"Vasile Alecsandri" University of Bacău
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CHARACTERIZATIONS OF COUNTABLY $\rho\mathcal{I}$-COMPACT IDEAL TOPOLOGICAL SPACES

SUMIT MITTAL, B. K. TYAGI

Abstract. The concept of countably $\rho\mathcal{I}$-compactness is introduced and several characterizations of this notion are obtained. It is shown that an ideal space $(X, \tau, \mathcal{I})$ is countably $\rho\mathcal{I}$-compact if and only if every countable locally finite modulo I, $I \in \mathcal{I}$, family of non-ideal sets is finite.

1. Introduction

The concept of countably compact topological spaces was introduced by Fréchet in 1906. Several characterizations of this notion are well known now. Newcomb [10] generalized the concept of compact and countably compact spaces with respect to an ideal in a topological space and called these spaces compact modulo an ideal or $\mathcal{I}$-compact and countably compact modulo an ideal or countably $\mathcal{I}$-compact respectively. These notions were also further studied by Hamlett, Jancovic and Rose [3, 4]. In this paper, we introduced the concept of countably $\rho\mathcal{I}$-compact spaces using open $\mathcal{I}$-covers which is stronger than the concept of countably $\mathcal{I}$-compact spaces. Then we obtain some characterizations of this notion. Motivated by Bacon [1] we give a sufficient condition for a countably $\rho\mathcal{I}$-compact topological space to be $\rho\mathcal{I}$-compact space.

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This paper is organised as follows. Section 2 contains preliminaries. In Section 3, the notion of countably $\rho\mathcal{I}$-compact spaces is introduced and some basic properties and characterizations are obtained. The behavior of these spaces under certain kind of mappings is also investigated. In Section 4, further characterizations of these spaces are given.

2. Preliminaries

An ideal on a set X is a nonempty collection $\mathcal{I}$ of subsets of X such that

(1) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
(2) if $B \in \mathcal{I}$ and $A \subseteq B \subseteq X$, then $A \in \mathcal{I}$.

If the collection $\mathcal{I}$ is also closed under countable unions, that is "countable additive", then $\mathcal{I}$ is called a σ-ideal.

If X is a set and $B \subseteq X$, then the collection $\mathcal{P}(B) = \{A \subseteq X : A \subseteq B\}$ is an ideal on X. Some other examples are $\mathcal{I}_f = \{A \subseteq X : A$ is finite$\}$ and $\mathcal{I}_c = \{A \subseteq X : A$ is countable$\}$. We denote a topological space (X, τ) with an ideal $\mathcal{I}$ defined on X by $(X, \tau, \mathcal{I})$ and call it an ideal space. If (X, τ) is a topological space then it is clear that the collections $\mathcal{N}(\tau)$ of nowhere dense subsets of X and $\mathcal{M}(\tau)$ of first category (meager) subsets of X are both ideals on X.

A subset A of a space (X, τ) is said to be g-closed [9] if $\overline{A} \subseteq U$ whenever $A \subseteq U$ for every $U \in \tau$. It is clear that every closed set is g-closed.

Given an ideal space $(X, \tau, \mathcal{I})$ and (Y, β) a topological space, for any function $f : X \to Y$, the collection $\mathcal{J} = \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{I}\}$ is an ideal on Y. Moreover if f is a bijection, then the collection $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is also an ideal on Y. We will use these ideals later in Section 3.

Recall that a subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be countably $\mathcal{I}$-compact [4] if for any countable open cover $\{V_\alpha\}_{\alpha\in\omega}$ of A, there exists a finite set $\omega_0 \subset \omega$ such that $A \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$. The ideal space $(X, \tau, \mathcal{I})$ is said to be countably $\mathcal{I}$-compact if X is countably $\mathcal{I}$-compact. Clearly (X, τ) is countably compact if and only if $(X, \tau, \{\emptyset\})$ is countably $\{\emptyset\}$-compact and that if (X, τ) is countably compact then $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact for $\mathcal{I} = \{\emptyset\}$.

In an ideal space $(X, \tau, \mathcal{I})$, a family $\mathcal{U}$ of open subsets of X is said to be an open $\mathcal{I}$-cover if $X \setminus \cup\mathcal{U} \in \mathcal{I}$. Note that every open cover is an open $\mathcal{I}$-cover for every ideal $\mathcal{I}$ on X. Nestor [12] introduced the concept of $\rho\mathcal{I}$-compact spaces.

Definition 1. *An ideal space $(X, \tau, \mathcal{I})$ is said to be $\rho\mathcal{I}$-compact if for any open $\mathcal{I}$-cover $\{U_\alpha\}$, there exists a finite $\mathcal{I}$-subcover, that is, a finite subcollection $\{U_{\alpha_i} : i = 1, 2, \ldots, m\}$ such that $X \setminus \cup_{i=1}^{m} U_{\alpha_i} \in \mathcal{I}$.*

In Section 4, we generalized the notion of $\mathcal{I}$-paracompact spaces to characterize the countably $\rho\mathcal{I}$-compact spaces.

Definition 2. [5] *An ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-paracompact or paracompact with respect to an ideal if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X \setminus \cup\mathcal{V} \in \mathcal{I}$.*

If (X, τ) is a topological space and $A \subseteq X$, then $\overline{A}$ will denote the closure of A in (X, τ).

3. Countably $\rho\mathcal{I}$-compact spaces

Definition 3. *If $(X, \tau, \mathcal{I})$ be an ideal space and $A \subseteq X$, then A is said to be countably $\rho\mathcal{I}$-compact if for any countable family $\{V_\alpha\}_{\alpha\in\omega}$ of open subsets of X such that $A \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$, there exists a finite set $\omega_0 \subset \omega$ such that $A \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$ or in other words, if for any countable open $\mathcal{I}$-cover, there exists a finite $\mathcal{I}$-subcover. The ideal space $(X, \tau, \mathcal{I})$ is said to be countably $\rho\mathcal{I}$-compact if X is countably $\rho\mathcal{I}$-compact.*

Clearly (X, τ) is countably compact if and only if $(X, \tau, \{\emptyset\})$ is countably $\rho\{\emptyset\}$-compact and that if $(X, \tau, \mathcal{I})$ is countably $\rho\mathcal{I}$-compact then $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact.

Example 4. *If $X = [0, +\infty)$, $\tau = \{\emptyset, X\} \cup \{(r, \infty) \mid r \geq 0\}$ and $\mathcal{I} = \mathcal{I}_f$ then the ideal space $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact but not countably $\rho\mathcal{I}$-compact as the open $\mathcal{I}$-cover $\mathcal{U} = \{(1/n, \infty) \mid n \in \mathbb{N}\}$ does not have any finite $\mathcal{I}$-subcover.*

Hamlett [2] introduced the notion of $\mathcal{I}$-Lindelöf ideal topological spaces. An ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-Lindelöf if for any open cover of X, there exists a countable $\mathcal{I}$-subcover. It is known that a countably compact space is compact if and only if it is Lindelöf. We give a more general result in countably $\rho\mathcal{I}$-compact spaces.

Theorem 5. *If an ideal space $(X, \tau, \mathcal{I})$ is countably $\rho\mathcal{I}$-compact and $\mathcal{I}$-Lindelöf, then $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-compact space.*

Proof. Suppose $\{V_\alpha\}_{\alpha\in\lambda}$ be any open cover of X, then X being an $\mathcal{I}$-Lindelöf space, there exists a countable subcover $\{V_\alpha\}_{\alpha\in\omega}$ such that $X \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$ and since X is countably $\rho\mathcal{I}$-compact, there exists a finite set $\omega_0 \subset \omega$ such that $X \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$. Hence X is an $\mathcal{I}$-compact space. □

Levine [9] and Jafari [7] defined the concept of g-closed sets and $\mathcal{I}g$-closed sets, respectively and it is showed that every g-closed set is $\mathcal{I}g$-closed. Nestor [13] introduced the concept of $\rho\mathcal{I}g$-closed subsets.

Definition 6. [13] *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\rho\mathcal{I}g$-closed set if for every $U \in \tau$, $A \setminus U \in \mathcal{I}$ implies $\overline{A} \setminus U \in \mathcal{I}$.*

Note that every g-closed set is a $\rho\mathcal{I}g$-closed set and every $\rho\mathcal{I}g$-closed set is $\mathcal{I}g$-closed set. In [4] it is showed that a closed subspace of a countably $\mathcal{I}$-compact space is countably $\mathcal{I}$-compact. More generally, we have the following result.

Theorem 7. *If an ideal space $(X, \tau, \mathcal{I})$ is countably $\rho\mathcal{I}$-compact and $A \subseteq X$ is $\rho\mathcal{I}g$-closed, then A is countably $\rho\mathcal{I}$-compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in\omega}$ be a countable family of open subsets of X such that $A \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$. Since A is $\rho\mathcal{I}g$-closed, so by definition $\overline{A} \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$. Then $X \setminus \{(X \setminus \overline{A}) \cup (\cup_{\alpha\in\omega} V_\alpha)\} = \overline{A} \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$. Given that X is countably $\rho\mathcal{I}$-compact, there exists a finite set $\omega_0 \subset \omega$ such that $X \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$ or $X \setminus \{(X \setminus \overline{A}) \cup (\cup_{\alpha\in\omega_0} V_\alpha)\} \in \mathcal{I}$.

In any case $\overline{A} \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$ and since $A \setminus \cup_{\alpha\in\omega_0} V_\alpha \subseteq \overline{A} \setminus \cup_{\alpha\in\omega_0} V_\alpha$, we have $A \setminus \cup_{\alpha\in\omega_0} V_\alpha \in \mathcal{I}$ and thus A is countably $\rho\mathcal{I}$-compact. $\square$

Theorem 8. *If A and B are countably $\rho\mathcal{I}$-compact subsets of an ideal space $(X, \tau, \mathcal{I})$ then $A \cup B$ is countably $\rho\mathcal{I}$-compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in\omega}$ be a countable family of open subsets such that $(A \cup B) \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$. This implies that $A \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$ and $B \setminus \cup_{\alpha\in\omega} V_\alpha \in \mathcal{I}$ and there exists finite sets $\omega_1 \subset \omega$ and $\omega_2 \subset \omega$ with $A \setminus \cup_{\alpha\in\omega_1} V_\alpha \in \mathcal{I}$ and $B \setminus \cup_{\alpha\in\omega_2} V_\alpha \in \mathcal{I}$. Then $A \setminus \cup_{\alpha\in\omega_1\cup\omega_2} V_\alpha \in \mathcal{I}$ and $B \setminus \cup_{\alpha\in\omega_1\cup\omega_2} V_\alpha \in \mathcal{I}$ and hence $(A \cup B) \setminus \cup_{\alpha\in\omega_1\cup\omega_2} V_\alpha \in \mathcal{I}$. $\square$

If $(X, \tau, \mathcal{I})$ is an ideal space, we denote by $\tau^*(\mathcal{I})$, the topology on X generated by the basis $\beta(\mathcal{I}, \tau) = \{U \setminus I : U \in \tau, I \in \mathcal{I}\}$ [8]. When there is no ambiguity we will simply write τ^* for $\tau^*(\mathcal{I})$ and β for $\beta(\mathcal{I}, \tau)$. When $\tau^* = \beta$, that is when β is a topology, we say that the topology τ is simple with respect to $\mathcal{I}$ or just simple when no ambiguity is present since all the τ^*-open sets are of simple form, that is, $V \in \tau^*$ means $V = U \setminus I$ for some $U \in \tau$ and $I \in \mathcal{I}$. As $\beta(\mathcal{I}, \tau) = \beta(\mathcal{I}, \tau^*)$, we have τ is simple if and only if τ^* is simple.

A condition which implies τ is simple is the following: Given a space $(X, \tau, \mathcal{I})$, Njåstad [11] defines the ideal $\mathcal{I}$ to be compatible with τ, denoted $\mathcal{I} \sim \tau$, if $A \subseteq X$ and if for every $x \in A$ there exists a $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$ then $A \in \mathcal{I}$. It is known that in any space (X, τ), the ideals $\mathcal{N}(\tau)$ and $\mathcal{M}(\tau)$ are compatible.

Theorem 9. *Let $(X, \tau, \mathcal{I})$ be an ideal space.*

(1) *If (X, τ^*) is countably $\rho\mathcal{I}$-compact, then (X, τ) is countably $\rho\mathcal{I}$-compact.*
(2) *If τ is simple and (X, τ) is countably $\rho\mathcal{I}$-compact then (X, τ^*) is countably $\rho\mathcal{I}$-compact.*

Proof. 1. The result is immediate from the observation that $\tau \subseteq \tau^*$.
2. Let $\{V_n : n \in \omega\}$ be a τ^*-open $\mathcal{I}$-cover of X. For every V_n in the cover, there exists $U_n \in \tau$ and $I_n \in \mathcal{I}$ such that $V_n = U_n \setminus I_n$. Now $\{U_n : n \in \omega\}$ will be a τ-open $\mathcal{I}$-cover and hence there exists a finite subcollection $\{U_{n_i} : i = 1, 2, \ldots, m\}$ such that $X \setminus \cup_{i=1}^m U_{n_i} = I \in \mathcal{I}$. Now we have $X \setminus \cup_{i=1}^m V_{n_i} \subseteq I \cup (\cup_{i=1}^m I_{n_i}) \in \mathcal{I}$. □

Corollary 10. *Let $(X, \tau, \mathcal{I})$ be a space with $\mathcal{I} \sim \tau$. Then (X, τ) is countably $\rho\mathcal{I}$-compact if and only if $(X, \tau*)$ is countably $\rho\mathcal{I}$-compact.*

A collection $\{A_\alpha : \alpha \in \lambda\}$ of subsets of X with an ideal $\mathcal{I}$ on X is said to have the finite intersection property modulo $\mathcal{I}$ or $\mathcal{I}$-FIP if for every finite subcollection $\{A_{\alpha_i} : i = 1, 2, \ldots, m\}$, we have $\cap_{i=1}^m A_{\alpha_i} \notin \mathcal{I}$. The following theorem contains some useful characterizations of countable $\rho\mathcal{I}$-compactness.

Theorem 11. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(1) *(X, τ) is countably $\rho\mathcal{I}$-compact space.*
(2) *For every countable family $\{F_n : n \in \omega\}$ of closed sets such that $\cap_{n=1}^\infty F_n \in \mathcal{I}$, there exists a finite subfamily $\{F_{n_i} : i = 1, 2, \ldots, m\}$ such that $\cap_{i=1}^m F_{n_i} \in \mathcal{I}$.*
(3) *For every countable family $\{F_n : n \in \omega\}$ of closed sets with $\mathcal{I}$-FIP, $\cap_{i=1}^\infty F_n \notin \mathcal{I}$.*

Proof. (1) $\Rightarrow$ (2). Let $\{F_n : n \in \omega\}$ be a countable collection of closed sets such that $\cap_{i=1}^\infty F_n \in \mathcal{I}$. Then $\{X \setminus F_n : n \in \omega\}$ is a countable open $\mathcal{I}$-cover of X, so there exists a finite subcollection $\{X \setminus F_{n_i} : i = 1, 2, \ldots, m\}$ such that $X \setminus \cup_{i=1}^m (X \setminus F_{n_i}) \in \mathcal{I}$ which implies $\cap_{i=1}^m F_{n_i} \in \mathcal{I}$.

(2) $\Rightarrow$ (3) These are contrapositive implications.

(3) $\Rightarrow$ (1) Let $\{U_n : n \in \mathbb{N}\}$ be a countable open $\mathcal{I}$-cover of X, that is $X \setminus \cup_{n=1}^\infty U_n \in \mathcal{I}$. Then $\{X \setminus U_n : n \in \mathbb{N}\}$ is a countable collection of closed sets such that $\cap_{n=1}^\infty (X \setminus U_n) \in \mathcal{I}$. By hypothesis this family does not have the property $\mathcal{I}$-FIP and so there exists a finite subcollection $\{U_{n_i} : i = 1, 2, \ldots, m\}$ such that $\cap_{i=1}^m X \setminus U_{n_i} \in \mathcal{I}$. Hence X is countably $\rho\mathcal{I}$-compact space. □

Now we study the behavior of countably $\rho\mathcal{I}$-compact spaces under certain types of maps.

Theorem 12. *If $(X,\tau,\mathcal{I})$ is countably $\rho\mathcal{I}$-compact space, $f:(X,\tau)\to(Y,\beta)$ be a continuous map and if $\mathcal{J}=\{B\subseteq Y \mid f^{-1}(B)\in\mathcal{I}\}$. Then $(Y,\beta,\mathcal{J})$ is countably $\rho\mathcal{J}$-compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in\omega}$ be a countable family of open subsets of Y such that $Y\setminus\cup_{\alpha\in\omega}V_\alpha\in\mathcal{J}$. Since $f^{-1}(Y\setminus\cup_{\alpha\in\omega}V_\alpha)=X\setminus\cup_{\alpha\in\omega}f^{-1}(V_\alpha)\in\mathcal{I}$, the countable family $\{f^{-1}(V_\alpha)\}$ is an open $\mathcal{I}$-cover of X. So there exists a finite set $\omega_0\subset\omega$ such that $f^{-1}(Y\setminus\cup_{\alpha\in\omega_0}V_\alpha)=X\setminus\cup_{\alpha\in\omega_0}f^{-1}(V_\alpha)\in\mathcal{I}$ which implies that $Y\setminus\cup_{\alpha\in\omega_0}V_\alpha\in\mathcal{J}$. □

Theorem 13. *If $(X,\tau,\mathcal{I})$ is countably $\rho\mathcal{I}$-compact, $f:(X,\tau)\to(Y,\beta)$ is a continuous bijection, then $(Y,\beta,f(\mathcal{I}))$ is countably $\rho f(\mathcal{I})$-compact.*

Proof. Let $\{V_\alpha\}_{\alpha\in\omega}$ be a countable family of open subsets of Y such that $Y\setminus\cup_{\alpha\in\omega}V_\alpha\in f(\mathcal{I})$. Then $f^{-1}(Y\setminus\cup_{\alpha\in\omega}V_\alpha)=X\setminus\cup_{\alpha\in\omega}f^{-1}(V_\alpha)\in\mathcal{I}$ so there exists a finite set $\omega_0\subset\omega$ such that $X\setminus\cup_{\alpha\in\omega_0}f^{-1}(V_\alpha)\in\mathcal{I}$. Thus $Y\setminus\cup_{\alpha\in\omega_0}V_\alpha\subset f(X\setminus\cup_{\alpha\in\omega_0}f^{-1}(V_\alpha))\in f(\mathcal{I})$. Hence Y is countably $\rho f(\mathcal{I})$-compact. □

If $f:X\to Y$ is an injective map and $\mathcal{J}$ is an ideal on Y, then the collection $f^{-1}(\mathcal{J})=\{f^{-1}(J)\mid J\in\mathcal{J}\}$ is an ideal on X.

Theorem 14. *If $f:(X,\tau)\to(Y,\beta)$ is an open bijective map and $(Y,\beta,\mathcal{J})$ is countably $\rho\mathcal{J}$-compact space then $(X,\tau,f^{-1}(\mathcal{J}))$ is countably $\rho f^{-1}(\mathcal{J})$-compact.*

Proof. Suppose $\{V_\alpha\}_{\alpha\in\omega}$ be a countable family of open subsets of X such that $X\setminus\cup_{\alpha\in\omega}V_\alpha\in f^{-1}(\mathcal{J})$. Then there exists $J\in\mathcal{J}$ with $X\setminus\cup_{\alpha\in\omega}V_\alpha=f^{-1}(J)$. So $Y\setminus\cup_{\alpha\in\omega}f(V_\alpha)=f(f^{-1}(J))=J\in\mathcal{J}$ and since $(Y,\beta,\mathcal{J})$ is countably $\rho\mathcal{J}$-compact, there exists a finite set $\omega_0\subset\omega$ with $f(X\setminus\cup_{\alpha\in\omega_0}V_\alpha)=Y\setminus\cup_{\alpha\in\omega_0}f(V_\alpha)\in\mathcal{J}$. This implies that $X\setminus\cup_{\alpha\in\omega_0}V_\alpha\in f^{-1}(\mathcal{J})$. □

4. Characterization of Countably $\rho\mathcal{I}$-compact Spaces

Bacon [1] gave a sufficient condition for a countably compact topological space to be compact space. We generalize this result to characterize the countably $\rho\mathcal{I}$-compact spaces. A property L-$\mathcal{I}$ is defined which is sufficient condition for a countably $\rho\mathcal{I}$-compact space to be a $\rho\mathcal{I}$-compact space.

In this section, the closure of A will be denoted by A^-, for any subset $A\subseteq X$.

Definition 15. *Let $\mathcal{D}$ and $\mathcal{E}$ be the collections of subsets of a space X. We say that $\mathcal{D}$ is weakly cushioned modulo $\mathcal{I}$ in $\mathcal{E}$ if there is a function $f : \mathcal{D} \to \mathcal{E}$ such that if $\mathcal{G}$ is a countable subcollection of $\mathcal{D}$ and for each $G \in \mathcal{G}$, x_G is a point of G, then $\{x_G : G \in \mathcal{G}\}^- \setminus f(\mathcal{G}) \in \mathcal{I}$ and for any finite subcollection $\mathcal{H}$ of $\mathcal{G}$, $\cup\mathcal{H} \subseteq \cup f(\mathcal{H})$.*

Let $\mathcal{E}$ be a subcollection of subsets of X, then $\omega(\mathcal{E})$ denote the collection of all countable (finite or infinite) unions of members of $\mathcal{E}$.

Definition 16. *An ideal space $(X, \tau, \mathcal{I})$ is said to have property L-$\mathcal{I}$ if whenever $\mathcal{E}$ is an open $\mathcal{I}$-cover of X, there is a sequence $\mathcal{D}_1$, $\mathcal{D}_2 \ldots$ such that for each n, $\mathcal{D}_n$ is a collection of subsets of X weakly cushioned modulo $\mathcal{I}$ in $\omega(\mathcal{E})$ and $X \setminus \cup\{U : U \in \mathcal{E}\} \subseteq X \setminus \cup_n\mathcal{D}_n \in \mathcal{I}$ and $\cup\mathcal{D}_n \cap I$ is finite for all $I \in \mathcal{I}$.*

Theorem 17. *Every countably $\rho\mathcal{I}$-compact space is $\rho\mathcal{I}$-compact if it has the property L-$\mathcal{I}$.*

Proof. Suppose X is countably $\rho\mathcal{I}$-compact space with property L-$\mathcal{I}$ and $\mathcal{E}$ is an open $\mathcal{I}$-cover of X. Let $\mathcal{D}_1$, $\mathcal{D}_2 \ldots$ be a sequence such that for each n, $\mathcal{D}_n$ is a collection of subsets of X such that $X \setminus \cup\{U : U \in \mathcal{E}\} \subseteq X \setminus \cup Z_n \in \mathcal{I}$, where $Z_n = \cup\{G : G \in \mathcal{D}_n\}$ with $Z_n \cap I$ being finite for each $I \in \mathcal{I}$ and for each n, $\mathcal{D}_n$ is weakly cushioned modulo $\mathcal{I}$ in $\omega(\mathcal{E})$. For each n, let $f_n : \mathcal{D}_n \to \omega(\mathcal{E})$ be a function such that if $\mathcal{G}$ is a countable subcollection of $\mathcal{D}_n$ and $x_G \in G$ for each $G \in \mathcal{G}$, then $\{x_G : G \in \mathcal{G}\}^- \setminus \cup f_n(\mathcal{G}) \in \mathcal{I}$ and for any finite subcollection $\mathcal{H}$ of $\mathcal{G}$, $\cup\mathcal{H} \subseteq \cup f_n(\mathcal{H})$.

Suppose that for some n, Z_n is not a subset of any element in $\omega(\mathcal{E})$. Let $\{x_1, \ldots, x_n\}$ is a subset of Z_n and for each $i \in \{1, \ldots, n\}$, G_i is an element of $\mathcal{D}_n$ that contains x_i. Define $A_k = \cup_{i=1}^k f_n(G_i)$. Since $A_k \in \omega(\mathcal{E})$, there is a point x_{k+1} in $Zn \setminus A_k$. Let G_{k+1} be an element of $\mathcal{D}_n$ that contains x_{k+1}. Since $\cup_{i=1}^k G_i$ is a subset A_k, G_{k+1} is not in $\{G_1, \ldots, G_k\}$. By induction there exist sequences (x_n), (G_k) and (A_k) such that for each k, G_k is an element of $\mathcal{D}_n$ different from G_j where $j \neq k$, x_k is in $G_k \cap Z_n$, $A_k = \cup_{i=1}^k f_n(G_i)$ and $x_{k+1} \in Z_n \setminus A_k$. Define $B = (X \setminus \{x_1, x_2 \ldots\}^-) \cap (\cup\mathcal{E})$. As $\mathcal{D}_n$ is weakly cushioned modulo $\mathcal{I}$ in $\omega(\mathcal{E})$, $\{x_1, x_2, \ldots\}^- \setminus \cup_{k=1}^\infty f_n(G_k) = \{x_1, x_2, \ldots\}^- \setminus \cup_{k=1}^\infty A_k \in \mathcal{I}$. Since $X \setminus \cup\mathcal{E} \in \mathcal{I}$, $X \setminus \{B, A_1, \ldots\} \in \mathcal{I}$ which implies that $\{B, A_1, \ldots\}$ is a countable open $\mathcal{I}$-cover of X and X being a countably $\rho\mathcal{I}$-compact space, there will be some k such that $X \setminus B \cup A_k \in \mathcal{I}$. Since $x_{k+1}, x_{k+2}, \ldots$ are infinite number of elements of Z_n contained in a member of $\mathcal{I}$, we arrive at a contradiction. Hence each Z_n is contained in some element of $\omega(\mathcal{E})$ and $\{Z_1, Z_2, \ldots\}$ is a countable $\mathcal{I}$-cover of X contained in $\mathcal{E}$. Hence X has a finite $\mathcal{I}$-subcover of X. □

Definition 18. *A topological space X is said to be $\rho\mathcal{I}$-isocompact if every closed countably $\rho\mathcal{I}$-compact subset of X is $\rho\mathcal{I}$-compact.*

Theorem 19. *If an ideal space $(X, \tau, \mathcal{I})$, where $\mathcal{I}$ is a σ-ideal, is the union of a countable collection of closed $\rho\mathcal{I}$-isocompact subsets then X is $\rho\mathcal{I}$-isocompact space.*

Proof. Suppose $X = \cup_{i=1}^{\infty} F_i$, where each F_i is closed and $\rho\mathcal{I}$-isocompact. Let M be a closed countably $\rho\mathcal{I}$-compact subset of X and $\mathcal{G}$ be an open $\mathcal{I}$-cover of M, that is $M \setminus \cup\mathcal{G} \in \mathcal{I}$. For each i, $M \cap F_i$ is a closed countably $\rho\mathcal{I}$-compact subset of F_i which implies that $M \cap F_i$ is $\rho\mathcal{I}$-compact. So there exists a finite $\mathcal{I}$-subcover H_i of $M \cap F_i$ that is $M \cap F_i \setminus \cup H_i \in \mathcal{I}$. Then $\cup_{i=1}^{\infty} H_i$ is a countable open $\mathcal{I}$-cover of M and so contains a finite subcollection that covers M modulo $\mathcal{I}$. □

A subset A of X is said to be a non-ideal set if $A \notin \mathcal{I}$. It is known that a space X is countably compact if and only if locally finite collections of nonempty subsets of X are finite. Here we prove a theorem which has this known fact as a corollary.

Definition 20. *Let $(X, \tau, \mathcal{I})$ be a space and $I \in \mathcal{I}$. A family $\mathcal{A}$ of subsets of X is said to be locally finite modulo I if for each $x \in X \setminus I$, there exists $U \in \tau(x)$ such that U intersects with at most finite number of elements in $\mathcal{A}$.*

Every locally finite family is locally finite modulo I for every $I \in \mathcal{I}$.

Theorem 21. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then X is countably $\rho\mathcal{I}$-compact if and only if for any $I \in \mathcal{I}$ and any countable locally finite modulo I family of non-ideal sets in X is finite.*

Proof. Let $(X, \tau, \mathcal{I})$ be a countably $\rho\mathcal{I}$-compact space and $\{A_n : n \in \mathbb{N}\}$ be a countable locally finite family of non-ideal sets modulo $I \in \mathcal{I}$. For every $n \in \mathbb{N}$, define $B_n = \cup_{i=n}^{\infty} A_i$. Then $B_{n+1} \subseteq B_n$ and $B_n \notin \mathcal{I}$ for every n. Let $x \in X \setminus I$, then there exists $U \in \tau(x)$ and $j \in \mathbb{N}$ such that $U \cap A_i = \emptyset$ for every $i \geq j$. Therefore $U \cap B_j = \emptyset$ and $x \notin \overline{B}_j$. This shows that $\cap_{n=1}^{\infty} \overline{B}_n \subseteq I \in \mathcal{I}$. And by Theorem 11, we have a finite subcollection such that $\cap_{i=1}^{m} \overline{B}_{n_i} \in \mathcal{I}$. Choosing $k \in \mathbb{N}$ such that $k > max\{n_1, \ldots, n_m\}$, we have

$$B_k \subseteq \cap_{i=1}^{m} B_{n_i} \subseteq \cap_{i=1}^{m} \overline{B}_{n_i}.$$

This implies $B_k \in \mathcal{I}$ which is a contradiction.

Conversely, assume that X is not a countably $\rho\mathcal{I}$-compact space. Then there exists a countable family of open sets $\{U_n : n \in \mathbb{N}\}$ such that $X \setminus \cup_{i=1}^{\infty} U_n \in \mathcal{I}$ and for any finite set $F \subset \mathbb{N}$, $X \setminus \cup_{k \in F} U_k \notin \mathcal{I}$. Let $A_n = X \setminus \cup_{k=1}^{n} U_k$. Then $A_n \notin \mathcal{I}$, $A_{n+1} \subseteq A_n$ and $\cap_{i=1}^{\infty} A_n = X \setminus \cup_{k=1}^{\infty} U_k = I \in \mathcal{I}$. Let $x \in X \setminus I$, since each A_n is closed, there exists $U \in \tau(x)$ and $m \in \mathbb{N}$ such that $U \cap A_n = \emptyset$ for every $n \geq m$. From this we conclude that $\{A_n : n \in \mathbb{N}\}$ is countably infinite locally finite modulo $I \in \mathcal{I}$ family of non-ideal sets which leads to a contradiction. $\square$

For $\mathcal{I} = \{\emptyset\}$, the Theorem 21 reduces to the following well known result [6].

Corollary 22. *A space X is countably compact if and only if every locally finite family of nonempty sets in X is finite.*

Definition 23. *An ideal space $(X, \tau, \mathcal{I})$ is said to be paracompact modulo $\mathcal{I}$ if each open $\mathcal{I}$-cover $\mathcal{U}$ of X has an open $\mathcal{I}$-cover $\mathcal{V}$ which is locally finite refinement modulo $I = X \setminus \cup\mathcal{U} \in \mathcal{I}$.*

Note that for any space $(X, \tau, \mathcal{I})$, the following implications hold:
paracompact modulo $\mathcal{I} \Rightarrow \mathcal{I}$-paracompact $\Leftarrow$ paracompact.

Theorem 24. *Let $(X, \tau, \mathcal{I})$ be paracompact modulo $\mathcal{I}$ space with $\mathcal{I} \cap \tau = \{\emptyset\}$. Then X is countably $\rho\mathcal{I}$-compact if and only if X is $\rho\mathcal{I}$-compact.*

Proof. Assume that $(X, \tau, \mathcal{I})$ is a countably $\rho\mathcal{I}$-compact and paracompact modulo $\mathcal{I}$ space. Let $\mathcal{U}$ be an open $\mathcal{I}$-cover of X, then $\mathcal{U}$ has an open $\mathcal{I}$-cover $\mathcal{V}$ which is locally finite refinement modulo $I = X \setminus \cup\mathcal{V} \in \mathcal{I}$. Since X is countably $\rho\mathcal{I}$-compact and $\mathcal{I} \cap \tau = \{\emptyset\}$, by Theorem 21, the family $\mathcal{V}$ is finite. Hence we can say that X is $\rho\mathcal{I}$-compact. $\square$

A space (X, τ) is a Baire space if and only if $M(\tau) \cap \tau = \{\emptyset\}$, where $M(\tau)$ denotes the σ-ideal of meager (first category) subsets of X.

Corollary 25. *Let $(X, \tau, M(\tau))$ be a Baire paracompact modulo $M(\tau)$ space. Then X is countably $\rho M(\tau)$-compact space if and only if X is $\rho M(\tau)$-compact space.*

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SUMIT MITTAL
University of Delhi,
Deparment of Mathematics,
New Delhi, India 110007
e-mail: sumitmittal1105@gmail.com

B. K. TYAGI
Atma Ram Sanatan Dharma College,
Department of Mathematics,
University of Delhi,
New Delhi, India 110021
e-mail: brijkishore.tyagi@gmail.com