}D_1 & \Rightarrow & m\text{-}D_0 \end{array}$$

Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. For any subset A of X, a set $\Lambda_m(A)$ is defined in [5] as follows: $\Lambda_m(A) = \cap\{U : A \subset U \in m_X\}$. A subset A is called a Λ_m-*set* if $A = \Lambda_m(A)$. It is shown in Theorem 3.1 of [5] that the pair (X, Λ_m) is an Alexandorff (topological) space. A subset A of X is said to be Λ_m-*closed* [5] if $A = U \cap F$, where U is a Λ_m-set and F is an m-closed set of (X, m_X).

3. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals has been introduced in [14] and [28] and further investigated in [12]

Definition 3.1. A nonempty collection I of subsets of a set X is called an *ideal on* X if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [12]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 3.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ implies $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ and $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 3.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [10] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [10] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [6] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(4) *b-I-open* [18] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,

(5) *β-I-open* [10] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [8] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [18] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [9] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

Among the sets defined in Definition 3.2, we have the following relation:

DIAGRAM 1

$$\begin{array}{ccccccc}
\text{open} & \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} \\
 & & & & \Downarrow & & \Uparrow \\
 & & & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open}
\end{array}$$

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\mathrm{IO}(X)$ (resp. $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Definition 3.3. By $mIO(X)$, we denote each one of the families $\tau^\star$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$.

Lemma 3.2. [22] *Let (X, τ, I) be an ideal topological space. Then $mIO(X)$ is a minimal structure and has property $\mathcal{B}$.*

Definition 3.4. Let (X, τ, I) be an ideal topological space. For a subset A of X, $\mathrm{mCl_I}(A)$ and $\mathrm{mInt_I}(A)$ as follows:
(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in mIO(\mathrm{X})\}$,
(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in mIO(\mathrm{X})\}$.

Let (X, τ, I) be an ideal topological space and $\mathrm{mIO}(X)$ the m_X-structure on X. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then we have

(1) $\mathrm{mCl_I}(A) = \tau^\star\mathrm{Cl_I}(A)$ (resp. $\alpha\mathrm{Cl_I}(A)$, $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$),

(2) $\mathrm{mInt_I}(A) = \tau^\star\mathrm{Int_I}(A)$ (resp. $\alpha\mathrm{Int_I}(A)$, $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$).

4. Separation axioms in ideal topological spaces

Definition 4.1. An ideal topological space (X, τ, I) is said to be *m-I-T_0* (resp. *m-I-T_1*, *m-I-T_2*) if $(X, mIO(X))$ is *m-T_0* (resp. *m-T_1*, *m-T_2*).

Then, for example, for m-I-T_0 we have the following: an ideal topological space (X, τ, I) is m-I-T_0 if for any pair of distinct points x, y of X, there exists an m-I-open set containing x but not y or an m-I-open set containing y but not x. Let $mIO(X)$ be $\beta\mathrm{IO}(X)$ (resp. $\mathrm{BIO}(X)$), then for m-I-T_0 we have the following definition:

Definition 4.2. An ideal topological space (X, τ, I) is said to be β-I-T_0 [7] (resp. b-I-T_0 [1]) if for any pair of distinct points x, y of X, there exists a β-I-open (resp. b-I-open) set containing x but not y or an m_X-open set containing y but not x,

Lemma 4.1. [5], [19] *For an m-space (X, m_X), the following properties are equivalent:*
(1) (X, m) is m-T_0;
(2) $\mathrm{mCl}(\{x\}) \neq \mathrm{mCl}(\{y\})$ for any pair of distinct points $x, y \in X$;
(3) The singleton $\{x\}$ is (Λ, m)-closed for each $x \in X$;
(4) (X, Λ_m) is T_0.

Theorem 4.1. *For an ideal topological space (X, τ, I), the following properties are equivalent:*
(1) (X, τ, I) is m-I-T_0;
(2) $\mathrm{mCl_I}(\{x\}) \neq \mathrm{mCl_I}(\{y\})$ for any pair of distinct points $x, y \in X$;
(3) The singleton $\{x\}$ is (Λ, mI)-closed for each $x \in X$;
(4) (X, Λ_{mI}) is T_0.

Remark 4.1. If $\mathrm{mIO}(X) = \beta\mathrm{IO}(X)$ (resp. $\mathrm{BIO}(X)$, $\tau^\star$), by Theorem 4.1 we obtain Theorem 3.2 of [7] (resp. Theorem 5 of [1], Theorem 3.3 of [17]).

Corollary 4.1. *For an ideal topological space (X, τ, I), the following properties are equivalent:*
(1) (X, τ, I) is pre-I-T_0;
(2) $\mathrm{pCl_I}(\{x\}) \neq \mathrm{pCl_I}(\{y\})$ for any pair of distinct points $x, y \in X$;
(3) The singleton $\{x\}$ is (Λ, pI)-closed for each $x \in X$;
(4) (X, Λ_{pI}) is T_0.

Lemma 4.2. [5], [19] *Let (X, m_X) be an m-space and m_X have property $(\mathcal{B})$. Then the following properties are equivalent:*
(1) (X, m_X) is m-T_1;
(2) for each points $x \in X$, the singleton $\{x\}$ is m-closed;
(3) The singleton $\{x\}$ is a Λ_m-set for each $x \in X$;
(3) (X, Λ_m) is discrete.

Theorem 4.2. *For an ideal topological space (X, τ, I), the following properties are equivalent:*

(1) (X, τ, I) *is* m-*I*-T_1;
(2) for each points $x \in X$, *the singleton* $\{x\}$ *is* m-*I-closed;*
(3) The singleton $\{x\}$ *is a* Λ_{mI}*-set for each* $x \in X$;
(3) (X, Λ_{mI}) *is discrete.*

Proof. The proof follows from Lemma 4.2 and the fact that $\mathrm{mIO}(X)$ has property $\mathcal{B}$.

Corollary 4.2. *For an ideal topological space* (X, τ, I), *the following properties are equivalent:*
(1) (X, τ, I) *is pre-I-*T_1;
(2) for each points $x \in X$, *the singleton* $\{x\}$ *is pre-I-closed;*
(3) The singleton $\{x\}$ *is a* Λ_{pI}*-set for each* $x \in X$;
(4) (X, Λ_{pI}) *is discete.*

Lemma 4.3. [20] *Let* (X, m_X) *be an m-space and* m_X *have property* $\mathcal{B}$. *Then the following properties are equivalent:*
(1) (X, m_X) *is m-*T_2;
(2) For any distinct points $x, y \in X$, *there exists* $U \in m_X$ *containing* x *such that* $y \notin \mathrm{mCl}(U)$;
(3) For each point $x \in X$, $\{x\} = \cap\{\mathrm{mCl}(U) : x \in U \in m_X\}$;
(4) For each pair of distinct points $x, y \in X$, *there exists an M-continuous function* f *of* (X, m_X) *into an m-*T_2*-space* (Y, m_Y) *such that* $f(x) \neq f(y)$.

A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be *M-I-continuous* if $f : (X, MIO(X)) \to (Y, mJO(Y))$ is M-continuous.

Theorem 4.3. *For an ideal topological space* (X, τ, I), *the following properties are equivalent:*
(1) (X, τ, I) *is m-I-*T_2;
(2) For any distinct points $x, y \in X$, *there exists* $U \in mIO(X)$ *containing* x *such that* $y \notin \mathrm{mCl_I}(U)$;
(3) For each point $x \in X$, $\{x\} = \cap\{\mathrm{mCl_I}(U) : x \in U \in mIO(X)\}$;
(4) For each pair of distinct points $x, y \in X$, *there exists an M-I-continuous function* f *of* (X, τ, I) *into an m-I-*T_2*-space* (Y, σ, J) *such that* $f(x) \neq f(y)$.

For example, if $\mathrm{mIO}(X) = \mathrm{PIO}(X)$, by Theorem 4.3 we obtain the following corollary.

Corollary 4.3. *For an ideal topological space* (X, τ, I), *the following properties are equivalent:*
(1) (X, τ, I) *is pre-I-*T_2;

(2) For any distinct points $x, y \in X$, there exists $U \in PIO(X)$ containing x such that $y \notin \mathrm{pCl_I}(U)$;

(3) For each point $x \in X$, $\{x\} = \cap\{\mathrm{pCl_I}(U) : x \in U \in PIO(X)\}$;

(4) For each pair of distinct points $x, y \in X$, there exists a pre-I-continuous function f of (X, τ, I) into an pre-I-T_2-space (Y, σ, J) such that $f(x) \neq f(y)$.

Definition 4.3. An m-space (X, m_X) is said to be *m-regular* [25] if for each m-closed set F and for each point $x \notin F$, there exist disjoint m-open sets U and V such that $x \in U$ and $F \subset V$.

Definition 4.4. An ideal topological space (X, τ, I) is said to be *m-I-regular* if for each m-I-closed set F and for each point $x \notin F$, there exist disjoint m-I-open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 4.4. [21] *If an m-space (X, m_X) is m-T_0 and m-regular, then it is m-T_2.*

Theorem 4.4. *If an ideal topological space (X, τ, I) is m-I-T_0 and m-I-regular, then it is m-I-T_2.*

Proof. The proof is obvious by Lemma 4.4.

Remark 4.2. If $\mathrm{mIO}(X) = \beta\mathrm{IO}(X)$ (resp. $\mathrm{BIO}(X)$), by Theorem 4.4 we obtain Theorem 5.8 of [7] (resp. Theorem 27 of [1]).

For example, if $\mathrm{mIO}(X) = \mathrm{PIO}(X)$, by Theorem 4.4 we obtain the following corollary.

Corollary 4.4. *If an ideal topological space (X, τ, I) is pre-I-T_0 and pre-I-regular, then it is pre-I-T_2.*

Definition 4.5. An m-space (X, m_X) is said to be *m-symmetric* [19] if for each point $x, y \in X$, $x \in \mathrm{mCl}(\{y\})$ implies $y \in \mathrm{mCl}(\{x\})$.

Definition 4.6. An ideal topological space (X, τ, I) is said to be *m-I-symmetric* if for each point $x, y \in X$, $x \in \mathrm{mCl_I}(\{y\})$ implies $y \in \mathrm{mCl_I}(\{x\})$.

Lemma 4.5. [19] *Let (X, m_X) be an m-space, where m_X has property $\mathcal{B}$. Then the following properties are equivalent:*

(1) (X, m_X) is m-symmetric and m-T_0;

(2) (X, m_X) is m-T_1.

Theorem 4.5. *For an ideal topological space (X, τ, I), the following properties are equivalent:*

(1) (X, τ, I) is m-I-symmetric and m-I-T_0;

(2) (X, τ, I) is m-I-T_1.

Proof. The proof follows from Lemma 4.5 and the fact that mIO(X) has property $\mathcal{B}$.

For example, if mIO(X) = PIO(X), by Theorem 4.5 we obtain the following corollary.

Corollary 4.5. *For an ideal topological space (X, τ, I), the following properties are equivalent:*
(1) (X, τ, I) is pre-I-symmetric and pre-I-T_0;
(2) (X, τ, I) is pre-I-T_1.

Lemma 4.6. [20] *Let $f : (X, m_X) \to (Y, m_Y)$ be an injective M-continuous function and m_X have property $\mathcal{B}$. If (Y, m_Y) is m-T_i, then (X, m_X) is m-T_i for i = 0, 1, 2.*

Theorem 4.6. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an injective M-I-continuous function. If (Y, σ, J) is m-J-T_i, then (X, τ, I) is m-I-T_i for i = 0, 1, 2.*

Proof. The proof follows from Lemma 4.6 and the fact that mIO(X) has property $\mathcal{B}$.

Remark 4.3. (1) If mIO(X) = βIO(X) and mJO(Y) = βJO(Y), by Theorem 4.6 we obtain Theorem 3.10 for $i = 0$ and Theorem 4.9 for $i = 1$ in [7].

(2) If mIO(X) = BIO(X) and mJO(Y) = BJO(Y), by Theorem 4.6 for $i = 1$ we obtain Theorem 20 of [1].

For example, if mIO(X) = PIO(X), by Theorem 4.6 we obtain the following corollary.

Corollary 4.6. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an injective pre-I-continuous function. If (Y, σ, J) is pre-J-T_i, then (X, τ, I) is pre-I-T_i for i = 0, 1, 2.*

Definition 4.7. A subset A of an ideal topological space (X, τ, I) is called an *m-I-D-set* if there exist two m-I-open sets U and V such that $U \neq X$ and $A = U - V$.

Definition 4.8. An ideal topological space (X, τ, I) is said to be *m-I-D_0* (resp. *m-I-D_1*, *m-I-D_2*) if $(X, mIO(X))$ is *m-D_0* (resp. *m-D_1*, *m-D_2*).

Lemma 4.7. [19] *An m-space (X, m_X) m-D_0 if and only if it is m-T_0.*

Theorem 4.7. *An ideal topological space (X, τ, I) is m-I-D_0 if and only if it is m-I-T_0.*

Remark 4.4. If mIO$(X) = \tau^\star$, by Theorem 4.7 we obtain Theorem 3.1 of [17].

Lemma 4.8. [19]. *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. Then (X, m_X) is m-D_1 if and only if it is m-D_2.*

Theorem 4.8. *An ideal topological space (X, τ, I) is m-I-D_1 if and only if it is m-I-D_2.*

Proof. The proof follows from Lemma 4.8 and the fact that mIO(X) has property $\mathcal{B}$.

Remark 4.5. If mIO$(X) = \tau^\star$, by Theorem 4.8 we obtain Theorem 3.2 of [17].

Definition 4.9. Let (X, m_X) be an m-space. A point $x \in X$ is called an *m-cc-point* if X is the unique m-open set that contains x.

Definition 4.10. Let (X, τ, I) be an ideal topological space. A point $x \in X$ is called an *mI-cc-point* if X is the unique mI-open set that contains x.

Lemma 4.9. [19]. *An m-T_0 m-space (X, m_X) is m-D_1 if and only if it does not have any mcc-point.*

Theorem 4.9. *An m-I-T_0 ideal topological space (X, τ, I) is m-I-D_1 if and only if it does not have any mI-cc-point.*

Lemma 4.10. [19] *Let $f : (X, m_X) \to (Y, m_Y)$ be an M-continuous bijection and m_X have property $\mathcal{B}$. If $(Y, m(Y))$ is m-D_1, then (X, m_X) is m-D_1.*

Theorem 4.10. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an M-I-continusou bijection. If (Y, σ, J) is m-J-D_1, then (X, τ, I) is m-I-D_1.*

Proof. The proof follows from Lemma 4.10 and the fact that mIO(X) has property $\mathcal{B}$.

Lemma 4.11. [19] *An m-space (X, m_X), where m_X has property $\mathcal{B}$, is m-D_1 if and only if, for each pair of distinct points x, y of X, there exists an M-continuous surjection f of (X, m_X) onto an m-D_1 m-space (Y, m_Y) such that $f(x) \neq f(y)$.*

Theorem 4.11. *An ideal topological space (X, τ, I) is m-I-D_1 if and only if, for each pair of distinct points x, y of X, there exists an M-I-continuous surjection f of (X, τ, I) onto an m-J-D_1 space (Y, σ, J) such that $f(x) \neq f(y)$.*

Proof. The proof follows from Lemma 4.11 and the fact that mIO(X) has property $\mathcal{B}$.

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