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# NEW SUBCLASSES OF ANALYTIC FUNCTIONS RELATED TO QUASI-CONVEX FUNCTIONS

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**Abstract.** This paper deals with the study of certain new subclasses of analytic functions related to quasi-convex functions in the open unit disc $E = \{z : |z| < 1\}$. We establish some geometric properties such as the coefficient estimates, distortion theorems and growth theorems for these classes. The results proved earlier will follow as special cases.

## 1. Introduction

Let $\mathcal{U}$ be the class of Schwarzian functions $w(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc $E = \{z : |z| < 1\}$ such that $w(0) = 0, |w(z)| < 1$.

By $\mathcal{A}$, we denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

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which are analytic in the unit disc $E$ and further normalized specifically by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$, consisting of functions of the form (1) and which are univalent in $E$, is denoted by $\mathcal{S}$.

For the functions $f$ and $g$ analytic in $E$, we say that $f$ is subordinate to $g$ (symbolically $f \prec g$) if a Schwarzian function $w(z) \in \mathcal{U}$ can be found for which $f(z) = g(w(z))$. Under the assumption that $g$ is univalent, we have $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. This is subordination principle.

The well known classes of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$ respectively and defined as

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

The classes $\mathcal{S}^*$ and $\mathcal{K}$ are related by the Alexander relation [1] as $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of close-to-convex functions if there exists a convex function $h$ such that $Re\left(\frac{f'(z)}{h'(z)}\right) > 0$ or equivalently $Re\left(\frac{zf'(z)}{g(z)}\right) > 0$ where $g = zh'$ is starlike. Clearly, $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. The concept of close-to-convex functions was established by Kaplan [3]. Subsequently, Noor [4] introduced the class $\mathcal{C}^*$ of quasi-convex functions as

$$\mathcal{C}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in \mathcal{K}, z \in E \right\}.$$

Every quasi-convex function is convex and close-to-convex and so is univalent. Also $f \in \mathcal{C}^*$ if and only if $zf' \in \mathcal{C}$.

By $\mathcal{C}_s^*$, we denote the subclass of quasi-convex functions defined as

$$\mathcal{C}_s^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0, g \in \mathcal{S}^*, z \in E \right\}.$$

This class was discussed by Singh and Singh [7]. For $-1 \leq B < A \leq 1$, Xiong and Liu [8] and Singh and Singh [7] studied the classes $\mathcal{C}^*(A, B)$

and $\mathcal{C}_s^*(A,B)$ respectively, which are the subclasses of quasi-convex functions defined as:

$$\mathcal{C}^*(A,B) = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{h'(z)} \prec \frac{1+Az}{1+Bz}, h \in \mathcal{K}, z \in E \right\}$$

and

$$\mathcal{C}_s^*(A,B) = \left\{ f : f \in \mathcal{A}, \frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}, g \in \mathcal{S}^*, z \in E \right\}.$$

Some more subclasses of quasi-convex functions were studied by several authors such as the classes $\mathcal{C}^*(\alpha,\beta)$ $(0 \leq \alpha < 1, 0 < \beta \leq 1)$ and $\mathcal{C}_s^*(\alpha,\beta)$ were studied respectively by Selvaraj and Stelin [5] and Selvaraj et al. [6] and more recently, Singh and Singh [7] studied the classes $\mathcal{C}^*(A,B;C,D)$ and $\mathcal{C}_s^*(A,B;C,D)$.

Using the concept of quasi-convex functions, Noor [4] introduced the class $\mathcal{K}_1$ which includes the functions $f \in \mathcal{A}$ such that $Re\left(\frac{f'(z)}{g'(z)}\right) > 0$ where $g \in \mathcal{C}^*$. Obviously, $\mathcal{C}^* \subset \mathcal{C} \subset \mathcal{K}_1$.

Throughout this paper, we assume that

$$-1 \leq D \leq B < A \leq C \leq 1, z \in E.$$

Getting inspired by the above work, we now introduce the following generalized subclasses of analytic functions related to quasi-convex functions:

**Definition 1** Let $\mathcal{K}_1(A,B;C,D)$ be the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\frac{f'(z)}{g'(z)} \prec \frac{1+Cz}{1+Dz},$$

where $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{C}^*(A,B)$.

The following observations are obvious:
(i) $\mathcal{K}_1(1,-1;C,D) \equiv \mathcal{K}_1(C,D)$.
(ii) $\mathcal{K}_1(1,-1;1,-1) \equiv \mathcal{K}_1$.

**Definition 2** Let $\mathcal{K}_1'(A,B;C,D)$ denote the class of functions $f \in \mathcal{A}$ and satisfying the condition

$$\frac{f'(z)}{h'(z)} \prec \frac{1+Cz}{1+Dz},$$

where $h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{C}_s^*(A, B)$.

We have the following observations:
(i) $\mathcal{K}_1'(1, -1; C, D) \equiv \mathcal{K}_1'(C, D)$.
(ii) $\mathcal{K}_1'(1, -1; 1, -1) \equiv \mathcal{K}_1'$.

The paper is concerned with the study of the classes $\mathcal{K}_1(A, B; C, D)$ and $\mathcal{K}_1'(A, B; C, D)$. We obtain the coefficient estimates, distortion theorems and growth theorems for the functions in these classes. The results established earlier by various authors follow as special cases by giving the particular values to the parameters $A, B, C$ and $D$.

## 2. Preliminary Results

**Lemma 1** [2] If $P(z) = \dfrac{1 + Cw(z)}{1 + Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, then

$$|p_n| \leq (C - D), n \geq 1.$$

**Lemma 2** [7] If $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{C}^*(A, B)$, then for $A - (n - 1)B \geq (n - 2), n \geq 3$,

$$|b_n| \leq \frac{1}{n} + \frac{(n-1)(A-B)}{2n}.$$

**Lemma 3** [7] If $g(z) \in \mathcal{C}^*(A, B)$, then for $|z| = r < 1$,
for $B \neq -1$,

$$\frac{A-B}{r(1+B)^2} log\left(\frac{1-Br}{1+r}\right) + \frac{1+A}{(1+B)(1+r)} \leq$$

$$\leq |g'(z)| \leq \frac{A-B}{r(1+B)^2} log\left(\frac{1-r}{1+Br}\right) + \frac{1+A}{(1+B)(1-r)},$$

and for $B = -1$,

$$-\frac{1+A}{2r(1+r)^2} + \frac{A}{r(1+r)} + \frac{1}{2r}(1-A) \leq |g'(z)| \leq \frac{1+A}{2r(1-r)^2} - \frac{A}{r(1-r)} + \frac{1}{2r}(A-1).$$

**Lemma 4** [7] If $h(z) = z + \sum_{k=2}^{\infty} d_k z^k \in \mathcal{C}_s^*(A, B)$, then for $A - (n - 1)B \geq (n - 2), n \geq 3$,

$$|d_n| \leq 1 + \frac{(A-B)(n-1)(2n-1)}{6n}.$$

**Lemma 5** [7] If $h(z) \in \mathcal{C}_s^*(A, B)$, then for $|z| = r < 1$,
for $B \neq -1$, $B \neq 0$,

$L_1 \leq |h'(z)| \leq L_2$, where

$$L_1 = \frac{(B-1)}{r(B+1)^3} log \left| \frac{1+r}{1-Br} \right| + \left[ \frac{(B-1)}{B(B+1)^2} - \frac{A}{B} \right] \frac{1}{1+r} +$$

$$+ \frac{1}{2} \left[ 1 + \frac{A}{B} - \frac{(B-1)}{B(B+1)} \right] \frac{(2+r)}{(1+r)^2},$$

$$L_2 = \frac{(B-1)}{r(B+1)^3} log \left| \frac{1+Br}{1-r} \right| + \left[ \frac{(B-1)}{B(B+1)^2} - \frac{A}{B} \right] \frac{1}{1-r} +$$

$$+ \frac{1}{2} \left[ 1 + \frac{A}{B} - \frac{(B-1)}{B(B+1)} \right] \frac{(2-r)}{(1-r)^2},$$

and for $B = -1$,

$$\frac{A}{1+r} - \frac{(3A+1)(2+r)}{2(1+r)^2} + \frac{2(A+1)(3+3r+r^2)}{3(1+r)^3}$$

$$\leq |h'(z)| \leq \frac{A}{1-r} - \frac{(3A+1)(2-r)}{2(1-r)^2} + \frac{2(A+1)(3-3r+r^2)}{3(1-r)^3}.$$

## 3. The class $\mathcal{K}_1(A, B; C, D)$

**Theorem 1** Let $f(z) \in \mathcal{K}_1(A, B; C, D)$, then for $A - (n-1)B \geq (n-2), n \geq 2$,

(2)

$$|a_n| \leq \frac{1}{n} + \frac{(n-1)(A-B)}{2n} + \frac{(C-D)(n-1)}{n} \left[ 1 + \frac{(A-B)(n-2)}{4} \right].$$

The bounds are sharp.

**Proof.** In Definition 1, using Principle of subordination, we have

$$f'(z) = g'(z) \left( \frac{1+Cw(z)}{1+Dw(z)} \right), w(z) \in \mathcal{U}. \tag{3}$$

On expanding (3), it yields

$1 + 2a_2z + 3a_3z^2 + ... + na_nz^{n-1} + ...$

(4)

$= (1+2b_2z+3b_3z^2+...+nb_nz^{n-1}+...)(1+p_1z+p_2z^2+...+p_{n-1}z^{n-1}+...)$.

Equating the coefficients of $z^{n-1}$ in (4), we have

$$na_n = nb_n + (n-1)p_1b_{n-1} + (n-2)p_2b_{n-2}... + 2p_{n-2}b_2 + p_{n-1}. \tag{5}$$

Applying triangle inequality and Lemma 1 in (5), it gives

(6)

$n|a_n| \leq n|b_n| + (C-D) \left[ (n-1)|b_{n-1}| + (n-2)|b_{n-2}|... + 2|b_2| + 1 \right].$

Using Lemma 2 in (6), the result (2) is obvious.
For $n = 2$, equality sign in (2) holds for the functions $f_n$ defined as
(7)
$$f_n' = \frac{1}{z}\left(\frac{1+C\delta_1 z^n}{1+D\delta_1 z^n}\right)\int_0^z \frac{1}{(1-\delta_2 z)^2}\left(\frac{1+A\delta_3 z^n}{1+B\delta_3 z^n}\right)dz, |\delta_1| = |\delta_2| = |\delta_3| = 1.$$

On putting $A = 1, B = -1$ in Theorem 1, we obtain the following result:

**Corollary 1** Let $f(z) \in \mathcal{K}_1(C, D)$, then
$$|a_n| \leq 1 + \frac{(n-1)(C-D)}{2}.$$

For $A = 1, B = -1, C = 1, D = -1$, Theorem 1 coincides with the following result due to Noor [4].

**Corollary 2** Let $f(z) \in \mathcal{K}_1$, then
$$|a_n| \leq n.$$

**Theorem 2** If $f(z) \in \mathcal{K}_1(A, B; C, D)$, then for $|z| = r, 0 < r < 1$, we have
for $B \neq -1$,
$$\left(\frac{1-Cr}{1-Dr}\right)\left[\frac{A-B}{r(1+B)^2}log\left(\frac{1-Br}{1+r}\right) + \frac{1+A}{(1+B)(1+r)}\right] \leq |f'(z)|$$
$$\leq \left(\frac{1+Cr}{1+Dr}\right)\left[\frac{A-B}{r(1+B)^2}log\left(\frac{1-r}{1+Br}\right) + \frac{1+A}{(1+B)(1-r)}\right]; \tag{8}$$
$$\int_0^r \left(\frac{1-Ct}{1-Dt}\right)\left[\frac{A-B}{t(1+B)^2}log\left(\frac{1-Bt}{1+t}\right) + \frac{1+A}{(1+B)(1+t)}\right]dt \leq |f(z)|$$
$$\leq \int_0^r \left(\frac{1+Ct}{1+Dt}\right)\left[\frac{A-B}{t(1+B)^2}log\left(\frac{1-t}{1+Bt}\right) + \frac{1+A}{(1+B)(1-t)}\right]dt, \tag{9}$$
and for $B = -1$,
$$\left(\frac{1-Cr}{1-Dr}\right)\left[-\frac{1+A}{2r(1+r)^2} + \frac{A}{r(1+r)} + \frac{1}{2r}(1-A)\right] \leq |f'(z)|$$
$$\leq \left(\frac{1+Cr}{1+Dr}\right)\left[\frac{1+A}{2r(1-r)^2} - \frac{A}{r(1-r)} + \frac{1}{2r}(A-1)\right]; \tag{10}$$
$$\int_0^r \left(\frac{1-Ct}{1-Dt}\right)\left[-\frac{1+A}{2t(1+t)^2} + \frac{A}{t(1+t)} + \frac{1}{2t}(1-A)\right]dt \leq |f'(z)|$$
$$\leq \int_0^r \left(\frac{1+Ct}{1+Dt}\right)\left[\frac{1+A}{2t(1-t)^2} - \frac{A}{t(1-t)} + \frac{1}{2t}(A-1)\right]dt. \tag{11}$$

Estimates are sharp.

**Proof.** From (3), we have

$$|f'(z)| = |g'(z)| \left| \frac{1 + Cw(z)}{1 + Dw(z)} \right|, w(z) \in \mathcal{U}. \tag{12}$$

It is easy to show that the transformation

$$\frac{f'(z)}{g'(z)} = \frac{1 + Cw(z)}{1 + Dw(z)}$$

maps $|w(z)| \leq r$ onto the circle

$$\left| \frac{f'(z)}{g'(z)} - \frac{1 - CDr^2}{1 - D^2r^2} \right| \leq \frac{(C - D)r}{(1 - D^2r^2)}, |z| = r.$$

This implies that

$$\frac{1 - Cr}{1 - Dr} \leq \left| \frac{1 + Cw(z)}{1 + Dw(z)} \right| \leq \frac{1 + Cr}{1 + Dr}. \tag{13}$$

As $g(z) \in \mathcal{C}^*(A, B)$, so using Lemma 3 and (13) in (12), the results (8) and (10) can be easily obtained. On integrating (8) and (10) from 0 to $r$, the results (9) and (11) are obvious.

Sharpness follows for the function $f_n$ defined in (7).

On putting $A = 1, B = -1$ in Theorem 2, it gives the following result:

**Corollary 3** Let $f(z) \in \mathcal{K}_1(C, D)$, then

$$\left( \frac{1 - Cr}{1 - Dr} \right) \left[ -\frac{2}{2r(1 + r)^2} + \frac{1}{r(1 + r)} \right] \leq$$

$$\leq |f'(z)| \leq \left( \frac{1 + Cr}{1 + Dr} \right) \left[ \frac{2}{2r(1 - r)^2} - \frac{1}{r(1 - r)} \right]$$

and

$$\int_0^r \left( \frac{1 - Ct}{1 - Dt} \right) \left[ -\frac{2}{2t(1 + t)^2} + \frac{1}{t(1 + t)} \right] dt \leq$$

$$\leq |f'(z)| \leq \int_0^r \left( \frac{1 + Ct}{1 + Dt} \right) \left[ \frac{2}{2t(1 - t)^2} - \frac{1}{t(1 - t)} \right] dt.$$

For $A = 1, B = -1, C = 1, D = -1$, Theorem 2 coincides with the following result due to Noor [4].

**Corollary 4** Let $f(z) \in \mathcal{K}_1$, then

$$\frac{(1 - r)}{(1 + r)^3} \leq |f'(z)| \leq \frac{(1 + r)}{(1 - r)^3}$$

and

$$\frac{r}{(1 + r)^2} \leq |f(z)| \leq \frac{r}{(1 - r)^2}.$$

## 4. The class $\mathcal{K}_1'(A,B;C,D)$

**Theorem 3** Let $f(z) \in \mathcal{K}_1'(A,B;C,D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$|a_n| \leq 1 + \frac{(n-1)(2n-1)(A-B)}{6n} + \frac{(C-D)(n-1)}{2} \left[1 + \frac{(n-2)(4n-3)(A-B)}{18n}\right]. \tag{14}$$

The results are sharp.

**Proof.** From Definition 2, using Lemma 1 and lemma 4 and following the procedure of Theorem 1, the result (14) can be easily derived.

For $n=2$, equality sign in (14) hold for the functions $f_n(z)$ defined by

(15)

$$f_n' = \frac{1}{z}\left(\frac{1+C\delta_1 z^n}{1+D\delta_1 z^n}\right)\int_0^z \frac{1+\delta_2 z}{(1-\delta_2 z)^3}\left(\frac{1+A\delta_3 z^n}{1+B\delta_3 z^n}\right)dz, |\delta_1| = |\delta_2| = |\delta_3| = 1.$$

On putting $A=1, B=-1$ in Theorem 3, it yields the following result:

**Corollary 5** Let $f(z) \in \mathcal{K}_1'(C,D)$, then

$$|a_n| \leq \frac{2n^2+1}{3n} + \frac{(C-D)(n-1)(2n^2-n+3)}{9n}.$$

For $A=1, B=-1, C=1, D=-1$, Theorem 3 gives the following result:

**Corollary 6** Let $f(z) \in \mathcal{K}_1'$, then

$$|a_n| \leq \frac{4n^3+8n-3}{9n}.$$

**Theorem 4** If $f(z) \in \mathcal{K}_1'(A,B;C,D)$, then for $|z|=r, 0<r<1$, we have

for $B \neq -1$, $B \neq 0$,

$$\left(\frac{1-Cr}{1-Dr}\right)L_1 \leq |f'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)L_2;$$

$$\int_0^r \left(\frac{1-Ct}{1-Dt}\right)L_1 dt \leq |f(z)| \leq \int_0^r \left(\frac{1+Ct}{1+Dt}\right)L_2,$$

where

$$L_1 = \frac{(B-1)}{r(B+1)^3} log\left|\frac{1+r}{1-Br}\right| + \left[\frac{(B-1)}{B(B+1)^2} - \frac{A}{B}\right]\frac{1}{1+r} + \\ + \frac{1}{2}\left[1 + \frac{A}{B} - \frac{(B-1)}{B(B+1)}\right]\frac{(2+r)}{(1+r)^2},$$

$$L_2 = \frac{(B-1)}{r(B+1)^3} log\left|\frac{1+Br}{1-r}\right| + \left[\frac{(B-1)}{B(B+1)^2} - \frac{A}{B}\right]\frac{1}{1-r} +$$

$$+\frac{1}{2}\left[1+\frac{A}{B}-\frac{(B-1)}{B(B+1)}\right]\frac{(2-r)}{(1-r)^2},$$

and for $B=-1$,

$$\left(\frac{1-Cr}{1-Dr}\right)\left[\frac{A}{1+r}-\frac{(3A+1)(2+r)}{2(1+r)^2}+\frac{2(A+1)(3+3r+r^2)}{3(1+r)^3}\right]$$

$$\leq |f'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)\left[\frac{A}{1-r}-\frac{(3A+1)(2-r)}{2(1-r)^2}+\frac{2(A+1)(3-3r+r^2)}{3(1-r)^3}\right];$$

$$\int_0^r\left(\frac{1-Ct}{1-Dt}\right)\left[\frac{A}{1+t}-\frac{(3A+1)(2+t)}{2(1+t)^2}+\frac{2(A+1)(3+3t+t^2)}{3(1+t)^3}\right]dt$$

$$\leq |f(z)| \leq \int_0^r\left(\frac{1+Ct}{1+Dt}\right)\left[\frac{A}{1-t}-\frac{(3A+1)(2-t)}{2(1-t)^2}+\frac{2(A+1)(3-3t+t^2)}{3(1-t)^3}\right]dt.$$

Estimates are sharp.

**Proof.** Using Lemma 5 and following the procedure of Theorem 2, the results of Theorem 4 can be easily established.

Sharpness follows for the function $f_n$ defined in (15).

On putting $A=1, B=-1$ in Theorem 4, it gives the following result:

**Corollary 7** Let $f(z)\in\mathcal{K}_1'(C,D)$, then

$$\left(\frac{1-Cr}{1-Dr}\right)\left[\frac{1}{1+r}-\frac{4(2+r)}{2(1+r)^2}+\frac{4(3+3r+r^2)}{3(1+r)^3}\right]$$

$$\leq |f'(z)| \leq \left(\frac{1+Cr}{1+Dr}\right)\left[\frac{1}{1-r}-\frac{4(2-r)}{2(1-r)^2}+\frac{4(3-3r+r^2)}{3(1-r)^3}\right];$$

$$\int_0^r\left(\frac{1-Ct}{1-Dt}\right)\left[\frac{1}{1+t}-\frac{4(2+t)}{2(1+t)^2}+\frac{4(3+3t+t^2)}{3(1+t)^3}\right]dt$$

$$\leq |f(z)| \leq \int_0^r\left(\frac{1+Ct}{1+Dt}\right)\left[\frac{1}{1-t}-\frac{4(2-t)}{2(1-t)^2}+\frac{4(3-3t+t^2)}{3(1-t)^3}\right]dt.$$

For $A=1, B=-1, C=1, D=-1$, Theorem 4 gives the following result:

**Corollary 8** Let $f(z)\in\mathcal{K}_1'$, then

$$\frac{(1-r)(r^2+3)}{3(1+r)^4}\leq |f'(z)| \leq \frac{(1+r)(r^2+3)}{3(1-r)^4}$$

and

$$\int_0^r\frac{(1-t)(t^2+3)}{3(1+t)^4}dt \leq |f(z)| \leq \int_0^r\frac{(1+t)(t^2+3)}{3(1-t)^4}dt.$$

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