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COMMON COUPLED FIXED POINT THEOREM FOR HYBRID PAIR OF MAPPINGS SATISFYING $\varphi - \psi$ CONTRACTION ON NONCOMPLETE METRIC SPACE

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Abstract. We establish a coupled coincidence and common coupled fixed point theorem for hybrid pair of mappings under $\varphi - \psi$ contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. We also give an example to validate our result. We improve and generalize several known results.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} max\left\{sup_{a\in A}D(a,\ B),\ sup_{b\in B}D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

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for all $A,\ B \in CL(X)$, where $D(x,\ A) = \inf_{a \in A} d(x,\ a)$ denote the distance from x to $A \subset X$. For simplicity, if $x \in X$, we denote $g(x)$ by gx.

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [13]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics. In 1969, Nadler [14] extended the famous Banach Contraction Principle [2] from single-valued mapping to multivalued mapping and proved the fixed point theorem for the multivalued contraction.

In [4], Bhaskar and Lakshmikantham established some coupled fixed point theorems in the setting of single-valued mappings and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Luong and Thuan [11] generalized the results of Bhaskar and Lakshmikantham [4]. Berinde [3] extended the results of Bhaskar and Lakshmikantham [4] and Luong and Thuan [11]. Lakshmikantham and Ciric [9] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [4]. Jain et al. [12] extended and generalized the results of Berinde [3], Bhaskar and Lakshmikantham [4], Lakshmikantham and Ciric [9] and Luong and Thuan [11]. For more details on coupled fixed point theory, we also refer the reader to ([5], [6], [7], [8], [16], [17], [18], [20]).

Recently Samet et al. [19] claimed that most of the coupled fixed point theorems in the setting of single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

Coupled fixed point theory for multivalued mappings was introduced by Abbas et al. [1] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces.

On the other hand, at present, very few papers were devoted to coupled fixed point problems for hybrid pair of mappings including ([1], [10]).

In [1], Abbas et al. introduced the following:

Definition 1. *Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x,\ y) \in X \times X$ is called*

(1) *a coupled coincidence point of hybrid pair* $\{F, g\}$ *if* $gx \in F(x, y)$ *and* $gy \in F(y, x)$.

(2) *a common coupled fixed point of hybrid pair* $\{F, g\}$ *if* $x = gx \in F(x, y)$ *and* $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F *and* g *by* $C(F, g)$. *Note that if* $(x, y) \in C(F, g)$, *then* (y, x) *is also in* $C(F, g)$.

Definition 2. *Let* $F : X \times X \to 2^X$ *be a multivalued mapping and* g *be a self-mapping on* X. *The hybrid pair* $\{F, g\}$ *is called* $w-$*compatible if* $gF(x, y) \subseteq F(gx, gy)$ *whenever* $(x, y) \in C(F, g)$.

Definition 3. *Let* $F : X \times X \to 2^X$ *be a multivalued mapping and* g *be a self-mapping on* X. *The mapping* g *is called* $F-$*weakly commuting at some point* $(x, y) \in X \times X$ *if* $g^2x \in F(gx, gy)$ *and* $g^2y \in F(gy, gx)$.

Lemma 4. [15]. *Let* (X, d) *be a metric space. Then, for each* $a \in X$ *and* $B \in K(X)$, *there is* $b_0 \in B$ *such that* $D(a, B) = d(a, b_0)$, *where* $D(a, B) = \inf_{b\in B} d(a, b)$.

In this paper, we establish a coupled coincidence and common coupled fixed point theorem for hybrid pair of mappings under $\varphi - \psi$ contraction on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Berinde [3], Bhaskar and Lakshmikantham [4], Lakshmikantham and Ciric [9], Luong and Thuan [11], Jain et al. [12] and many others. An example is furnished which demonstrate the validity of the hypotheses and degree of generality of our main result.

2. **Main results**

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is continuous and (strictly) increasing,

(ii_φ) $\varphi(t) < t$ for all $t > 0$,

(iii_φ) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s > 0$.

Note that, by (i_φ) and (ii_φ) we have that $\varphi(t) = 0$ if and only if $t = 0$.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies

(i_ψ) $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+} \psi(t) = 0$.

Now, we prove the following theorem:

Theorem 5. *Let (X, d) be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exist some $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$(1) \qquad \varphi\left(\frac{H(F(x, y), F(u, v)) + H(F(y, x), F(v, u))}{2}\right) \leq \varphi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right) - \psi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right),$$

for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x, y) \in C(F, g)$, gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.

(d) $g(C(F, g))$ is a singleton subset of $C(F, g)$.

Proof. Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$, because $F(X \times X) \subseteq g(X)$. Since $F : X \times X \to K(X)$, therefore by Lemma 4, there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$\begin{aligned} d(gx_1, z_1) &\leq H(F(x_0, y_0), F(x_1, y_1)), \\ d(gy_1, z_2) &\leq H(F(y_0, x_0), F(y_1, x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist $x_2, y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1, gx_2) &\leq H(F(x_0, y_0), F(x_1, y_1)), \\ d(gy_1, gy_2) &\leq H(F(y_0, x_0), F(y_1, x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n, y_n)$ and $gy_{n+1} \in F(y_n, x_n)$

such that

$$\begin{aligned}
&\varphi\left(\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right) \\
\leq\ & \varphi\left(\frac{H(F(x_n, y_n),\ F(x_{n-1}, y_{n-1})) + H(F(y_n, x_n),\ F(y_{n-1}, x_{n-1}))}{2}\right) \\
\leq\ & \varphi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right) - \psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\varphi\left(\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right) \\
\leq\ & \varphi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right) - \psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right),
\end{aligned} \tag{2}$$

which, by the fact that $\psi \geq 0$, implies

$$\varphi\left(\frac{d(gx_{n+1},\ gx_n) + d(gy_{n+1},\ gy_n)}{2}\right) \leq \varphi\left(\frac{d(gx_n,\ gx_{n-1}) + d(gy_n,\ gy_{n-1})}{2}\right),$$

this shows, by the monotony of φ, that the sequence $\{\delta_n\}_{n=0}^{\infty}$ given by

$$\delta_n = \frac{d(gx_{n+1},\ gx_n) + d(gy_{n+1},\ gy_n)}{2},\ \forall n \geq 0,$$

is non-increasing. Therefore, there exists some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{d(gx_{n+1},\ gx_n) + d(gy_{n+1},\ gy_n)}{2} = \delta.$$

We shall prove that $\delta = 0$. Assume that $\delta > 0$. Then by letting $n \to \infty$ in (2), by using (i_φ) and (i_ψ), we get

$$\begin{aligned}
\varphi(\delta) &= \lim_{n\to\infty} \varphi(\delta_{n+1}) \\
&\leq \lim_{n\to\infty} \varphi(\delta_n) - \lim_{n\to\infty} \psi(\delta_n) \\
&\leq \varphi(\delta) - \lim_{\delta_n \to \delta+} \psi(\delta_n) \\
&< \varphi(\delta),
\end{aligned}$$

which is a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{d(gx_{n+1},\ gx_n) + d(gy_{n+1},\ gy_n)}{2} = 0. \tag{3}$$

We now prove that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are Cauchy sequences in $(X,\ d)$. Suppose, to the contrary, that at least one of the sequences $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}_{n=0}^{\infty}$ such that

$$\frac{d(gx_{n(k)},\ gx_{m(k)}) + d(gy_{n(k)},\ gy_{m(k)})}{2} \geq \varepsilon,\ k = 1,\ 2,\ \ldots \tag{4}$$

We can choose $n(k)$ to be the smallest positive integer satisfying (4). Then

$$\frac{d(gx_{n(k)-1},\ gx_{m(k)}) + d(gy_{n(k)-1},\ gy_{m(k)})}{2} < \varepsilon. \tag{5}$$

By (4), (5) and triangle inequality, we have

$$\begin{aligned}
\varepsilon \leq r_k &= \frac{d(gx_{n(k)},\ gx_{m(k)}) + d(gy_{n(k)},\ gy_{m(k)})}{2} \\
&\leq \frac{d(gx_{n(k)},\ gx_{n(k)-1}) + d(gy_{n(k)},\ gy_{n(k)-1})}{2} \\
&\quad + \frac{d(gx_{n(k)-1},\ gx_{m(k)}) + d(gy_{n(k)-1},\ gy_{m(k)})}{2} \\
&< \frac{d(gx_{n(k)},\ gx_{n(k)-1}) + d(gy_{n(k)},\ gy_{n(k)-1})}{2} + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \frac{d(gx_{n(k)},\ gx_{m(k)}) + d(gy_{n(k)},\ gy_{m(k)})}{2} = \varepsilon. \tag{6}$$

By the triangle inequality, we have

$$\begin{aligned}
& d(gx_{n(k)},\ gx_{m(k)}) \\
\leq\ & d(gx_{n(k)},\ gx_{n(k)+1}) + d(gx_{n(k)+1},\ gx_{m(k)+1}) + d(gx_{m(k)+1},\ gx_{m(k)}),
\end{aligned}$$

and similarly

$$\begin{aligned}
& d(gy_{n(k)},\ gy_{m(k)}) \\
\leq\ & d(gy_{n(k)},\ gy_{n(k)+1}) + d(gy_{n(k)+1},\ gy_{m(k)+1}) + d(gy_{m(k)+1},\ gy_{m(k)}).
\end{aligned}$$

This shows that

$$r_k \leq \delta_{n(k)} + \delta_{m(k)} + \frac{d(gx_{n(k)+1},\ gx_{m(k)+1}) + d(gy_{n(k)+1},\ gy_{m(k)+1})}{2}. \tag{7}$$

Now, since $gx_{n(k)+1} \in F(x_{n(k)},\ y_{n(k)})$, $gx_{m(k)+1} \in F(x_{m(k)},\ y_{m(k)})$, $gy_{n(k)+1} \in F(y_{n(k)},\ x_{n(k)})$ and $gy_{m(k)+1} \in F(y_{m(k)},\ x_{m(k)})$. Therefore by using (1) and (i_φ), we get

$$\begin{aligned}
& \varphi\left(\frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right) \\
\leq\ & \varphi\left(\frac{H(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) + H(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))}{2}\right) \\
\leq\ & \varphi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right) \\
& -\psi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right) \\
\leq\ & \varphi(r_k) - \psi(r_k).
\end{aligned}$$

Thus
(8)
$$\varphi\left(\frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right) \leq \varphi(r_k) - \psi(r_k).$$

On the other hand, by (7) and (iii_φ), we get
(9)
$$\varphi(r_k) \leq \varphi(\delta_{n(k)}) + \varphi(\delta_{m(k)}) + \varphi\left(\frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right).$$

By (8) and (9), we get

$$\varphi(r_k) \leq \varphi(\delta_{n(k)}) + \varphi(\delta_{m(k)}) + \varphi(r_k) - \psi(r_k). \tag{10}$$

Letting $k \to \infty$ in (10), by using (3), (6), (i_φ), (ii_φ) and (i_ψ), we get

$$\begin{aligned}
\varphi(\varepsilon) &\leq \varphi(0) + \varphi(0) + \varphi(\varepsilon) - \lim_{k\to\infty} \psi(r_k) \\
&\leq \varphi(\varepsilon) - \lim_{r_k \to \varepsilon+} \psi(r_k) \\
&< \varphi(\varepsilon),
\end{aligned}$$

which is a contradiction. This shows that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are indeed Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist x, $y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{11}$$

Now, since $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$. Therefore by using condition (1) and (i_φ), we get

$$\begin{aligned}
&\varphi\left(\frac{D(gx_{n+1},\ F(x,\ y)) + D(gy_{n+1},\ F(y,\ x))}{2}\right) \\
\leq\ &\varphi\left(\frac{H(F(x_n,\ y_n),\ F(x,\ y)) + H(F(y_n,\ x_n),\ F(y,\ x))}{2}\right) \\
\leq\ &\varphi\left(\frac{d(gx_n,\ gx) + d(gy_n,\ gy)}{2}\right) - \psi\left(\frac{d(gx_n,\ gx) + d(gy_n,\ gy)}{2}\right).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (11), (i_φ), (ii_φ) and (i_ψ), we obtain

$$\varphi\left(\frac{D(gx,\ F(x,\ y)) + D(gy,\ F(y,\ x))}{2}\right) \leq \varphi(0) - 0 = 0 - 0 = 0,$$

which, by (i_φ) and (ii_φ), implies

$$D(gx,\ F(x,\ y)) = 0 \text{ and } D(gy,\ F(y,\ x)) = 0,$$

it follows that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and g. Hence $C(F,\ g)$ is nonempty.

Suppose now that (a) holds. Assume that for some $(x,\ y) \in C(F,\ g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{12}$$

where $u,\ v \in X$. Since g is continuous at u and v. We have, by (12), that u and v are fixed points of g, that is,

$$gu = u \text{ and } gv = v. \tag{13}$$

As F and g are $w-$compatible, so

$$(g^n x,\ g^n y) \in C\{F, g\}, \text{ for all } n \geq 1,$$

that is, for all $n \geq 1$,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x). \tag{14}$$

Now, by using (1) and (14), we obtain

$$\begin{aligned}
&\varphi\left(\frac{D(g^n x,\ F(u,\ v))+D(g^n y,\ F(v,\ u))}{2}\right)\\
\leq\ &\varphi\left(\frac{H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v))+H(F(g^{n-1}y,\ g^{n-1}x),\ F(v,\ u))}{2}\right)\\
\leq\ &\varphi\left(\frac{d(g^n x,\ gu)+d(g^n y,\ gv)}{2}\right)-\psi\left(\frac{d(g^n x,\ gu)+d(g^n y,\ gv)}{2}\right).
\end{aligned}$$

On taking limit as $n\to\infty$ in the above inequality, by using (12), (13), (i_φ), (ii_φ) and (i_ψ), we get

$$\varphi\left(\frac{D(gu,\ F(u,\ v))+D(gv,\ F(v,\ u))}{2}\right)\leq\varphi(0)-0=0-0=0,$$

which shows, by (i_φ) and (ii_φ), that

$$D(gu,\ F(u,\ v))=0 \text{ and } D(gv,\ F(v,\ u))=0,$$

which implies that

$$gu\in F(u,\ v) \text{ and } gv\in F(v,\ u). \tag{15}$$

Now, from (13) and (15), we have

$$u=\ gu\in F(u,\ v) \text{ and } v=gv\in F(v,\ u),$$

that is, $(u,\ v)$ is a common coupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x,\ y)\in C(F,\ g)$, g is $F-$weakly commuting, that is, $g^2x\in F(gx,\ gy)$, $g^2y\in F(gy,\ gx)$ and $g^2x=gx$, $g^2y=gy$. Thus, $gx=g^2x\in F(gx,\ gy)$ and $gy=g^2y\in F(gy,\ gx)$, that is, $(gx,\ gy)$ is a common coupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x,\ y)\in C(F,\ g)$ and for some $u,\ v\in X$,

$$\lim_{n\to\infty} g^n u=x \text{ and } \lim_{n\to\infty} g^n v=y. \tag{16}$$

Since g is continuous at x and y. We have, by (16), that x and y are fixed points of g, that is,

$$gx=x \text{ and } gy=y. \tag{17}$$

Since $(x,\ y)\in C(F,\ g)$, therefore by (17), we obtain

$$x=gx\in F(x,\ y) \text{ and } y=gy\in F(y,\ x),$$

that is, $(x,\ y)$ is a common coupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C(F,\ g)) = \{(x,\ x)\}$. Then $\{x\} = \{gx\} = F(x,\ x)$. Hence $(x,\ x)$ is a common coupled fixed point of F and g. ∎

Put $g = I$ (the identity mapping) in Theorem 5, we get the following result:

Corollary 6. *Let $(X,\ d)$ be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume there exist some $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} &\varphi\left(\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2}\right) \\ \leq\ &\varphi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right) - \psi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right), \end{aligned}$$

for all $x,\ y,\ u,\ v \in X$. Then F has a coupled fixed point.

Corollary 7. *Let $(X,\ d)$ be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Assume that there exists some $\psi \in \Psi$ such that*

$$\begin{aligned} &H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u)) \\ \leq\ &d(gx,\ gu) + d(gy,\ gv) - 2\psi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right). \end{aligned} \tag{18}$$

for all $x,\ y,\ u,\ v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. Then F and g have a coupled coincidence point. Moreover, F and g have a common coupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x,\ y) \in C(F,\ g)$ and for some $u,\ v \in X$ and g is continuous at u and v.

(b) g is $F-$weakly commuting for some $(x,\ y) \in C(F,\ g)$, gx and gy are fixed points of g, that is, $g^2x = gx$ and $g^2y = gy$.

(c) g is continuous at x and y. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x,\ y) \in C(F,\ g)$ and for some $u,\ v \in X$.

(d) $g(C(F,\ g))$ is a singleton subset of $C(F,\ g)$.

Proof. If $\psi \in \Psi$, then for all $r > 0$, $r\psi \in \Psi$. Now divide (18) by 4 and take $\varphi(t) = \frac{1}{2}t$, $t \in [0,\ \infty)$, then condition (18) reduces to (1) with $\psi_1 = \frac{1}{2}\psi$ and hence by Theorem 5 we get Corollary 7. ∎

Put $g = I$ (the identity mapping) in Corollary 7, we get the following result:

Corollary 8. *Let (X, d) be a complete metric space, $F : X \times X \to K(X)$ be a mapping. Assume that there exists some $\psi \in \Psi$ such that*

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u)) \\ \leq\ & d(x,\ u) + d(y,\ v) - 2\psi\left(\frac{d(x,\ u) + d(y,\ v)}{2}\right), \end{aligned}$$

for all $x, y, u, v \in X$. Then F has a coupled fixed point.

Example 9. *Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \to K(X)$ be defined as*

$$F(x,\ y) = \begin{cases} \{0\},\ \textit{for } x,\ y = 1, \\ \left[0,\ \frac{x^2+y^2}{4}\right],\ \textit{for } x,\ y \in [0,\ 1), \end{cases}$$

and $g : X \to X$ be defined as

$$gx = x^2 \textit{ for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{t}{2},\ \textit{for all } t > 0,$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} \frac{t}{4},\ \textit{for } t \neq 1, \\ 0,\ \textit{for } t = 1. \end{cases}$$

Now, for all $x, y, u, v \in X$ with $x, y, u, v \in [0, 1)$, we have

Case (a). *If* $x^2+y^2=u^2+v^2$, *then*

$$\begin{aligned}
&\varphi\left(\frac{H(F(x,\ y),\ F(u,\ v))+H(F(y,\ x),\ F(v,\ u))}{2}\right)\\
=\ &\frac{1}{4}\left(H(F(x,\ y),\ F(u,\ v))+H(F(y,\ x),\ F(v,\ u))\right)\\
=\ &\frac{1}{4}\left[\frac{u^2+v^2}{4}+\frac{v^2+u^2}{4}\right]\\
=\ &\frac{1}{4}\left[\frac{u^2+v^2}{2}\right]\\
\leq\ &\frac{1}{4}\left[\frac{\max\{x^2,\ u^2\}+\max\{y^2,\ v^2\}}{2}\right]\\
\leq\ &\frac{1}{4}\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right)\\
\leq\ &\varphi\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right)-\psi\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right).
\end{aligned}$$

Case (b). *If* $x^2+y^2\neq u^2+v^2$ *with* $x^2+y^2<u^2+v^2$, *then*

$$\begin{aligned}
&\varphi\left(\frac{H(F(x,\ y),\ F(u,\ v))+H(F(y,\ x),\ F(v,\ u))}{2}\right)\\
=\ &\frac{1}{4}\left(H(F(x,\ y),\ F(u,\ v))+H(F(y,\ x),\ F(v,\ u))\right)\\
=\ &\frac{1}{4}\left[\frac{u^2+v^2}{4}+\frac{v^2+u^2}{4}\right]\\
=\ &\frac{1}{4}\left[\frac{u^2+v^2}{2}\right]\\
\leq\ &\frac{1}{4}\left[\frac{\max\{x^2,\ u^2\}+\max\{y^2,\ v^2\}}{2}\right]\\
\leq\ &\frac{1}{4}\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right)\\
\leq\ &\varphi\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right)-\psi\left(\frac{d(gx,\ gu)+d(gy,\ gv)}{2}\right).
\end{aligned}$$

Similarly, we obtain the same result for $u^2+v^2<x^2+y^2$. Thus the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v\in X$ with $x,\ y,\ u,\ v\in[0,\ 1)$. Again, for all $x,\ y,\ u,\ v\in X$ with $x,\ y\in[0,\ 1)$ and u,

$v = 1$, we have

$$\begin{aligned}
&\varphi\left(\frac{H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))}{2}\right)\\
= &\ \frac{1}{4}\left(H(F(x,\ y),\ F(u,\ v)) + H(F(y,\ x),\ F(v,\ u))\right)\\
= &\ \frac{1}{4}\left[\frac{x^2+y^2}{4} + \frac{y^2+x^2}{4}\right]\\
= &\ \frac{1}{4}\left[\frac{x^2+y^2}{2}\right]\\
\leq &\ \frac{1}{4}\left[\frac{\max\{x^2,\ u^2\} + \max\{y^2,\ v^2\}}{2}\right]\\
\leq &\ \frac{1}{4}\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right)\\
\leq &\ \varphi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right) - \psi\left(\frac{d(gx,\ gu) + d(gy,\ gv)}{2}\right).
\end{aligned}$$

Thus the contractive condition (1) is satisfied for all $x, y, u, v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 5 are satisfied and $z = (0, 0)$ is a common coupled fixed point of hybrid pair $\{F, g\}$. The function $F : X \times X \to K(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.

Remark 10. *We improve, extend and generalize the result of Jain et al.* [12] *in the following sense:*

(*i*) *We prove our result in the settings of multivalued mapping and for hybrid pair of mappings.*

(*ii*) *To prove our result we consider non complete metric space and the space is also not partially ordered.*

(*iii*) *The multivalued mapping $F : X \times X \to K(X)$ is discontinuous and not satisfying mixed g-monotone property.*

(*iv*) *The function $\psi : [0,\ +\infty) \to [0,\ +\infty)$ involved in our theorem and example is discontinuous.*

References

[1] M. Abbas, L. Ciric, B. Damjanovic and M. A. Khan, **Coupled coincidence point and common fixed point theorems for hybrid pair of mappings**, Fixed Point Theory Appl. 1687-1812-2012-4.

[2] S. Banach, **Sur les Operations dans les Ensembles Abstraits et leur. Applications aux Equations Integrales**, Fund. Math. 3 (1922), 133-181.

[3] V. Berinde, **Coupled fixed point theorems for $\varphi-$contractive mixed monotone mappings in partially ordered metric spaces**, Nonlinear Anal. 75 (2012), 3218-3228.

[4] T. G. Bhaskar and V. Lakshmikantham, **Fixed point theorems in partially ordered metric spaces and applications**, Nonlinear Anal. 65 (7) (2006), 1379-1393.

[5] B. S. Choudhury and A. Kundu, **A coupled coincidence point results in partially ordered metric spaces for compatible mappings**, Nonlinear Anal. 73 (2010), 2524-2531.

[6] L. Ciric, B. Damjanovic, M. Jleli and B. Samet, **Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications**, Fixed Point Theory Appl. 2012, 51.

[7] B. Deshpande and A. Handa, **Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations,** Afr. Mat. Vol. 26 (3-4) (2015), 317-343.

[8] H. S. Ding, L. Li and S. Radenovic, **Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces**, Fixed Point Theory Appl. 2012, 96.

[9] V. Lakshmikantham and L. Ciric, **Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces**, Nonlinear Anal. 70 (12) (2009), 4341-4349.

[10] W. Long, S. Shukla and S. Radenovic, **Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces**, Fixed Point Theory Appl. 2013, 145.

[11] N. V. Luong and N. X. Thuan, **Coupled fixed points in partially ordered metric spaces and application**, Nonlinear Anal. 74 (2011), 983-992.

[12] M. Jain, K. Tas, S. Kumar and N. Gupta, **Coupled common fixed point results involving a $\varphi-\psi$ contractive condition for mixed g-monotone operators in partially ordered metric spaces**, J. Inequal. Appl. 2012, 285.

[13] J. T. Markin, **Continuous dependence of fixed point sets**, Proc. Amer. Math. Soc. 38 (1947), 545-547.

[14] S. B. Nadler, **Multivalued contraction mappings**, Pacific J. Math. 30 (1969), 475-488.

[15] J. Rodriguez-Lopez and S. Romaguera, **The Hausdorff fuzzy metric on compact sets**, Fuzzy Sets Syst. 147 (2004), 273–283.

[16] B. Samet, **Coupled fixed point theorems for generalized Meir-Keeler contraction in partially ordered metric spaces**, Nonlinear Anal. 72 (2010), 4508-4517.

[17] B. Samet and H. Yazidi, **Coupled fixed point theorems in partially ordered F-chainable metric spaces**, TJMCS 1 (2010), 142-151.

[18] B. Samet and C. Vetro, **Coupled fixed point, F-invariant set and fixed point of N-order**, Ann. Funct. Anal. 1 (2010), 46-56.

[19] B. Samet, E. Karapinar, H. Aydi and V. C. Rajic, **Discussion on some coupled fixed point theorems**, Fixed Point Theory Appl. 2013, 50.

[20] W. Sintunavarat, P. Kumam and Y. J. Cho, **Coupled fixed point theorems for nonlinear contractions without mixed monotone property**, Fixed Point Theory Appl. 2012, 170.

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