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ON SOME WEAKER FORMS OF HUREWICZ PROPERTY IN BITOPOLOGICAL SPACES

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Abstract. We introduce mildly Hurewicz property in the setting of bitopological spaces and study this concept along with that of almost Hurewicz property introduced by A. E. Eysen and S. Özcağ in [4]. Some connections between these properties and Hurewicz property are highlighted. We investigate the preservation of almost Hurewicz property and of mildly Hurewicz property under various types of mappings between bitopological spaces.

1. Introduction

Hurewicz introduced in 1925 [8] a covering property in topological spaces, that now bears his name. A topological space X has the Hurewicz property if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \cup\mathcal{V}_n$ for all but finitely many n. Clearly every Hurewicz space is Lindelöf. As a generalization of Hurewicz spaces, Y. K. Song and R. Li [17] defined a topological space to be almost Hurewicz if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open

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covers of X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{\overline{V} : V \in \mathcal{V}_n\}$ for all but finitely many n. In [10], Lj. D. R. Kočinac defined a mildly Hurewicz space to be a topological space such that for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of clopen covers of X, there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n, n \in \mathbb{N}$ such that each x belongs to $\cup\mathcal{V}_n$ for all but finitely many n.

In the last two decades, many papers on weaker forms of the covering properties like Menger, Hurewicz and Rothberger properties have been published. Recently, H. V. S. Chauhan and B. Singh in [2] mentions that the concept of almost Hurewicz property of a bitopological space has been introduced by A. E. Eysen and S. Öscağ in [4]. In the present paper, we mainly deal with the notions of almost Hurewiczness and mildly Hurewiczness in bitopological settings. In Section 2, we first recall an (i, j)-almost Hurewicz bitopological space and investigate its properties. In section 3, we define an (i, j)-mildly Hurewicz bitopological space and study its properties. Both of these properties are weaker than (i, j)-Hurewicz property. We then seek conditions under which these two properties are equivalent with (i, j)-Hurewicz property, where a bitopological space (X, τ_1, τ_2) is called (i, j)-Hurewicz if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of X by τ_i-open sets, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n, where $i \neq j$ and $i, j \in \{1, 2\}$.

Throughout the paper, (X, τ_1, τ_2) always denotes a bitopological space. For a subset $A \subseteq X$, we denote the closure of A and the interior of A with respect to τ_i by $cl_{\tau_i} A$ and $int_{\tau_i} A$ respectively for $i = 1, 2$. Always $i, j \in \{1, 2\}$ and $i \neq j$.

2. Almost Hurewicz bispaces

In this section we first recall the notion of an (i, j)-almost Hurewicz space and then discuss the relation between an (i, j)-Hurewicz space and an (i, j)-almost Hurewicz space.

Definition 1. [2] *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-almost Hurewicz $(i, j = 1, 2, i \neq j)$ if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of X by τ_i-open sets, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_j} V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

Note 2. Note that if (X, τ_1) is almost Hurewicz and $\tau_2 \leq \tau_1$, then the bitopological space (X, τ_1, τ_2) is $(1, 2)$-almost Hurewicz.

Proposition 2. *Let (X, τ_1, τ_2) be a bitopological space. If (X, τ_1) is Hurewicz, then (X, τ_1, τ_2) is $(1, 2)$-almost Hurewicz.*

Proof. Obvious. ∎

Example 3. *Consider the set $\mathbb{R}$ of real numbers endowed with the Euclidean topology τ_1 and the discrete topology τ_2. Since $(\mathbb{R}, \tau_1)$ is Hurewicz, the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is $(1, 2)$-almost Hurewicz, but the space $(\mathbb{R}, \tau_2)$ is not Hurewicz.*

Example 4. *Consider the Euclidean plane X endowed with the lower limit topology τ_1 and the usual topology τ_2. Then the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is $(1, 2)$-almost Hurewicz, but the space $(\mathbb{R}, \tau_1)$ is not Hurewicz.*

To get an answer to the converse problem of Proposition 3, let us recall the notion of an (i, j)-regular bitopological space.

Definition 5. [15] *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular $(i, j = 1, 2, i \neq j)$ if for each point $x \in X$ and each τ_i-closed set F with $x \notin F$, there exist a τ_i-open set U and a τ_j-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \phi$.*

Proposition 6. [15] *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular if for each point $x \in X$ and each τ_i-open set U with $x \in U$, there exists a τ_i-open set V such that $x \in V \subseteq cl_{\tau_j}(V) \subseteq U$.*

We now have the following theorem.

Theorem 7. [2] *If (X, τ_1, τ_2) is an (i, j)-regular, (i, j)-almost Hurewicz bitopological space, then (X, τ_i) is Hurewicz.*

We next give a characterization of an (i, j)-almost Hurewicz bitopological space in terms of (i, j)-regular open sets. Let us recall the following.

Definition 8. [15] *Let (X, τ_1, τ_2) be a bitopological space. A set $A \subseteq X$ is called (i, j)-regular open ((i, j)-regular closed) $(i \neq j, i, j = 1, 2)$ if $A = int_{\tau_i} cl_{\tau_j}(A)$(resp., $A = cl_{\tau_i} int_{\tau_j}(A)$).*

Theorem 9. [2] *A bitopological space(X, τ_1, τ_2) is (i, j)-almost Hurewicz if and only if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of X by (i, j)-regular open sets, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_j} V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

Next we consider the preservation of (i,j)-almost Hurewicz property under subspaces. Recall that a subset A of a bitopological space (X,τ_1,τ_2) is called an (i,j)-clopen set [13] if $A \in \tau_i \cap co\tau_j$, where $i,j \in \{1,2\}$, $i \neq j$ and $A \in co\tau_j$ implies that A is a τ_j-closed set.

Lemma 10. *Let A be a τ_j-open subset of a bitopological space (X,τ_1,τ_2). Then the subspace $(A,\tau_{1_A},\tau_{2_A})$ is (i,j)-almost Hurewicz if and only if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of A by τ_i-open sets in X, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each $x \in A$, $x \in \cup\{cl_{\tau_j} V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

Proof. We consider only the case $i = 1, j = 2$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of A by τ_1-open sets in X. Let $\{\mathcal{U}^*_n : n \in \mathbb{N}\}$ be the sequence defined by $\mathcal{U}^*_n = \{A \cap U : U \in \mathcal{U}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}^*_n$ is a cover of A by τ_{1_A}-open sets of A. Now by the given condition, there exists a sequence $\{\mathcal{V}^*_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}^*_n$ is a finite subset of $\mathcal{U}^*_n$ and for each $x \in A$, $x \in \cup\{cl_{\tau_{2_A}} V : V \in \mathcal{V}^*_n\}$ for all but finitely many n. For each $n \in \mathbb{N}$ and each $V \in \mathcal{V}^*_n$, there is a set $U_V \in \mathcal{U}_n$ such that $V = A \cap U_V$. Therefore, for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{U_V : V \in \mathcal{V}^*_n\}$ is a finite subset of $\mathcal{U}_n$ and for each $V \in \mathcal{V}^*_n$ we have $cl_{\tau_{2_A}}(V) \subseteq cl_{\tau_2}(U_V)$. Hence for each $x \in A$, $x \in \cup\{cl_{\tau_2}(U_V) : V \in \mathcal{V}^*_n\}$ for all but finitely many n.

Conversely, let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of A by τ_{1_A}-open sets in A. Then for each $n \in \mathbb{N}$ and each $U \in \mathcal{U}_n$, there exists a set $V_U \in \tau_1$ such that $U = A \cap V_U$. For each $n \in \mathbb{N}$, put $\mathcal{V}_n = \{V_U : U \in \mathcal{U}_n\}$. Then $\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a sequence of covers of A by τ_1-open sets in X. By the given condition, there exists a sequence $\{\mathcal{V}'_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $x \in A$, $x \in \cup\{cl_{\tau_2}(V_U) : V_U \in \mathcal{V}'_n\}$ for all but finitely many n. Then for each $n \in \mathbb{N}$, $\mathcal{U}'_n = \{A \cap V_U : V_U \in \mathcal{V}'_n\}$ is a finite subset of $\mathcal{U}_n$. Also for each $U \in \mathcal{U}'_n$, $cl_{\tau_{2_A}}(U) = cl_{\tau_2}(V_U) \cap A$, so that for each $x \in A$, $x \in \cup\{cl_{\tau_{2_A}}(U) : U \in \mathcal{U}'_n\}$ for all but finitely many n. ∎

Theorem 11. *Every τ_i-closed and τ_j-open subset of an (i,j)-almost Hurewicz space is (i,j)-almost Hurewicz.*

Proof. We consider only the case $i = 1, j = 2$. Let F be a τ_1-closed and τ_2-open subset of an $(1,2)$-almost Hurewicz bitopological space (X,τ_1,τ_2). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of F by τ_1-open sets in X. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \mathcal{U}_n \cup \{X \setminus F\}$ is a cover of

X by τ_1-open sets. Since X is $(1,2)$-almost Hurewicz, there exists a sequence $\{\mathcal{V}'_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_2} V : V \in \mathcal{V}'_n\}$ for all but finitely many n. Then for each $n \in \mathbb{N}$, $\mathcal{U}'_n = \{V \in \mathcal{V}'_n : V \neq X \setminus F\}$ is a finite subset of $\mathcal{U}_n$. Thus for each $x \in F$, $x \in \cup\{cl_{\tau_2}(V) : V \in \mathcal{U}'_n\}$ for all but finitely many n (using Lemma 11). Hence F is (i,j)-almost Hurewicz. ∎

Next we study the preservation of an (i,j)-almost Hurewicz space under different types of mappings.

Recall that a mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be
(i) pairwise continuous [9] if both $f : (X, \tau_1) \to (Y, \sigma_1)$ and $f : (X, \tau_2) \to (Y, \sigma_2)$ are continuous.
(ii) (i,j)-almost continuous [12] if $f^{-1}(B)$ is a τ_i-open set in X for every (i,j)-regular open set B in Y. In addition, f is called pairwise almost continuous if it is $(1,2)$ and $(2,1)$-almost continuous.
(iii) (i,j)-θ-continuous [13] if for each σ_i-open set V in Y, there exists a τ_i-open set U in X such that $f(cl_{\tau_i}(U)) \subseteq cl_{\sigma_i}(V)$. f is called pairwise θ-continuous if it is $(1,2)$ and $(2,1)$-θ-continuous.
(iv) (i,j)-contra continuous [16] if $f^{-1}(V) \in \tau_i$, for each $V \in co\tau_j$. f is called pairwise contra continuous if it is $(1,2)$ and $(2,1)$-contra continuous.
(v) (i,j)-precontinuous [11] if $f^{-1}(V) \subseteq int_{\tau_i} cl_{\tau_j}(f^{-1}(V))$, for all σ_i-open set V in Y. f is called pairwise precontinuous if it is $(1,2)$ and $(2,1)$-precontinuous.
(vi) (i,j)-weakly continuous [1] if for each $x \in X$ and each σ_i-open set V of Y containing $f(x)$, there exists a τ_i-open set U containing x such that $f(U) \subseteq cl_{\sigma_j}(V)$. f is called pairwise weakly continuous if it is $(1,2)$ and $(2,1)$-weakly continuous.
(vii) (i,j)-strongly θ-continuous [13] if for each σ_i-open set V in Y, there exists a τ_i-open set U in X such that $f(cl_{\tau_i}(U)) \subseteq V$. f is called pairwise strongly θ-continuous if it is $(1,2)$ and $(2,1)$-strongly θ-continuous.
(viii) (i,j)-almost open if for each σ_i-open set V in Y, $f^{-1}(cl_{\sigma_i}(V)) \subseteq cl_{\tau_i}(f^{-1}(V))$, for $i = 1, 2$. f is called pairwise almost open if it is $(1,2)$ and $(2,1)$-almost open.
(ix) (i,j)-open if for each τ_i-open set U in X, $f(U)$ is σ_i-open in Y, for $i = 1, 2$. f is called pairwise open if it is $(1,2)$ and $(2,1)$-open.
(x) pairwise perfect [3] if f is pairwise continuous, image of every τ_1

(τ_2)-closed set of X is σ_1 (resp., σ_2)-closed in Y, inverse image of every point of Y is compact w.r.t. τ_1 and also w.r.t. τ_2 in X.

Theorem 12. *Let (X, τ_1, τ_2) be an (i, j)-almost Hurewicz bitopological space and let (Y, σ_1, σ_2) be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise almost continuous or a pairwise continuous surjection, then (Y, σ_1, σ_2) is also (i, j)-almost Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. We prove the result for a pairwise continuous surjection, the proof for the almost pairwise continuous map is almost the same (see [2] for details). Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by σ_1-open sets. Let $\{\mathcal{U}'_n : n \in \mathbb{N}\}$ be a sequence defined by $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$, $n \in \mathbb{N}$. Since f is $(1, 2)$-continuous, each $\mathcal{U}'_n$ is a cover of X by τ_1-open sets. Now as X is $(1, 2)$-almost Hurewicz, there exists a sequence $\{\mathcal{V}'_n : n \in \mathbb{N}\}$ of finite families such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n \subseteq \mathcal{U}'_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_2} V : V \in \mathcal{V}'_n\}$ for all but finitely many n. For $V \in \mathcal{V}'_n$, choose $U_V \in \mathcal{U}_n$ such that $V = f^{-1}(U_V)$. Let $\mathcal{V}_n = \{U_V : V \in \mathcal{V}'_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and clearly $\{\mathcal{V}_n : n \in \mathbb{N}\}$ witnesses the (i, j)-almost Hurewiczness of (Y, σ_1, σ_2) for $\{\mathcal{U}_n : n \in \mathbb{N}\}$. ∎

Theorem 13. *Let (X, τ_1, τ_2) be an (i, j)-almost Hurewicz bitopological space and let (Y, σ_1, σ_2) be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise θ-continuous surjection, then (Y, σ_1, σ_2) is also (i, j)-almost Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by σ_1-open sets. Fix $x \in X$. For each $n \in \mathbb{N}$, there is a set $V_{n,x} \in \mathcal{V}_n$ containing $f(x)$. Since f is pairwise θ-continuous, there is a τ_1-open set $U_{n,x}$ in X containing x such that $f(cl_{\tau_1}(U_{n,x})) \subseteq cl_{\sigma_1}(V_{n,x})$. Then for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,x} : x \in X\}$ is a cover of X by τ_1-open sets of X. As X is $(1, 2)$-almost Hurewicz, there is a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{H}_n = \{U_{n,x_i} : i \leq k_n\} \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_2} H : H \in \mathcal{H}_n\}$ for all but finitely many n. For each $U_{n,x_i} \in \mathcal{H}_n$, choose a set $W_{n,x_i} \in \mathcal{V}_n$ such that $f(cl_{\tau_2}(U_{n,x_i})) \subseteq cl_{\sigma_2}(W_{n,x_i})$ and set $\mathcal{W}_n = \{W_{n,x_i} : i \leq k_n\}$. Then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the (i, j)-almost Hurewiczness of (Y, σ_1, σ_2) for $\{\mathcal{V}_n : n \in \mathbb{N}\}$. ∎

Remark 14. *As each pairwise almost continuous mapping is a pairwise θ-continuous mapping, the above theorem extends and generalizes Theorem 13.*

Theorem 15. *Let (X, τ_1, τ_2) be an (i,j)-almost Hurewicz bitopological space and let (Y, σ_1, σ_2) be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise strongly θ-continuous surjection, then (Y, σ_1, σ_2) is also (i,j)-Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by σ_1-open sets. Fix $x \in X$. For each $n \in \mathbb{N}$, there is a set $V_{n,x} \in \mathcal{V}_n$ containing $f(x)$. Since f is pairwise strongly θ-continuous, there is a τ_1-open set $U_{n,x}$ in X containing x such that $f(cl_{\tau_1}(U_{n,x})) \subseteq V_{n,x}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,x} : x \in X\}$ is a cover of X by τ_1-open sets. As X is $(1,2)$-almost Hurewicz, there is a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{H}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_2} H : H \in \mathcal{H}_n\}$ for all but finitely many n. For each $H \in \mathcal{H}_n$, choose a set $W_H \in \mathcal{V}_n$ such that $f(cl_{\tau_2}(H)) \subseteq W_H$ and set $\mathcal{W}_n = \{W_H : H \in \mathcal{H}_n\}$. Then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the (i,j)-Hurewiczness of (Y, σ_1, σ_2) for $\{\mathcal{V}_n : n \in \mathbb{N}\}$. ∎

Theorem 16. *Let (X, τ_1, τ_2) be an (i,j)-almost Hurewicz bitopological space and let (Y, σ_1, σ_2) be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise contra continuous, pairwise precontinuous surjection, then (Y, σ_1, σ_2) is also (i,j)-Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by σ_1-open sets. Since f is pairwise contra continuous, for each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$, $f^{-1}(V)$ is τ_2-closed in X. As f is pairwise precontinuous, $f^{-1}(V) \subseteq int_{\tau_1} cl_{\tau_2}(f^{-1}(V))$ so that $f^{-1}(V) = int_{\tau_1}(f^{-1}(V))$. Then for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a cover of X by τ_1-open sets. As X is $(1,2)$-almost Hurewicz, there is a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{H}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl_{\tau_2} H : H \in \mathcal{H}_n\}$ for all but finitely many n. Then $\mathcal{W}_n = \{f(H) : H \in \mathcal{H}_n\}$ is a finite subset of $\mathcal{V}_n$ and $\{\mathcal{W}_n : n \in \mathbb{N}\}$ witnesses the (i,j)-Hurewiczness of (Y, σ_1, σ_2) for $\{\mathcal{V}_n : n \in \mathbb{N}\}$. ∎

Theorem 17. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise almost open, pairwise perfect and pairwise continuous mapping and (Y, σ_1, σ_2) be (i,j)-almost Hurewicz. Then (X, τ_1, τ_2) is also (i,j)-almost Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of X by τ_1-open sets. Then for each $y \in Y$ and each $n \in \mathbb{N}$, there is a finite subfamily $\mathcal{U}_{n_y}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subseteq \mathcal{U}_{n_y}$.

Let $U_{n_y} = \cup \mathcal{U}_{n_y}$. Then $V_{n_y} = Y \backslash f(X \backslash U_{n_y})$ is a σ_1-open set containing y. Then for each $n \in \mathbb{N}$, $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$ is a cover of Y by σ_1-open sets. As (Y, σ_1, σ_2) is (i,j)-almost Hurewicz, there is a sequence $\{\mathcal{V}'_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in Y$, $y \in \cup\{cl_{\sigma_2}(V) : V \in \mathcal{V}'_n\}$ for all but finitely many n. Without loss of generality, we may assume that $\mathcal{V}'_n = \{V_{n_{y_i}} : i \leq n'\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\mathcal{U}'_n = \cup_{i \leq n'} \mathcal{U}_{n y_i}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Hence the sequence $\{\mathcal{U}'_n : n \in \mathbb{N}\}$ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ which shows that (X, τ_1, τ_2) is also (i,j)-almost Hurewicz. In fact, let $x \in X$, $f(x) \in \cup\{cl_{\sigma_2}(V_{n y_i}) : i \leq n'\}$ for all but finitely many n. For $n \in \mathbb{N}$, if $f(x) \in \cup\{cl_{\sigma_2}(V_{n y_i}) : i \leq n'\}$, then there exists some $i \leq n'$ such that $f(x) \in cl_{\sigma_2}(V_{n y_i})$. Hence $x \in f^{-1}(f(x)) \in f^{-1}(cl_{\sigma_2}(V_{n y_i})) \subseteq cl_{\tau_2}(f^{-1}(V_{n y_i})) \subseteq cl_{\tau_2}(U_{n y_i}) \subseteq cl_{\tau_2}(\cup \mathcal{U}_{n y_i})$. Therefore $x \in \cup\{cl_{\tau_2}(U) : U \in \mathcal{U}'_n\}$ for all but finitely many n, which completes the proof . ∎

3. Mildly Hurewicz bispaces

We define and study in this section, a version of the classical Hurewicz covering property in a bitopological space (X, τ_i, τ_2) by using covers by sets which are both τ_i-open and τ_j-closed (or, simply called (i,j)-clopen), for $i \neq j, i, j = 1, 2$. We call this property (i,j)-mildly Hurewicz.

Definition 18. *A bitopological space (X, τ_1, τ_2) is said to be (i,j)-mildly Hurewicz ($i, j = 1, 2, i \neq j$) if for each sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of covers of X by (i,j)-clopen sets, there exists a sequence $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of finite families such that for each n, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup \mathcal{V}_n$ for all but finitely many n.*

Note 20. Note that if (X, τ_1) is Hurewicz, then the bitopological space (X, τ_1, τ_2) is $(1,2)$-mildly Hurewicz.

Example 19. *Consider the set $\mathbb{R}$ of real numbers endowed with the cofinite topology τ_1 and the usual topology τ_2. Since $(\mathbb{R}, \tau_1)$ is Hurewicz, the bitopological space $(\mathbb{R}, \tau_1, \tau_2)$ is $(1,2)$-mildly Hurewicz, but the space $(\mathbb{R}, \tau_2)$ is not Hurewicz.*

It is also evident that each (i,j)-clopen subset of an (i,j)-mildly Hurewicz space is also (i,j)-mildly Hurewicz, and that any pairwise continuous image of an (i,j)-mildly Hurewicz space is (i,j)-mildly Hurewicz. Also every (i,j)-Hurewicz space is (i,j)-mildly Hurewicz.

We next discuss the behaviour of the (i, j)-mildly Hurewicz property under some classes of mappings.

Theorem 20. *A pairwise contra-continuous, pairwise precontinuous image $Y = f(X)$ of an (i, j)-mildly Hurewicz space (X, τ_1, τ_2) is (i, j)-Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a cover of Y by $(1, 2)$-clopen sets. Since f is pairwise contra continuous, for each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$, $f^{-1}(V) \in co\tau_2$. As f is pairwise precontinuous, $f^{-1}(V) \subseteq int_{\tau_1} cl_{\tau_2} f^{-1}(V) = int_{\tau_1} f^{-1}(V)$, which implies that $f^{-1}(V)$ is τ_1-open, so that $f^{-1}(V)$ is $(1, 2)$-clopen. Hence for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a cover of X by $(1, 2)$-clopen sets. As X is $(1, 2)$-mildly Hurewicz, there exists a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in \cup\mathcal{G}_n$, for all but finitely many n. Let $\mathcal{W}_n = \{f(G) : G \in \mathcal{G}_n\}$ for $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $y = f(x) \in Y$. As $x \in \cup\mathcal{G}_n$ for all but finitely many n, $y \in \cup\mathcal{W}_n$ for all but finitely many n. Hence Y is $(1, 2)$-Hurewicz. ∎

Theorem 21. *Let (X, τ_1, τ_2) be an (i, j)-Hurewicz bitopological space and let (Y, σ_1, σ_2) be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise weakly continuous surjection, then (Y, σ_1, σ_2) is (i, j)-mildly Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{V}_n : n \in \mathbb{N}\}$ be a sequence of covers of Y by $(1, 2)$-clopen sets. Let $x \in X$. Then for each $n \in \mathbb{N}$, there is a $V_{n,x} \in \mathcal{V}_n$ such that $f(x) \in V_{n,x}$. As f is weakly continuous, there is a τ_1-open set $U_{n,x}$ in X containing x such that $f(U_{n,x}) \subseteq cl_{\sigma_2}(V_{n,x}) = V_{n,x}$. Then for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,x} : x \in X\}$ is a cover of X by τ_1-open sets. As X is $(1, 2)$-Hurewicz, there is a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{H}_n \subseteq \mathcal{U}_n$ and for each $x \in X$, $x \in \cup \mathcal{H}_n$ for all but finitely many n. Let $\mathcal{W}_n = \{V_{n,x} : f(H) \subset V_{n,x}, H \in \mathcal{H}_n\}$. Then for each $n \in \mathbb{N}$, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$. Let $y = f(x) \in Y$. As $x \in \cup \mathcal{H}_n$ for all but finitely many n, $y \in \cup\mathcal{W}_n$ for all but finitely many n. Hence Y is (i, j)-mildly Hurewicz. ∎

Definition 22. [14] *A bitopological space (X, τ_1, τ_2) is said to be τ_1-zero dimensional w. r. t. τ_2 if τ_1 has a base of τ_2-closed sets, i.e. if for each point $x \in X$ and each τ_1-open set U containing x, there exists a τ_2-closed set G such that $x \in G \subseteq U$.*

(X, τ_1, τ_2) is said to be pairwise zero dimensional if X is τ_1-zero dimensional w. r. t. τ_2 as well as τ_2-zero dimensional w. r. t. τ_1.

We next recall two notions.

Definition 23. *(i) A cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called pairwise open, if $\mathcal{U} \subset \tau_1 \cup \tau_2$ and if furthermore $\mathcal{U}$ is contained in a nonempty member of τ_1 as well as in τ_2. X is called pairwise compact* [7] *if every pairwise open cover has a finite subcover.*
(ii) A bitopological space (X, τ_1, τ_2) is said to be pairwise T_0 [7]*, if for each pair x, y of distinct points of X, there is either a τ_1-open set U such that $x \in U$ and $y \notin U$ or a τ_2-open set V such that $y \in V$ and $x \notin V$.*

Theorem 24. *Let (X, τ_1, τ_2) be a pairwise zero dimensional, pairwise T_0 and pairwise compact bitopological space. Then X is (i, j)-mildly Hurewicz if and only if X is (i, j)-Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of τ_1-open covers of X. As X is τ_1-zero dimensional w. r. t. τ_2, pairwise T_0 and pairwise compact, there exists a cover $\{\mathcal{V}_n : n \in \mathbb{N}\}$ of X consisting of $(1,2)$-clopen basic sets. As X is $(1,2)$-mildly Hurewicz, for each $n \in \mathbb{N}$, there exists a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that $x \in \cup \mathcal{W}_n$ for all but finitely many n. Then $\mathcal{H}_n = \{U_W \in \mathcal{U}_n : W \subset U_W, W \in \mathcal{W}_n\}$, $n \in \mathbb{N}$ is a finite subset of $\mathcal{U}_n$ for $n \in \mathbb{N}$ and $\{\mathcal{H}_n : n \in \mathbb{N}\}$ witnesses for $\{\mathcal{U}_n : n \in \mathbb{N}\}$ so that X is $(1,2)$-Hurewicz. The other part is obvious. ∎

Theorem 25. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise open, pairwise perfect, pairwise continuous mapping and (Y, σ_1, σ_2) be (i, j)-Hurewicz. Then (X, τ_1, τ_2) is (i, j)-mildly Hurewicz.*

Proof. We consider the case $i = 1, j = 2$. Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of covers of X by $(1,2)$-clopen sets. For each $y \in Y$, $f^{-1}(y)$ is compact w.r.t. τ_1 and also w.r.t. τ_2, so that for each $n \in \mathbb{N}$, there exists a finite set $\mathcal{V}_{n,y} \subseteq \mathcal{U}_n$ which covers $f^{-1}(y)$. Let $V_{n,y} = \cup \mathcal{V}_{n,y}$. Since f is $(1,2)$-closed, for each $n \in \mathbb{N}$ and each $y \in Y$, there exists $W_{n,y} \in \sigma_1$ such that $y \in W_{n,y}$ and $f^{-1}(W_{n,y}) \subseteq V_{n,y}$. Set $\mathcal{W}_n = \{W_{n,y} : y \in Y\}$, $n \in \mathbb{N}$. Then $\{\mathcal{W}_n : n \in \mathbb{N}\}$ is a cover of Y by σ_1-open sets. As Y is $(1,2)$-Hurewicz, there exists a sequence $\{\mathcal{H}_n : n \in \mathbb{N}\}$ such that $\mathcal{H}_n$ is a finite subset of $\mathcal{W}_n$ for each $n \in \mathbb{N}$, and for each $y \in Y$, $y \in \cup\, \mathcal{H}_n$ for all but finitely many n. For each $n \in \mathbb{N}$ and each $H \in \mathcal{H}_n$, there exists finite $\mathcal{U}_{n,H} \in \mathcal{U}_n$ with $f^{-1}(H) \subset \cup\, \mathcal{U}_{n,H}$. If $\mathcal{G}_n = \{U \in \mathcal{U}_n : U \in \mathcal{U}_{n,H}, H \in \mathcal{H}_n\}$, then $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$. Let $y = f(x) \in Y$. Then $y \in \cup\, \mathcal{H}_n$ for all $n \geq n_0$, which implies that $x \in f^{-1}(\cup\, \mathcal{H}_n) \subset \cup\, \mathcal{G}_n$ for all $n \geq n_0$. Hence (X, τ_1, τ_2) is $(1,2)$-mildly Hurewicz. ∎

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References

[1] S. Bose and D. Sinha, **Pairwise almost continuous map and weakly continuous map in bitopological spaces**, Bull. Calcutta Math. Soc. 74 (1982), 195–206.

[2] H. V.S. Chauhan and B. Singh, **On almost Hurewicz property in bitopologicl spaces**, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 70(1) (2021), 74-81.

[3] M. C. Datta, **Projective bitopological space II**, J. Austral. Math. Soc. 14 (1972), 119–128.

[4] A. E. Eysen and S. Özçağ, **Weaker forms of the Menger property in bitopological spaces**, Quaestiones Mathematicae Vol. 41 (7) (2018), 877–888.

[5] Irakli Dochviri and Takashi Noiri, **Asymmetric clopen sets in the bitopological spaces**, Italian Journal of Pure and Applied Mathematics 33 (2014), 263-272.

[6] R. Engelking, **General Topology**, PWW, Warszawa, 1977.

[7] P. Fletcher, H.B. Hoyle III, C.W. Patty, **The comparison of topologies**, Duke Math. Journal 36 (1969), 325-331.

[8] W. Hurewicz, ***Ü*ber die Verallgemeinerung des Borelschen Theorems**, Math. Z. 24 (1925), 401-421.

[9] J. C. Kelley, **Bitopological spaces**, Proc. London Math. Soc. 9 (13) (1963), 71-89.

[10] Lj. D. R. Kočinac, **On Mildly Hurewicz Spaces**, Inter. Math. Forum 11 (12) (2016), 573-582.

[11] F. H. Khedr, S. M. Al-Areefi and T. Noiri, **Precontinuity and Semi-precontinuity in bitopological spaces**, Indian J. Pure Appl. Math. 23 (9) (1992), 625-633.

[12] S. N. Maheshwari and R. Prasad, **Semi open sets and semi continuous function in bitopological spaces**, Math. Notes 26 (1977/78), 29-37.

[13] G. Di Maio and T. Noiri, **Remarks on two weak forms of connectedness in bitopological spaces**, Bull. of the Institute of the Mathematics Academia Sinica 16 (2) (1988), 135-146.

[14] I. L. Reilly, **Zero dimensional bitopological spaces**, Indagationes Mathematicae 76 (2) (1973), 127-131.

[15] A.R. Singal and S. P. Arya, **On pairwise almost regular spaces**, Glasnik Math. 26 (6) (1971), 335-343.

[16] P. Singh, **Continuous and Contra Continuous Functions in Bitopological Spaces**, International Journal of Mathematics And its Applications Volume 3 Issue 4–E (2015), 63–66.

[17] Yan-Kui sond and Rui Li, **On Almost Hurewicz Spaces**, Questions and Answers in General Topology 31 (2013), 131-136.

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