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A NEW VIEW POINT IN THE STUDY OF $\star$-CONTINUITY FOR MULTIFUNCTIONS

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Abstract. We introduce and study a unified form (called $m^\star$-I-continuity) of $\star$-continuity [9], $\alpha(\star)$-continuity [10], $\beta(\star)$-continuity [11] and other continuity for multifunctions in ideal topological spaces.

1. Introduction

Semi-open sets, preopen sets, α-sets, β-open sets and b-open sets play an important part in the researches of generalizations of continuity for functions and multifunctions in topological spaces. By using these sets, various types of continuous multifunctions are introduced and studied. The notions of minimal structures, m-continuity, and M-continuity are introduced in [34] and [35]. By using these notions, the present authors unified the theory of continuity for multifunctions in [30], [31] and other papers.

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The notion of ideal topological spaces were introduced in [24] and [39]. As generalizations of open sets, the notion of I-open sets, semi-I-open sets, pre-I-open sets, α-I-open sets, β-I-open sets and b-I-open sets are introduced. The notion of I-continuous multifunctions is introduced in [2]. Quite recently, other generalizations of continuous multifunctions in ideal topological spaces are introduced and studied in [3], [4], [6], [7], [38].

Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and $F : (X, \tau) \to (Y, \sigma)$ presents a multi-valued function. For a multifunction $F : (X, \tau) \to (Y, \sigma)$, we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Recently, Boonpok introduced and studied $\star$-continuity [9], $\alpha(\star)$-continuity [10], and $\beta(\star)$-continuity [11] for multifunctions in ideal topological spaces. In this paper, we introduce the notion of upper and lower $m^\star$-I-continuous multifunctions and generalize upper and lower $\star$-continuous, upper and lower $\alpha(\star)$-continuous, upper and lower $\beta(\star)$-continuous multifunctions. We deduce the study of $m^\star$-I-continuity to the study of m-continuity for multifunctions.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [29] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [25] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [27] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *b-open* [5] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(5) *β-open* [1] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all α-open (resp. semi-open, preopen, b-open, β-open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\beta(X)$).

Definition 2.2. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [34], [35] if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an m-space. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed.*

Definition 2.3. Let (X, m) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [26] as follows:
(1) $\mathrm{mCl}(A) = \bigcap\{F : A \subset F, X - F \in m\}$,
(2) $\mathrm{mInt}(A) = \bigcup\{U : U \subset A, U \in m\}$.

Lemma 2.1. ([26]) *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$*,*
(2) If $(X \setminus A) \in m$*, then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$*, then* $\mathrm{mInt}(A) = A$*,*
(3) $\mathrm{mCl}(\emptyset) = \emptyset$ *and* $\mathrm{mCl}(X) = X$*,* $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$*,*
(4) If $A \subset B$*, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*
(5) $\mathrm{mInt}(A) \subset A \subset \mathrm{mCl}(A)$*,*
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$*.*

Definition 2.4. A minimal structure m on a nonempty set X is said to have *property $\mathcal{B}$* [26] if the union of any family of subsets belonging to m belongs to m.

Remark 2.1. Let (X, τ) be a topological space. Then the families $\alpha(X)$, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, and $\beta(X)$ are all minimal structures with property $\mathcal{B}$.

Lemma 2.2. ([36]) *Let (X, m) be an m-space and m have property $\mathcal{B}$. For a subset A of X, the following properties hold:*
(1) $A \in m$ *if and only if* $\mathrm{mInt}(A) = A$*,*
(2) A is m-closed if and only if $\mathrm{mCl}(A) = A$*,*
(3) $\mathrm{mInt}(A) \in m$ *and* $\mathrm{mCl}(A)$ *is m-closed.*

Definition 2.5. ([30]) Let (X, m) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m) \to (Y, \sigma)$ is said to be
(1) *upper m-continuous* at $x \in X$ if for each open set V of Y containing $F(x)$, there exists an m-open set U of X containing x such that $F(U) \subset V$,
(2) *lower m-continuous* at $x \in X$ if for each open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists $U \in m$ containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.
(3) F is *upper/lower m-continuous* if it has the property at each point $x \in X$.

If m has property $\mathcal{B}$, then by Lemma 2.2 we obtain the following theorems.

Theorem 2.1. ([30]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, where (X, m) has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is upper m-continuous;
(2) $F^+(V)$ is m-open for every $V \in \sigma$;
(3) $F^-(K)$ is m-closed for every closed set K of Y;
(4) $\mathrm{mCl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(B))$ for every subset B of Y.

Theorem 2.2. ([30]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, where (X, m) has property $\mathcal{B}$, the following properties are equivalent:*
(1) F is lower m-continuous;
(2) $F^-(V)$ is m-open for every open set V of Y;
(3) $F^+(K)$ is m-closed for every closed set K of Y;
(4) $\mathrm{mCl}(F^+(B)) \subset F^+(\mathrm{Cl}(B))$ for every subset B of Y;
(5) $F(\mathrm{mCl}(A)) \subset \mathrm{Cl}(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(B))$ for every subset B of Y.

For a multifunction $F : (X, m) \to (Y, \sigma)$, $D^+_m(F)$ and $D^-_m(F)$ are defined in [31] as follows:

$$D^+_m(F) = \{x \in X : F \text{ is not upper } m\text{-continuous at } x\},$$
$$D^-_m(F) = \{x \in X : F \text{ is not lower } m\text{-continuous at } x\}.$$

Theorem 2.3. ([31]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^+_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^+(G) - \mathrm{mInt}(F^+(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^+(\mathrm{Int}(B)) - \mathrm{mInt}(F^+(B))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl}(F^-(B)) - F^-(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}(F^-(H)) - F^-(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

Theorem 2.4. ([31]) *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D^-_m(F) &= \textstyle\bigcup_{G\in\sigma}\{F^-(G) - \mathrm{mInt}(F^-(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^-(\mathrm{Int}(B)) - \mathrm{mInt}(F^-(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl}(F^+(B)) - F^+(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mCl}(A) - F^+(\mathrm{Cl}(F(A)))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl}(F^+(H)) - F^+(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

3. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals has been introduced in [24] and [39] and further investigated in [21]

Definition 3.1. A nonempty collection I of subsets of a set X is called an *ideal* on X [24], [39] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [20], [21]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 3.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ implies $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ and $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 3.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *I-open* [22] if $A \subset \mathrm{Int}(A^\star)$,
(2) *α-I-open* [18] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(3) *semi-I-open* [18] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(4) *pre-I-open* [13] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(5) *b-I-open* [12] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(6) *β-I-open* [19] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(7) *weakly semi-I-open* [16] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *weakly b-I-open* [28] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(9) *strongly β-I-open* [17] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$
(10) *semi$^\star_I$-open* [15] if $A \subset \mathrm{Cl}(\mathrm{Int}^\star(A))$,
(11) *pre$^\star_I$-open* [14] if $A \subset \mathrm{Int}^\star(\mathrm{Cl}(A))$,
(12) *$\beta^\star_I$-open* [14] if $A \subset \mathrm{Cl}(\mathrm{Int}^\star(\mathrm{Cl}(A)))$.

The family of all $\tau^\star$-open (resp. I-open, α-I-open, semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open, semi$^\star_I$-open, pre$^\star_I$-open, $\beta^\star_I$-open) sets in an ideal topological space (X, τ, I) is denoted by $\tau^\star$ (resp. $\mathrm{IO}(X)$, $\alpha\mathrm{IO}(X)$,

SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X), S*IO(X), P*IO(X), $\beta^\star$IO(X)).

Definition 3.3. By mIO(X), we denote each one of the families $\tau^\star$, IO(X), αIO(X), SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X), S*IO(X), P*IO(X), and $\beta^\star$IO(X).

Lemma 3.2. *Let* (X, τ, I) *be an ideal topological space. Then* mIO*(X) is a minimal structure and has property* $\mathcal{B}$.

Proof. The proofs of (3)-(9) follow from Lemma 3.2 of [37]. Since $\tau^\star$ is a topology, it is obvious. The proofs of (10), (11) and (12) are similar to Lemma 3.2 of [37].

Definition 3.4. Let (X, τ, I) be an ideal topological space. For a subset A of X, $\mathrm{mCl_I}(A)$ and $\mathrm{mInt_I}(A)$ as follows:
(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO(X)}\}$,
(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO(X)}\}$.

Let (X, τ, I) be an ideal topological space and mIO(X) the m-structure on X. If mIO(X) $= \tau^\star$ (resp. IO(X), αIO(X), SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X), S*IO(X), P*IO(X), $\beta^\star$IO(X)), then we have
(1) $\mathrm{mCl_I}(A) = \mathrm{Cl}^\star(A)$ (resp. $\mathrm{Cl_I}(A)$, $\alpha\mathrm{Cl_I}(A)$, $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$), $s^\star\mathrm{Cl_I}(A)$, $p^\star\mathrm{Cl_I}(A)$, $\beta^\star\mathrm{Cl_I}(A)$,
(2) $\mathrm{mInt_I}(A) = \mathrm{Int}^\star(A)$ (resp. $\mathrm{Int_I}(A)$, $\alpha\mathrm{Int_I}(A)$ (resp. $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$), $s^\star\mathrm{Int_I}(A)$, $p^\star\mathrm{Int_I}(A)$, $\beta^\star\mathrm{Int_I}(A)$.

4. m-I-continuous multifunctions

Definition 4.1. A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be
(1) *upper $m^\star$-I-continuous* at $x \in X$ if for each $\sigma^\star$-open set V of Y such that $F(x) \subset V$ there exists $U \in$ mIO(X) containing x such that $F(U) \subset V$,
(2) *lower $m^\star$-I-continuous* at $x \in X$ if for each $\sigma^\star$-open set V of Y such that $V \cap F(x) \neq \emptyset$, there exists $U \in$ mIO(X) containing x such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.
(3) F is *upper/lower $m^\star$-I-continuous* if it has the property at each point $x \in X$.

Lemma 4.1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties hold: $F : (X, \tau, I) \to (Y, \sigma, J)$ is upper/lower $m^\star$-I-continuous at $x \in X$ if and only if $F : (X, \mathrm{mIO}(X)) \to (Y, \sigma^\star)$ is upper/lower m-continuous at $x \in X$.*

Proof. The proof is obvious from the definition.

Remark 4.1. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\alpha\mathrm{IO}(X)$, $\beta\mathrm{IO}(X)$), then we obtain Definition 3.1 of [9] (resp. Definition 3.1 of [10], Definition 3.1 of [11]),

By Lemma 4.1, Theorems 2.1 and 2.2, we obtain the following theorems:

Theorem 4.1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is upper $m^\star$-I-continuous;
(2) $F^+(V)$ is m-I-open for every $\sigma^\star$-open set V of Y;
(3) $F^-(K)$ is m-I-closed for every $\sigma^\star$-closed set K of Y;
(4) $\mathrm{mCl_I}(F^-(B)) \subset F^-\mathrm{Cl}^\star(B))$ for every subset B of Y;
(5) $F^+(\mathrm{Int}^\star(B)) \subset \mathrm{mInt_I}(F^+(B))$ for every subset B of Y.

If $\mathrm{mIO}(X) = \tau^\star$ in Theorem 4.1, then we obtain the following corollary.

Corollary 4.1. ([9]) *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is upper $\star$-I-continuous;
(2) $F^+(V)$ is $\tau^\star$-open in X for every $\sigma^\star$-open set V of Y;
(3) $F^-(K)$ is $\tau^\star$-closed in X for every $\sigma^\star$-closed set K of Y;
(4) $\mathrm{Cl}^\star(F^-(B)) \subset F^-\mathrm{Cl}^\star(B))$ for every subset B of Y;
(5) $F^+(\mathrm{Int}^\star(B)) \subset \mathrm{Int}^\star(F^+(B))$ for every subset B of Y.

Theorem 4.2. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is lower $m^\star$-I-continuous;
(2) $F^-(V)$ is m-I-open for every $\sigma^\star$-open set V of Y;
(3) $F^+(K)$ is m-I-closed for every $\sigma^\star$-closed set K of Y;
(4) $\mathrm{mCl_I}(F^+(B)) \subset F^+(\mathrm{Cl}^\star(B))$ for every subset B of Y;
(5) $F(\mathrm{mCl_I}(A)) \subset \mathrm{Cl}^\star(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{Int}^\star(B)) \subset \mathrm{mInt_I}(F^-(B))$ for every subset B of Y.

Remark 4.2. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\alpha\mathrm{IO}(X)$, $\beta\mathrm{IO}(X)$), then by Theorem 4.2 we obtain characterizations of lower $\star$-continuity in Theorem 3.2 of [9] (resp. lower $\alpha(\star)$-continuity in Theorem 3.4 (1), (2),

(3) of [10], lower $\beta(\star)$-continuity in Theorem 3.9 (1), (2), (3), (4), (8) of [11]).

For a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$, we define $D^{+}_{m^{\star}IO}(F)$ and $D^{-}_{m^{\star}IO}(F)$ as follows:

$D^{+}_{m^{\star}IO}(F)=\{x\in X: F$ is not upper $m^{\star}$-I-continuous at $x\}$,
$D^{-}_{m^{\star}IO}(F)=\{x\in X: F$ is not lower $m^{\star}$-I-continuous at $x\}$.

By Lemma 3.2, Theorems 2.3 and 2.4, we obtain the following theorem:

Theorem 4.3. *For a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$, the following properties hold:*

$$\begin{aligned} D^{+}_{m^{\star}IO}(F) &= \textstyle\bigcup_{G\in\sigma^{\star}}\{F^{+}(G)-\mathrm{mInt_I}(F^{+}(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^{+}(\mathrm{Int}^{\star}(B))-\mathrm{mInt_I}(F^{+}(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl_I}(F^{-}(B))-F^{-}(\mathrm{Cl}^{\star}(B))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl_I}(F^{-}(H))-F^{-}(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of $\sigma^{\star}$-closed sets of (Y,σ,J).

Theorem 4.4. *For a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$, the following properties hold:*

$$\begin{aligned} D^{-}_{m^{\star}IO}(F) &= \textstyle\bigcup_{G\in\sigma^{\star}}\{F^{-}(G)-\mathrm{mInt_I}(F^{-}(G))]\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{F^{-}(\mathrm{Int}^{\star}(B))-\mathrm{mInt_I}(F^{-}(B))\} \\ &= \textstyle\bigcup_{B\in\mathcal{P}(Y)}\{\mathrm{mCl_I}(F^{+}(B))-F^{+}(\mathrm{Cl}^{\star}(B))\} \\ &= \textstyle\bigcup_{A\in\mathcal{P}(X)}\{\mathrm{mCl_I}(A)-F^{+}(\mathrm{Cl}^{\star}(F(A)))\} \\ &= \textstyle\bigcup_{H\in\mathcal{F}}\{\mathrm{mCl_I}(F^{+}(H))-F^{+}(H)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of $\sigma^{\star}$-closed sets of (Y,σ,J).

5. Some properties of $m^{\star}$-I-continuous multifunctions

Let (X,τ) be a topological space and A a subset of X. A point $x\in X$ is called a θ-closure point of A [40] if $\mathrm{Cl}(U)\cap A\neq\emptyset$ for every open set U containing x. The set of all θ-closure points of A is called the *θ-closure* of A and is denoted by $\mathrm{Cl}_{\theta}(A)$. If $A=\mathrm{Cl}_{\theta}(A)$, then A is said to be *θ-closed*. The complement of a θ-closed set is said to be θ-open. It follows from [40] that the collection of all θ-open sets is a topology for X.

Lemma 5.1. ([32]) *Let (Y,σ) be a regular space and m a minimal structure having property $\mathcal{B}$. For a multifunction $F:(X,m)\to(Y,\sigma)$, the following properties are equivalent:*

(1) F is upper m-continuous;

(2) $F^{-}(\mathrm{Cl}_{\theta}(B))$ is m-closed for every subset B of Y;

(3) $F^-(K)$ *is m-closed for every* θ*-closed set* K *of* Y*;*
(4) $F^+(V)$ *is m-open for every* θ*-open set* V *of* Y*.*

Lemma 5.2. ([32]) *Let* (Y, σ) *be a regular space and* m *a minimal structure having property* $\mathcal{B}$*. For a multifunction* $F : (X, m) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) F *is lower m-continuous;*
(2) $F^+(\mathrm{Cl}_\theta(B))$ *is m-closed for every subset* B *of* Y*;*
(3) $F^+(K)$ *is m-closed for every* θ*-closed set* K *of* Y*;*
(4) $F^-(V)$ *is m-open for every* θ*-open set* V *of* Y*.*

Definition 5.1. An ideal topological space (Y, σ, J) is said to be $\sigma^\star$-*regular* if for each $\sigma^\star$-closed set F and each point $x \notin F$, there exist disjoint $\sigma^\star$-open sets U and V such that $x \in U$ and $F \subset V$.

Theorem 5.1. *Let* (Y, σ, J) *be a* $\sigma^\star$*-regular space. For a multifunction* $F : (X, \tau, I) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) F *is upper* $m^\star$*-I-continuous;*
(2) $F^-(\mathrm{Cl}^\star{}_\theta(B))$ *is m-I-closed for every subset* B *of* Y*;*
(3) $F^-(K)$ *is m-I-closed for every* $\theta^\star$*-closed set* K *of* Y*;*
(4) $F^+(V)$ *is m-I-open for every* $\theta^\star$*-open set* V *of* Y*.*

Proof. The proof follows from Lemma 5.1 and the fact that $\mathrm{mIO}(X)$ has property $\mathcal{B}$).

Theorem 5.2. *Let* (Y, σ, J) *be a* $\sigma^\star$*-regular space. For a multifunction* $F : (X, \tau, I) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) F *is lower* $m^\star$*-I-continuous;*
(2) $F^+(\mathrm{Cl}^\star{}_\theta(B))$ *is m-I-closed for every subset* B *of* Y*;*
(3) $F^+(K)$ *is m-I-closed for every* $\theta^\star$*-closed set* K *of* Y*;*
(4) $F^-(V)$ *is m-I-open for every* $\theta^\star$*-open set* V *of* Y*.*

Proof. The proof follows from Lemma 5.2 and the fact that $\mathrm{mIO}(X)$ has property $\mathcal{B}$).

Definition 5.2. A subset A of a topological space (Y, σ) is said to be
(1) α*-regular* [23] if for each $a \in A$ and each open set U containing a, there exists an open set G of Y such that $a \in G \subset \mathrm{Cl}(G) \subset U$,
(2) α*-paracompact* [41] if every Y-open cover of A has a Y-open refinement which covers A and is locally finite for each point of Y.

Definition 5.3. A subset A of an ideal topological space (Y, σ, J) is said to be
(1) $\alpha^\star$*-regular* if for each $a \in A$ and each $\sigma^\star$-open set U containing a, there exists a $\sigma^\star$-open set G of Y such that $a \in G \subset \mathrm{Cl}^\star(G) \subset U$,

(2) *$\alpha^\star$-paracompact* if every $\sigma^\star$-open cover of A has a $\sigma^\star$-open refinement which covers A and is $\sigma^\star$-locally finite for each point of Y.

For a multifunction $F : (X, \tau) \to (Y, \sigma)$, by $\mathrm{Cl}(F) : (X, \tau) \to (Y, \sigma)$ [8] we denote a multifunction defined as follows: $\mathrm{Cl}(F)(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we define $\mathrm{sCl}(F)$, $\mathrm{pCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $b\mathrm{Cl}(F)$. For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, similarly we define $\mathrm{Cl}^\star$, $\mathrm{sCl}^\star(F)$, $\mathrm{pCl}^\star(F)$, $\alpha\mathrm{Cl}^\star(F)$, $\beta\mathrm{Cl}^\star(F)$, and $\mathrm{bCl}^\star(F)$.

Lemma 5.3. ([33]) *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper m-continuous if and only if G is upper m-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$.*

Lemma 5.4. ([33]) *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction. Then F is lower m-continuous if and only if G is lower m-continuous, where $G = \mathrm{Cl}(F)$, $\mathrm{pCl}(F)$, $\mathrm{sCl}(F)$, $\alpha\mathrm{Cl}(F)$, $\beta\mathrm{Cl}(F)$, $\mathrm{bCl}(F)$.*

Theorem 5.3. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction such that $F(x)$ is $\alpha^\star$-regular and $\alpha^\star$-paracompact for each $x \in X$. Then F is upper $m^\star$-I-continuous if and only if G is upper $m^\star$-I-continuous, where $G = \mathrm{Cl}^\star(F)$, $\mathrm{pCl}^\star(F)$, $\mathrm{sCl}^\star(F)$, $\alpha\mathrm{Cl}^\star(F)$, $\beta\mathrm{Cl}^\star(F)$, $\mathrm{bCl}^\star(F)$.*

Theorem 5.4. *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction. Then F is lower $m^\star$-I-continuous if and only if G is lower $m^\star$-I-continuous, where $G = \mathrm{Cl}^\star(F)$, $\mathrm{pCl}^\star(F)$, $\mathrm{sCl}^\star(F)$, $\alpha\mathrm{Cl}^\star(F)$, $\beta\mathrm{Cl}^\star(F)$, $\mathrm{bCl}^\star(F)$.*

Definition 5.4. Let (X, τ, I) be an ideal topological space and A a subset of X. The *mI-frontier* of A, $\mathrm{mIFr}(A)$, is defined as follows: $\mathrm{mIFr}(A) = \mathrm{mCl_I}(A) \cap \mathrm{mCl_I}(X \setminus A)$.

Lemma 5.5. ([33]) *The set of all points $x \in X$ at which a multifunction $F : (X, m) \to (Y, \sigma)$ is not upper (resp. lower) m-continuous is identical with the union of the m-frontiers of the upper (resp. lower) inverse images of open sets containing (resp. meeting) F(x).*

Theorem 5.5. *The set of all points $x \in X$ at which a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is not upper (resp. lower) $m^\star$-I-continuous is identical with the union of the mI-frontiers of the upper (resp. lower) inverse images of $\sigma^\star$-open sets containing (resp. meeting) F(x).*

Definition 5.5. An m-space (X, m) is said to be *m-connected* [35] if X can not be written as the union of two nonempty disjoint m-open sets.

Definition 5.6. An ideal topological space $(Y.\sigma, J)$ is said to be $\sigma^\star$-*connected* if Y can not be written as the union of two nonempty disjoint $\sigma^\star$-open sets.

Lemma 5.6. ([30]) *Let (X, m) be an m-space, m have property $\mathcal{B}$ and (Y, σ) a topological space. If $F : (X, m_X) \to (Y, \sigma)$ is an upper or lower m-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m) is m-connected, then (Y, σ) is connected.*

Theorem 5.6. *If $F : (X, \tau, I) \to (Y, \sigma, J)$ is an upper or lower $m^\star$-I-continuous surjective multifunction such that $F(x)$ is $\sigma^\star$-connected for each $x \in X$ and $(X, \mathrm{mIO}(X))$ is mI-connected, then (Y, σ, J) is $\sigma^\star$-connected.*

Proof. The proof follows from Lemma 5.6 and the fact that $\mathrm{mIO}(X)$ has property $\mathcal{B}$.

Remark 5.1. If $\mathrm{mIO}(X) = \tau^\star$, then we obtain Theorem 3.15 of [9].

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