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SOME RESULTS ON THE HYPER-ORDER OF SOLUTIONS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. The purpose of this paper is the study of the growth of solutions of higher order linear differential equations

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(D_1 + B_1 e^{Q_1(z)}\right) f'$$
$$+ \left(D_0 + A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right) f = 0,$$

where $A_i(z)$ $(\not\equiv 0)$ $(i = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, ..., k-1)$, $D_m(z)$ $(m = 0, ..., k-1)$ are meromorphic functions of finite order less than n, $P_i(z) = a_{i,n}z^n + \cdots + a_{i,0}$ and $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ are polynomials with degree $n \geq 1$ such that $a_{i,q}, b_{j,q}$ $(i = 1, 2;\ j = 1, ..., k-1;\ q = 0, 1, ..., n)$ are complex numbers. Our results extend the previous results due to Habib and Belaïdi [3], [11], [12] and Beddani and Hamani [4].

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1. Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory, see [14], [20]. Let $\rho(f)$ denote the order of growth of a meromorphic function f and the hyper-order of f is defined by

$$\rho_2(f) := \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f, see [14], [15], [20].

Definition 1.1 ([16], [18]) Let f be a meromorphic function. Then, the convergence exponent of the zero-sequence of a meromorphic function f is defined by

$$\lambda(f) := \limsup_{r\to+\infty} \frac{\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of f in $\{z : |z| \le r\}$, and the exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}(f) := \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z : |z| \le r\}$. The exponent of convergence of the pole-sequence of f is denoted by

$$\lambda\left(\frac{1}{f}\right) := \limsup_{r\to+\infty} \frac{\log N(r,f)}{\log r},$$

where $N(r,f)$ is the integrated counting function of poles of f in $\{z : |z| \le r\}$. The hyper convergence exponents of zero-sequence and the distinct zeros of f are defined respectively by

$$\lambda_2(f) := \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},\ \overline{\lambda}_2(f) := \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

Definition 1.2 ([7]) Let f be a meromorphic function. Then, the exponent of convergence of the sequence of distinct fixed points of f

is defined by

$$\overline{\tau}(z) = \overline{\lambda}(f - z) := \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-z}\right)}{\log r}.$$

We also define

$$\overline{\tau}(\varphi) = \overline{\lambda}(f - \varphi) := \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f-\varphi}\right)}{\log r}$$

for any meromorphic function φ.

In [11], Habib and Belaïdi have investigated the order and the hyper-order of solutions of some higher order linear differential equations and obtained the following result.

Theorem A ([11]) *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$, $D_m(z)$ $(m = 0, ..., k-1)$ *be entire functions with*

$$\max\{\rho(A_j), \rho(B_l), \rho(D_m)\} < 1,$$

b_l $(l = 1, ..., k-1)$ *be complex constants such that:*
(i) $\arg b_l = \arg a_1$ *and* $b_l = c_l a_1$ $(0 < c_l < 1)\,(l \in I_1)$ *and* (ii) b_l *is a real constant such that* $b_l \le 0$ $(l \in I_2)$, *where* $I_1 \ne \emptyset$, $I_2 \ne \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, ..., k-1\}$ *and* a_1, a_2 *are complex numbers such that* $a_1 a_2 \ne 0$, $a_1 \ne a_2$ *(suppose that* $|a_1| \le |a_2|$*). If* $\arg a_1 \ne \pi$ *or* a_1 *is a real number such that* $a_1 < \frac{b}{1-c}$, *where* $c = \max\{c_l : l \in I_1\}$ *and* $b = \min\{b_l : l \in I_2\}$, *then every solution* $f \not\equiv 0$ *of the equation*

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{b_{k-1}z}\right) f^{(k-1)} + \cdots + \left(D_1 + B_1 e^{b_1 z}\right) f'$$

$$+ \left(D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.1}$$

satisfies $\rho(f) = +\infty$ *and* $\rho_2(f) = 1$.

In [3], they consider the relation between small functions with meromorphic solutions and their derivatives to complex higher order linear differential equations whose coefficients are meromorphic functions. Indeed, they obtained the following result.

Theorem B ([3]) *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, ..., k-1)$ *be meromorphic functions with*

$$\max\{\rho(A_j)\ \ (j = 1, 2), \rho(B_l)\ \ (l = 1, ..., k-1)\} < 1,$$

b_l $(l = 1, ..., k-1)$ be complex constants such that (i) $b_l = c_l a_1$ $(0 < c_l < 1)$ *$(l \in I_1)$ and* (ii) *b_l is a real constant such that $b_l < 0$ $(l \in I_2)$, where $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, ..., k-1\}$, and a_1, a_2 are complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or a_1 is a real number such that $a_1 < \frac{b}{1-c}$, where $c = \max\{c_l : l \in I_1\}$ and $b = \min\{b_l : l \in I_2\}$. If $\varphi(\not\equiv 0)$ is a meromorphic function with order $\rho(\varphi) < 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of equation*

$$f^{(k)} + B_{k-1} e^{b_{k-1} z} f^{(k-1)} + \cdots + B_1 e^{b_1 z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0 \tag{1.2}$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty.$

In the paper [12], the authors have investigated the fixed points of solutions, their first and second derivatives and proved:

Theorem C ([12]) *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, 2)$ and $B_l(z)$ $(l = 2, ..., k-1)$ be meromorphic functions with*

$$\max\{\rho(A_j) \ (j = 0, 1, 2), \rho(B_l)\,(l = 2, ..., k-1)\} < 1$$

and a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$), let a_0 be a constant satisfying $a_0 < 0$ such that $\arg a_1 \neq \pi$ *or $a_1 < a_0$. If $f\,(\not\equiv 0)$ is any meromorphic solution whose poles are of uniformly bounded multiplicities of equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f'' + A_0 e^{a_0 z} f' + \left(A_1 e^{a_1 z} + A_2 e^{a_2 z}\right) f = 0,$$

then f, f', f'' all have infinitely many fixed points and $\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \infty$.

Recently, Beddani and Hamani [4] have investigated the growth of solutions of more general higher order linear differential equations and obtained the following result.

Theorem D ([4]) *Let $k \geq 2$ be an integer, $P_s(z) = \sum_{i=0}^{n} a_{s,i} z^i$ $(s = 1, 2)$, $Q_j(z) = \sum_{i=0}^{n} b_{j,i} z^i$ $(j = 1, ..., k-1)$ be polynomials with degree $n \geq 1$, where $a_{s,0}, ..., a_{s,n}$ $(s = 1, 2)$, $b_{j,0}, ..., b_{j,n}$ $(j = 1, ..., k-1)$ are complex numbers such that $a_{s,n} = |a_{s,n}| e^{i\theta_s} \neq 0$ $(s = 1, 2)$, $\theta_s \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $a_{1,n} \neq a_{2,n}$ (suppose that $|a_{1,n}| \leq |a_{2,n}|$). Let $A_s(z)$ $(\not\equiv 0)$ $(s = 1, 2)$,*

$B_j(z)$ $(\not\equiv 0)$ $(j = 1, ..., k-1)$, $D_m(z)\,(m = 0, ..., k-1)$ *be meromorphic functions with*

$$\max\{\rho(A_s), \rho(B_j), \rho(D_m)\} < n.$$

Let I *and* J *be two sets satisfying* $I \neq \emptyset$, $J \neq \emptyset$, $I \cap J = \emptyset$, $I \cup J = \{1, ..., k-1\}$ *such that for* $j \in I$, $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)$ *and for* $j \in J$, $b_{j,n} < 0$. *If* $\arg a_{1,n} = \theta_1 \neq \pi$ *or* $a_{1,n}$ *is a real number such that* $a_{1,n} < \frac{b}{1-c}$, *where* $c = \max\{c_j : j \in I\}$ *and* $b = \min\{b_{j,n} : j \in J\}$, *then every meromorphic solution* $f \not\equiv 0$ *of equation*

$$f^{(k)} + \left(D_{k-1} + B_{k-1}e^{Q_{k-1}(z)}\right) f^{(k-1)} + \cdots + \left(D_1 + B_1 e^{Q_1(z)}\right) f'$$

$$+\left(D_0 + A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right) f = 0 \tag{1.3}$$

is of infinite order and satisfies $\rho_2(f) \geq n$. *Furthermore, if* $\lambda\left(\frac{1}{f}\right) < +\infty$, *then* $\rho_2(f) = n$.

The main purpose of this paper is to extend and improve the results of theorems A, B, C and D to some higher order linear differential equations. In fact we will prove the following results.

Theorem 1.1 *Let* $A_i(z)$ $(\not\equiv 0)$ $(i = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, ..., k-1)$, $D_m(z)\,(m = 0, ..., k-1)$ *be meromorphic functions with*

$$\max\{\rho(A_i), \rho(B_j), \rho(D_m)\} < n,$$

and $P_i(z) = a_{i,n}z^n + \cdots + a_{i,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials with degree* $n \geq 1$, *where* $a_{i,q}$, $b_{j,q}$ $(i = 1, 2;\ j = 1, ..., k-1;\ q = 0, 1, ..., n)$ *are complex numbers with* $a_{i,n} \neq 0$ *such that:*
(i) $\arg b_{j,n} = \arg a_{1,n}$ *and* $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)\,(j \in I_1)$ *and* (ii) $b_{j,n}$ *be real constants such that* $b_{j,n} \leq 0$ $(j \in I_2)$, *where* $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, ..., k-1\}$ *and* $a_{j,n}$ *are complex numbers such that* $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ *(suppose that* $|a_{1,n}| \leq |a_{2,n}|$ *). If* $\arg a_{1,n} \neq \pi$ *or* $a_{1,n}$ *is a real number such that* $a_{1,n} < \frac{b}{1-c}$, *where* $c = \max\{c_j : j \in I_1\}$ *and* $b = \min\{b_{j,n} : j \in I_2\}$, *then every meromorphic solution* $f \not\equiv 0$ *of the equation* (1.3) *whose poles are of uniformly bounded multiplicities satisfies* $\rho(f) = +\infty$ *and* $\rho_2(f) = n$.

Remark 1.1 Clearly, Theorem 1.1 is an extension of Theorem A from entire solutions of equation (1.1) to the case of meromorphic solutions of equation (1.3) with meromorphic coefficients instead of entire coefficients. Furthermore, we have changed the conditions "

$b_{j,n} < 0$ " and " $\lambda\left(\frac{1}{f}\right) < +\infty$ " in Theorem D by " $b_{j,n} \leq 0$ " and " every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities " in Theorem 1.1.

Corollary 1.1 *Let* $A_i(z)$ $(\not\equiv 0)$ $(i = 1, 2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1, ..., k-1)$, $D_m\,(m = 0, ..., k-1)$ *be entire functions with*

$$\max\{\rho(A_i), \rho(B_j), \rho(D_m)\} < n,$$

and $P_i(z) = a_{i,n}z^n + \cdots + a_{i,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *be polynomials with degree* $n \geq 1$, *where* $a_{i,q}$, $b_{j,q}$ $(i = 1, 2; j = 1, ..., k-1; q = 0, 1, ..., n)$ *are complex numbers with* $a_{i,n} \neq 0$ *such that:*
(i) $\arg b_{j,n} = \arg a_{1,n}$ *and* $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)$ $(j \in I_1)$ *and* (ii) $b_{j,n}$ *be real constants such that* $b_{j,n} \leq 0$ $(j \in I_2)$, *where* $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, ..., k-1\}$ *and* $a_{j,n}$ *are complex numbers such that* $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ *(suppose that* $|a_{1,n}| \leq |a_{2,n}|$ *). If* $\arg a_{1,n} \neq \pi$ *or* $a_{1,n}$ *is a real number such that* $a_{1,n} < \frac{b}{1-c}$, *where* $c = \max\{c_j : j \in I_1\}$ *and* $b = \min\{b_{j,n} : j \in I_2\}$, *then every solution* $f \not\equiv 0$ *of the equation* (1.3) *satisfies* $\rho(f) = +\infty$ *and* $\rho_2(f) = n$.

Example 1.1 Consider the following differential equation

$$f^{(3)} - 2ize^{iz^2}f'' + \left(4z^2 + 6i + ze^{-4z^2}\right)f'$$

$$-\left(\left(16iz^3 + 8z\right)e^{2iz^2} + 2iz^2e^{(-4+i)z^2}\right)f = 0. \tag{1.4}$$

Set

$$\begin{cases} A_1(z) = -16iz^3 - 8z, A_2(z) = -2iz^2, \\ B_1(z) = z, B_2(z) = -2iz, \\ D_0(z) \equiv 0, D_1(z) = 4z^2 + 6i, D_2(z) \equiv 0 \end{cases}$$

and

$$\begin{cases} P_1(z) = 2iz^2, \\ P_2(z) = (-4+i)z^2, \\ Q_1(z) = -4z^2, \\ Q_2(z) = iz^2. \end{cases}$$

We have $a_{12} = 2i, a_{22} = -4 + i, b_{12} = -4, b_{22} = i$, we can see that

$$\begin{cases} \arg a_{12} = \arg b_{22} = \frac{\pi}{2}, b_{22} = i = \frac{1}{2}a_{12}, c_1 = \frac{1}{2}, 0 < c_1 < 1, \\ b_{12} < 0 \end{cases}$$

and

$$\max\{\rho(A_i)(i = 1, 2), \rho(B_j)(j = 1, 2), \rho(D_m)(m = 0, 1, 2)\} = 0 < 2.$$

Then, according to Corollary 1.1, every solution $f \not\equiv 0$ of the equation (1.4) satisfies $\rho(f) = +\infty$ and $\rho_2(f) = 2$. We can see that $f(z) = e^{e^{iz^2}}$ represents a solution of equation (1.4) which verifies $\rho(f) = +\infty$ and $\rho_2(f) = 2$.

Example 1.2 Consider the following differential equation

$$\begin{aligned} f^{(3)} - \left(z + 1 + \frac{i}{z}e^{iz}\right) f'' + \Bigg[4z^2 + \frac{\cos z}{z^2} - \frac{4}{z}\sin z + \frac{z+1}{z} \\ +i\left(-6 + \frac{4}{z} + \frac{\sin z}{z^2} + z^2 + z\right) + (z^2+1)e^{iz^2}\Bigg] f' \\ + \Big[\left(-4z^3 - 4z^2 - 4\cos z + 12z + i\left(-4\sin z + 18z^3 + 2z\right)\right) e^{2iz^2} \\ +8iz^3 e^{3iz^2}\Big] f = 0. \end{aligned} \tag{1.5}$$

Set

$$\begin{cases} A_1(z) = -4z^3 - 4z^2 - 4\cos z + 12z + i\left(-4\sin z + 18z^3 + 2z\right), \\ A_2(z) = 8iz^3, \\ B_1(z) = z^2 + 1, B_2(z) = -\frac{i}{z}, \\ D_0(z) \equiv 0, \\ D_1(z) = 4z^2 + \frac{\cos z}{z^2} - \frac{4}{z}\sin z + \frac{z+1}{z} + i\left(-6 + \frac{4}{z} + \frac{\sin z}{z^2} + z^2 + z\right), \\ D_2(z) = -z - 1 \end{cases}$$

and

$$\begin{cases} P_1(z) = 2iz^2, \\ P_2(z) = 3iz^2, \\ Q_1(z) = iz^2, \\ Q_2(z) = iz. \end{cases}$$

We have $a_{12} = 2i, a_{22} = 3i, b_{12} = i, b_{22} = 0$, we can see that

$$\begin{cases} \arg a_{12} = \arg b_{12} = \frac{\pi}{2}, b_{12} = i = \frac{1}{2}a_{12}, c_1 = \frac{1}{2}, 0 < c_1 < 1, \\ b_{22} = 0 \end{cases}$$

and

$$\max\{\rho(A_i)(i=1,2), \rho(B_j)(j=1,2), \rho(D_m)(m=0,1,2)\} = 1 < 2.$$

Then, according to Theorem 1.1, every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of the equation (1.5) satisfies $\rho(f) = +\infty$ and $\rho_2(f) = 2$. We can see that $f(z) = e^{e^{iz^2}}$ represents a solution of equation (1.5) which verifies $\rho(f) = +\infty$ and $\rho_2(f) = 2$. Note that in this case we cannot apply Theorem D because $b_{22} = 0$.

Corollary 1.2 *Let $A_i(z)$ $(\not\equiv 0)$ $(i = 1,2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1,...,k-1)$, $D_m(z)$ $(m = 0,...,k-1)$ be meromorphic functions with*

$$\max\{\rho(A_i), \rho(B_j), \rho(D_m)\} < n$$

and $P_i(z) = a_{i,n}z^n + \cdots + a_{i,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials with degree $n \geq 1$, where $a_{i,q}$, $b_{j,q}$ $(i = 1,2;\ j = 1,...,k-1;\ q = 0,1,...,n)$ are complex numbers with $a_{i,n} \neq 0$ such that $\arg b_{j,n} = \arg a_{1,n}$ and $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)$ $(j = 1,...,k-1)$, where $a_{j,n}$ are complex numbers such that $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ (suppose that $|a_{1,n}| \leq |a_{2,n}|$). If $\arg a_{1,n} \neq \pi$ or $a_{1,n}$ is a real number such that $a_{1,n} < 0$, then every meromorphic solution $f \not\equiv 0$ of the equation (1.3) whose poles are of uniformly bounded multiplicities satisfies $\rho(f) = +\infty$ and $\rho_2(f) = n$.

Corollary 1.3 *Let $A_i(z)$ $(\not\equiv 0)$ $(i = 1,2)$, $B_j(z)$ $(\not\equiv 0)$ $(j = 1,...,k-1)$, $D_m(z)$ $(m = 0,...,k-1)$ be meromorphic functions with*

$$\max\{\rho(A_i), \rho(B_j), \rho(D_m)\} < n$$

and $P_i(z) = a_{i,n}z^n + \cdots + a_{i,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ be polynomials with degree $n \geq 1$, where $a_{i,q}$, $b_{j,q}$ $(i = 1,2;\ j = 1,...,k-1;\ q = 0,1,...,n)$ are complex numbers with $a_{i,n} \neq 0$ such that $b_{j,n}$ $(j = 1,...,k-1)$ are real constants satisfying $b_{j,n} \leq 0$, $a_{j,n}$ are complex numbers such that $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ (suppose that $|a_{1,n}| \leq |a_{2,n}|$). If $\arg a_{1,n} \neq \pi$ or $a_{1,n}$ is a real number such that $a_{1,n} < b$, where $b = \min\{b_{j,n},\ j = 1,...,k-1\}$, then every meromorphic solution $f \not\equiv 0$ of the equation (1.3) whose poles are of uniformly bounded multiplicities satisfies $\rho(f) = +\infty$ and $\rho_2(f) = n$.

Theorem 1.2 *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1,2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1,...,k-1)$, $D_m(z)$ $(m = 0,...,k-1)$ satisfy the additional hypotheses of Theorem 1.1. If $\varphi(\not\equiv 0)$ is a meromorphic function with order $\rho(\varphi) < n$, then every meromorphic solution $f \not\equiv 0$ whose poles are of uniformly bounded multiplicities of equation (1.3) satisfies*

$$\overline{\lambda}(f-\varphi) = \overline{\lambda}(f'-\varphi) = \overline{\lambda}(f''-\varphi) = \infty$$

and

$$\overline{\lambda}_2(f-\varphi) = \overline{\lambda}_2(f'-\varphi) = \overline{\lambda}_2(f''-\varphi) = n.$$

Remark 1.2 Obviously, Theorem 1.2 is an extension of Theorem B from meromorphic solutions of equation (1.2) to the case of meromorphic solutions of equation (1.3).

Corollary 1.4 *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1,2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1,...,k-1)$, $D_m(z)\,(m = 0,...,k-1)$ *satisfy the additional hypotheses of Theorem 1.1. If* $f(\not\equiv 0)$ *is a meromorphic solution whose poles are of uniformly bounded multiplicities of equation* (1.3)*, then* f, f', f'' *all have infinitely many fixed points and satisfy* $\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \infty$, $\overline{\tau}_2(f) = \overline{\tau}_2(f') = \overline{\tau}_2(f'') = n$.

2. Lemmas for the proofs of the theorems

First, we recall the following definitions. The linear measure of a set $E \subset [0,+\infty)$ is defined as $m(E) = \int_0^{+\infty} \chi_E(t)\,dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ is defined by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where $\chi_H(t)$ is the characteristic function of a set H.

Lemma 2.1 ([1]) *Let* $P_j(z)$ $(j = 0,1,...,k)$ *be polynomials with* $\deg P_0 = n$ $(n \geq 1)$ *and* $\deg P_j \leq n$ $(j = 1,2,...,k)$*. Let* $A_j(z)(j = 0,1,...,k)$ *be meromorphic functions with finite order and*

$$\max\{\rho(A_j) : j = 0,1,...,k\} < n$$

such that $A_0(z) \not\equiv 0$*. We denote*

$$F(z) = A_k e^{P_k(z)} + A_{k-1}e^{P_{k-1}(z)} + \cdots + A_1 e^{P_1(z)} + A_0 e^{P_0(z)}.$$

If $\deg(P_0(z) - P_j(z)) = n$ *for all* $j = 1,...,k$*, then* F *is a nontrivial meromorphic function with finite order and satisfies* $\rho(F) = n$*.*

Lemma 2.2 ([6]) *Let* f *be a meromorphic function of order* $\rho(f) = \rho < \infty$*. Then, for any given* $\varepsilon > 0$*, there exists a set* $E_1 \subset (1,+\infty)$ *that has finite linear measure and finite logarithmic measure, such that*

$$|f(z)| \leq \exp\left\{r^{\rho+\varepsilon}\right\}$$

holds for $|z| = r \notin [0,1] \cup E_1$, $r \to +\infty$*.*

Lemma 2.3 ([9]) *Let* f *be a transcendental meromorphic function of finite order* ρ*. Let* $\varepsilon > 0$ *be a constant,* k *and* j *be integers satisfying* $k > j \geq 0$*. Then, the following statements hold:*

(i) *There exists a set* $E_2 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ *with linear measure zero, such that, if* $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E_2$*, then there is a constant* $R_0 = R_0(\theta) > 1$*, such that for all* z *satisfying* $\arg z = \theta$ *and* $|z| = r \geq R_0$*, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

(ii) *There exists a set $E_3 \subset [1, +\infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_3 \cup [0, 1]$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)} .$$

(iii) *There exists a set $E_4 \subset [0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| \notin E_4$ for all $k > j \geq 0$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho+\varepsilon)} .$$

Lemma 2.4 ([17]) *Suppose that $n \geq 0$ is an integer. Let $P_j(z) = a_{jn} z^n + \cdots$ $(j = 1, 2)$ be nonconstant polynomials, where a_{jq} $(q = 0, 1, ..., n)$ are complex numbers and $a_{1n} a_{2n} \neq 0$. Set $z = re^{i\theta}, a_{jn} = |a_{jn}| e^{i\theta_j}$, $\theta_j \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\delta(P_j, \theta) = |a_{j,n}| \cos(n\theta + \theta_j)$, then there is a set $E_5 \subset [-\frac{\pi}{2n}, \frac{3\pi}{2n})$ that has linear measure zero. If $\theta_1 \neq \theta_2$, then there exists a ray $\arg z = \theta$, $\theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}) \backslash (E_5 \cup E_6)$, such that*

$$\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$$

or

$$\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0,$$

where $E_6 = \left\{\theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}) : \delta(P_j, \theta) = 0\right\}$ is a finite set, which has linear measure zero.

Remark 2.1 ([17]) *In Lemma 2.4, if $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \backslash (E_5 \cup E_6)$ is replaced by $\theta \in [\frac{\pi}{2n}, \frac{3\pi}{2n}) \backslash (E_5 \cup E_6)$, then we obtain the same result.*

Lemma 2.5 ([13]) *Let $P(z) = (\alpha + i\beta) z^n + \cdots$ $(\alpha, \beta$ are real numbers, $|\alpha| + |\beta| \neq 0)$ be a polynomial with degree $n \geq 1$ and $A(z)$ be a meromorphic function with $\rho(A) < n$. Set $f(z) = A(z)e^{P(z)}$, $\left(z = re^{i\theta}\right)$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then, for any given $\varepsilon > 0$, there is a set $E_7 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \backslash$ $(E_7 \cup E_8)$ for is $R > 0$, such that for $|z| = r > R$, we have:*

(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon) \delta(P, \theta) r^n\} \leq |f\left(re^{i\theta}\right)| \leq \exp\{(1 + \varepsilon) \delta(P, \theta) r^n\},$$

(ii) *if* $\delta(P, \theta) < 0$, *then*

$$\exp\{(1 + \varepsilon) \delta(P, \theta) r^n\} \leq |f\left(re^{i\theta}\right)| \leq \exp\{(1 - \varepsilon) \delta(P, \theta) r^n\},$$

where $E_8 = \{\theta \in [0, 2\pi) : (P, \theta) = 0\}$ is a finite set.

Lemma 2.6 ([8]) *Suppose that* $k \geq 2$ *and* A_0, $A_1, ..., A_{k-1}$ *are meromorphic functions such that* $\rho = \max\{\rho(A_j) : j = 0, 1, ...k-1\} < \infty$. *Let* f *be a transcendental meromorphic solution with all poles of* f *are of uniformly bounded multiplicities of equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0.$$

Then, $\rho_2(f) \leq \rho$.

Lemma 2.7 ([9]) *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *and* $\varepsilon > 0$ *be given constants. Then, there exists a set* $E_9 \subset (1, +\infty)$ *having finite logarithmic measure and a constant* $B > 0$, *that depends only on* α *and* (n, m) $(n, m$ *positive integers with* $n > m \geq 0)$ *such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E_9$, *we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{n-m}.$$

Lemma 2.8 ([10]) *Let* $\varphi : [0, +\infty) \to \mathbb{R}$ *and* $\psi : [0, +\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leq \psi(r)$ *for all* $r \notin (E_{10} \cup [0, 1])$, *where* E_{10} *is a set of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then, there exists an* $r_1 = r_1(\alpha) > 0$ *such that* $\varphi(r) \leq \psi(\alpha r)$ *for all* $r > r_1$.

Lemma 2.9 ([2],[5]) *Let* $A_j(z)(\not\equiv 0)$, $j = 0, 1, ..., k-1$, $F(z) \not\equiv 0$ *be finite order meromorphic functions.*
(i) *If* f *is a meromorphic solution of equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F \tag{2.1}$$

with $\rho(f) = +\infty$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

(ii) *If* f *is a meromorphic solution of equation* (2.1) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Lemma 2.10 *Let* $F_j(z)$ $(j = 0, 1, ..., k-1)$ *be meromorphic functions such that* $F_0(z) \not\equiv 0$. *Suppose that* f *is a solution of*

$$f^{(k)} + F_{k-1}(z) f^{(k-1)} + \cdots + F_1(z) f' + F_0(z) f = 0, \tag{2.2}$$

Then, $g_i = f^{(i)}$ *is a solution of the equation*

$$g_i^{(k)} + F_{k-1}^i(z) g_i^{(k-1)} + \cdots + F_1^i(z) g_i' + F_0^i(z) g_i = 0, \tag{2.3}$$

where $F_j^i(z), (j = 0, 1, ..., k), i \in \mathbb{N}$ *are defined by the following sequence of functions:*

$$\left\{ \begin{array}{c} F_j^0(z) = F_j(z),\ j = 0, 1, ..., k-1, \\ F_j^i(z) = \left(F_{j+1}^{i-1}(z)\right)' + F_j^{i-1}(z) - F_{j+1}^{i-1}(z) \frac{\left(F_0^{i-1}(z)\right)'}{F_0^{i-1}(z)}, \\ i = 1, 2, ..., j = 0, 1, ..., k-1, \\ F_k^i(z) = F_k^{i-1}(z) = \cdots = F_k^0(z) = F_k(z) = 1,\ \text{for all } i \in \mathbb{N}. \end{array} \right. \tag{2.4}$$

Proof. Assume that f is a solution of equation (2.2) and let $g_i = f^{(i)}$. We prove that g_i is a solution of the equation (2.3) : Our proof is by induction.

For $i = 1$, differentiating (2.2), we obtain

$$\begin{aligned} f^{(k+1)} + F_{k-1}(z) f^{(k)} + \left(F'_{k-1}(z) + F_{k-2}(z)\right) f^{(k-1)} \\ + \left(F'_{k-2}(z) + F_{k-3}(z)\right) f^{(k-2)} \\ + \cdots + (F'_1 + F_0(z)) f' + F'_0(z) f = 0. \end{aligned} \tag{2.5}$$

By (2.2), we get

$$f = -\frac{f^{(k)} + F_{k-1}(z) f^{(k-1)} + \cdots + F_1(z) f'}{F_0(z)}. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$\begin{aligned} f^{(k+1)} + \left(F_{k-1}(z) - \frac{F'_0(z)}{F_0(z)}\right) f^{(k)} \\ + \left(F'_{k-1}(z) + F_{k-2}(z) - F_{k-1}(z) \frac{F'_0(z)}{F_0(z)}\right) f^{(k-1)} \\ + \cdots + \left(F'_2(z) + F_1(z) - F_2(z) \frac{F'_0(z)}{F_0(z)}\right) f'' \\ + \left(F'_1(z) + F_0(z) - F_1(z) \frac{F'_0(z)}{F_0(z)}\right) f' = 0. \end{aligned} \tag{2.7}$$

Using (2.4), since $F_k(z) = 1$, then (2.7) becomes

$$\begin{aligned} g_1^{(k)} + \left(F'_k(z) + F_{k-1}(z) - F_k(z) \frac{F'_0(z)}{F_0(z)}\right) g_1^{(k-1)} \\ + F_{k-2}^1(z) g_1^{(k-2)} + \cdots + F_0^1(z) g_1 = 0. \end{aligned} \tag{2.8}$$

That is

$$g_1^{(k)} + F_{k-1}^1(z) g_1^{(k-1)} + F_{k-2}^1(z) g_1^{(k-2)} + \cdots + F_0^1(z) g_1 = 0. \tag{2.9}$$

Suppose that the assertion is true for the values which are strictly smaller than a certain i. We suppose g_{i-1} is a solution of the equation

$$g_{i-1}^{(k)} + F_{k-1}^{i-1}(z)\, g_{i-1}^{(k-1)} + F_{k-2}^{i-1}(z)\, g_{i-1}^{(k-2)}$$

$$+ \cdots + F_1^{i-1}(z)\, g_{i-1}' + F_0^{i-1}(z)\, g_{i-1} = 0. \tag{2.10}$$

Differentiating of (2.10), we can write

$$g_{i-1}^{(k+1)} + F_{k-1}^{i-1}(z)\, g_{i-1}^{(k)} + \left(\left(F_{k-1}^{i-1}(z)\right)' + F_{k-2}^{i-1}(z)\right) g_{i-1}^{(k-1)}$$

$$+ \cdots + \left(\left(F_1^{i-1}(z)\right)' + F_0^{i-1}(z)\right) g_{i-1}' + \left(F_0^{i-1}(z)\right)' g_{i-1} = 0. \tag{2.11}$$

From (2.10), we have

$$g_{i-1} = -\frac{g_{i-1}^{(k)} + F_{k-1}^{i-1}(z)\, g_{i-1}^{(k-1)} + F_{k-2}^{i-1}(z)\, g_{i-1}^{(k-2)} + \cdots + F_1^{i-1}(z)\, g_{i-1}'}{F_0^{i-1}(z)}. \tag{2.12}$$

Substituting (2.12) into (2.11), and using the fact that $F_k^{i-1}(z) = 1$, we get

$$g_{i-1}^{(k+1)} + \left(\left(F_k^{i-1}(z)\right)' + F_{k-1}^{i-1}(z) - F_k^{i-1}(z)\frac{\left(F_0^{i-1}(z)\right)'}{F_0^{i-1}(z)}\right) g_{i-1}^{(k)}$$

$$+ \left(\left(F_{k-1}^{i-1}(z)\right)' + F_{k-2}^{i-1}(z) - F_{k-1}^{i-1}(z)\frac{\left(F_0^{i-1}(z)\right)'}{F_0^{i-1}(z)}\right) g_{i-1}^{(k-1)}$$

$$+ \cdots + \left(\left(F_2^{i-1}(z)\right)' + F_1^{i-1}(z) - F_2^{i-1}(z)\frac{\left(F_0^{i-1}(z)\right)'}{F_0^{i-1}(z)}\right) g_{i-1}''$$

$$+ \left(\left(F_1^{i-1}(z)\right)' + F_0^{i-1}(z) - F_1^{i-1}(z)\frac{\left(F_0^{i-1}(z)\right)'}{F_0^{i-1}(z)}\right) g_{i-1}' = 0. \tag{2.13}$$

By (2.4) and (2.13), we have

$$g_i^{(k)} + F_{k-1}^{i}(z)\, g_i^{(k-1)} + F_{k-2}^{i}(z)\, g_i^{(k-2)} + \cdots + F_0^{i}(z)\, g_i = 0.$$

Thus, Lemma 2.10 is proved.

Lemma 2.11 ([20]) *Suppose that $f_1(z), f_2(z), ..., f_n(z)\ (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), ..., g_n(z)\ (n \geq 2)$ are entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z)\, e^{g_j(z)} = 0$;

(ii) $g_j(z) - g_k(z)$ *are no constants for* $1 \le j < k \le n$;
(iii) *for* $1 \le j \le n, 1 \le h < k \le n$,

$$T(r, f_j) = o\left\{T\left(r, e^{g_h - g_k}\right)\right\}$$

$(r \to +\infty,\ r \notin E_{11})$, *where* E_{11} *is a set with finite linear measure. Then,* $f_j(z) \equiv 0$, $j = 1, 2, ..., n$.

Lemma 2.12 ([19]) *Suppose that* $f_1(z), f_2(z), ..., f_n(z)\,(n \ge 2)$ *are meromorphic functions and* $g_1(z), g_2(z), ..., g_n(z)\,(n \ge 2)$ *are entire functions satisfying the following conditions:*
(i) $\sum\limits_{j=1}^{n} e^{g_j(z)} f_j(z) \equiv f_{n+1}$;
(ii) *If* $1 \le j \le n+1$ *and* $1 \le k \le n$, *then the order of* f_j *is less than the order of* $e^{g_k(z)}$. *If* $n \ge 2$, $1 \le j \le n+1$ *and* $1 \le h < k \le n$, *then the order of* f_j *is less than the order of* $e^{g_h - g_k}$. *Then,* $f_j(z) \equiv 0, j = 1, 2, ..., n+1$.

Lemma 2.13 *Let* $P(z) = a_{1,n}z^n + \cdots + a_{1,0}$ *and* $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ $(j = 1, ..., q)$ *be polynomials with degree* $n \ge 1$, *where* $a_{1,i}$ $(i = 0, 1, ..., n)$ *are complex numbers such that* $a_{1,n} \ne 0$, $b_{j,i}$ $(j = 1, ..., q; i = 0, 1, ..., n)$ *are real constants with* $b_{j,n} \le 0$ $(j = 1, ..., q)$. *Let* $m \ge 2$ *be an integer and* α, γ_j $(j = 1, ..., q)$ *be real numbers such that* $0 < \alpha < m$, $\gamma_j \ge 0$ *and* $0 < \alpha + \gamma_1 + \gamma_2 + \cdots + \gamma_q \le m$. *If* $\arg a_{1,n} \ne \pi$ *or* $a_{1,n} < b$, *where* $b = \min\{b_{j,n} : j = 1, ..., q\}$, *then* $ma_{1,n} \ne \alpha a_{1,n} + \gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_q b_{q,n}$.

Proof. Suppose that $ma_{1,n} = \alpha a_{1,n} + \gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_q b_{q,n}$. Then, we have

$$a_{1,n} = \frac{\gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_q b_{q,n}}{m - \alpha}.$$

Since $b = \min\{b_{j,n} : j = 1, ..., q\}$, then there exist constants c_j such that $0 \le c_j \le 1\,(j = 1, ..., q)$ and $b_{j,n} = c_j b$. By this, we obtain $a_{1,n} = Kb$, where

$$K = \frac{\gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_q c_q}{m - \alpha}.$$

Since $0 \le \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_q c_q \le \gamma_1 + \gamma_2 + \cdots + \gamma_q \le m - \alpha$, then we get $0 \le K \le 1$.
(i) If $K = 0$, then $a_{1,n} = 0$, which is a contradiction.
(ii) When $0 < K \le 1$ we have $a_{1,n} = Kb$. If $\arg a_{1,n} \ne \pi$ or $a_{1,n} < b$, then $a_{1,n} \ne cb$ $(0 < c \le 1)$ which is a contradiction. So,

$$ma_{1,n} \ne \alpha a_{1,n} + \gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_q b_{q,n}.$$

Lemma 2.14 *Let* $P_1(z) = a_{1,n}z^n + \cdots + a_{1,0}$, $P_2(z) = a_{2,n}z^n + \cdots + a_{2,0}$, $Q_j(z) = b_{j,n}z^n + \cdots + b_{j,0}$ *and* $Q'_{j'}(z) = b'_{j',n}z^n + \cdots + b'_{j',0}$ *be polynomials with degree* $n \geq 1$, *where* $a_{i,q}$ $(i = 1, 2;\ q = 0, 1, ..., n)$ *are complex numbers such that* $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ *(suppose that* $|a_{1,n}| \leq |a_{2,n}|$*),* $b_{j,q}, b_{j',q}$ $(j = 1, ..., l;\ j' = 1, ..., l';\ q = 0, 1, ..., n)$ *be real constants such that* $b_{j,n} \leq 0, b'_{j',n} \leq 0$ $(j = 1, ..., l;\ j' = 1, ..., l')$. *Let* $m \geq 2$ *be an integer and* $\alpha > 0, \beta > 0, \alpha' > 0, \beta' > 0, \gamma_j \geq 0$ $(j = 1, ..., l)$ *and* $\gamma'_{j'} \geq 0$ $(j' = 1, ..., l')$ *be real numbers such that* $0 < \alpha + \beta + \gamma_1 + \gamma_2 + \cdots + \gamma_l \leq m$ *and* $0 < \alpha' + \beta' + \gamma'_1 + \gamma'_2 + \cdots + \gamma'_{l'} \leq m$, $\max\{\alpha, \beta, \alpha', \beta'\} < m$. *If* $\arg a_{1,n} \neq \pi$ *or* $a_{1,n} < b$, *where* $b = \min\left\{b_{j,n}\ (j = 1, ..., l), b'_{j',n}\ (j' = 1, ..., l')\right\}$ *and* $ma_{1,n} = \alpha a_{1,n} + \beta a_{2,n} + \gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_l b_{l,n}$, *then* $ma_{2,n} \neq \alpha' a_{1,n} + \beta' a_{2,n} + \gamma'_1 b'_{1,n} + \gamma'_2 b'_{2,n} + \cdots + \gamma'_{l'} b'_{l',n}$.

Proof. Suppose that $ma_{2,n} = \alpha' a_{1,n} + \beta' a_{2,n} + \gamma'_1 b'_{1,n} + \gamma'_2 b'_{2,n} + \cdots + \gamma'_{l'} b'_{l',n}$. Then, we have the system

$$(2.14) \quad \begin{cases} ma_{1,n} = \alpha a_{1,n} + \beta a_{2,n} + \gamma_1 b_{1,n} + \gamma_2 b_{2,n} + \cdots + \gamma_l b_{l,n}, \\ ma_{2,n} = \alpha' a_{1,n} + \beta' a_{2,n} + \gamma'_1 b'_{1,n} + \gamma'_2 b'_{2,n} + \cdots + \gamma'_{l'} b'_{l',n}. \end{cases}$$

Since $b = \min\left\{b_{j,n}\ (j = 1, ..., l), b'_{j',n}\ (j' = 1, ..., l')\right\}$, then there exist constants c_j $(0 \leq c_j \leq 1)$ and c'_j $\left(0 \leq c'_{j'} \leq 1\right)$ $(j = 1, ..., l; j' = 1, ..., l')$ such that $b_{j,n} = c_j b, b'_{j',n} = c'_{j'} b$. Thus, by (2.14) we obtain

$$(2.15) \quad \begin{cases} (m - \alpha)\, a_{1,n} - \beta a_{2,n} = \gamma b, \\ -\alpha' a_{1,n} + (m - \beta')\, a_{2,n} = \gamma' b, \end{cases}$$

where

$$\begin{cases} \gamma = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_l c_l, \\ \gamma' = \gamma'_1 c'_1 + \gamma'_2 c'_2 + \cdots + \gamma'_{l'} c'_{l'}. \end{cases}$$

Set

$$(2.16) \quad \begin{cases} \delta = \alpha + \beta + \gamma, \\ \delta' = \alpha' + \beta' + \gamma'. \end{cases}$$

Since

$$\begin{cases} 0 \leq \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_l c_l \leq \gamma_1 + \gamma_2 + \cdots + \gamma_l \leq m - \alpha, \\ 0 \leq \gamma'_1 c_1 + \gamma'_2 c_2 + \cdots + \gamma'_{l'} c_{l'} \leq \gamma'_1 + \gamma'_2 + \cdots + \gamma'_{l'} \leq m - \alpha', \end{cases}$$

then

$$\begin{cases} \delta = \alpha + \beta + \gamma \leq \alpha + \beta + \gamma_1 + \gamma_2 + \cdots + \gamma_l \leq m, \\ \delta' = \alpha' + \beta' + \gamma' \leq \alpha' + \beta' + \gamma'_1 + \gamma'_2 + \cdots + \gamma'_{l'} \leq m. \end{cases}$$

The system (2.15) has the following determinant Δ

$$\Delta = \begin{vmatrix} m-\alpha & -\beta \\ -\alpha' & m-\beta' \end{vmatrix} = (m-\alpha)(m-\beta') - \alpha'\beta. \tag{2.17}$$

Case 1: If $\alpha'\beta = 0$, then $\Delta = (m-\alpha)(m-\beta') > 0$.
Case 2: If $\alpha'\beta \neq 0$. Using $\delta \leq m$, $\delta' \leq m$, we get $m-\alpha \geq \beta+\gamma, m-\beta' \geq \alpha'+\gamma'$. Thus, $\Delta \geq L$, where $L = (\beta+\gamma)(\alpha'+\gamma') - \alpha'\beta$.
Subcase 2.1: If $\gamma \neq 0$ and $\gamma' \neq 0$ or $\gamma = 0$ and $\gamma' \neq 0$ or $\gamma \neq 0$ and $\gamma' = 0$, then, $\Delta > L > 0$.
Subcase 2.2: If $\gamma = \gamma' = 0$, then

$$\begin{cases} \delta = \alpha+\beta, \\ \delta' = \alpha'+\beta'. \end{cases}$$

(i) If $\delta < m, \delta' < m$, then $m-\alpha > \beta$ and $m-\beta' > \alpha'$and so $\Delta = (m-\alpha)(m-\beta') - \alpha'\beta > \beta\alpha' - \alpha'\beta = 0$.
(ii) If $\delta = m, \delta' = m$, then $m-\alpha = \beta$ and $m-\beta' = \alpha'$, hence $\Delta = (m-\alpha)(m-\beta') - \alpha'\beta = \beta\alpha' - \alpha'\beta = 0$.
(iii) If $\delta = m, \delta' < m$, then $m-\alpha = \beta$ and $m-\beta' > \alpha'$, hence $\Delta = \beta(m-\beta') - \alpha'\beta > \beta\alpha' - \alpha'\beta = 0$.
(iv) If $\delta < m, \delta' = m$, then $m-\alpha > \beta$ and $m-\beta' = \alpha'$, hence $\Delta = (m-\alpha)\alpha' - \alpha'\beta > \beta\alpha' - \alpha'\beta > 0$.
a) For the Subcase 2.2 (ii), we have $\Delta = 0$, by the system (2.15), we get $a_{1,n} = a_{2,n}$, which is a contradiction.
b) For **Case 1** and **Subcase** (i), (ii) and (iv) of **Case 2,** when $\Delta > 0$, by the system (2.15), we get

$$a_{1,n} = b\frac{\begin{vmatrix} \gamma & -\beta \\ \gamma' & m-\beta' \end{vmatrix}}{\Delta} = b\frac{\gamma(m-\beta') + \beta\gamma'}{\Delta} = bS.$$

We have $\gamma(m-\beta') + \beta\gamma' \geq 0$, by (2.16), we can write

$$\begin{aligned} \gamma(m-\beta') + \beta\gamma' - \Delta &= (\delta-\alpha-\beta)(m-\beta') \\ &+\beta(\delta'-\alpha'-\beta') - (m-\alpha)(m-\beta') + \alpha'\beta \\ &= (m-\beta')(\delta-\beta-m) + \beta(\delta'-\beta') \\ &= (m-\beta')\delta - m\beta - m^2 + \beta'm + \beta\delta' \\ &\leq (m-\beta')m - m\beta - m^2 + \beta'm + \beta m = 0. \end{aligned}$$

Thus, $0 \leq \frac{\gamma(m-\beta')+\beta\gamma'}{\Delta} = S \leq 1$.
(i) When $S = 0$, then $a_{1,n} = 0$ which is a contradiction.
(ii) When $0 < S \leq 1$ we have $a_{1,n} = bS$. If $\arg a_{1,n} \neq \pi$ or $a_{1,n} < b$, then $a_{1,n} \neq cb\,(0 < c \leq 1)$ which is a contradiction. Hence, $ma_{2,n} \neq \alpha' a_{1,n} + \beta' a_{2,n} + \gamma_1' b_{1,n}' + \gamma_2' b_{2,n}' + \cdots + \gamma_{l'}' b_{l',n}'$.

3. Proof of the Theorem 1.1

Assume that $f\ (\not\equiv 0)$ is a meromorphic solution of equation (1.3) with all poles are of uniformly bounded multiplicities.

First step. We prove that $\rho(f) = +\infty$. First of all, we prove that equation (1.3) can not have a meromorphic solution $f\ (\not\equiv 0)$ with $\rho(f) < n$. Assume that there exists a meromorphic solution with $\rho(f) < n$. We can rewrite (1.3) in the following form

$$B_{k-1}e^{Q_{k-1}(z)}f^{(k-1)} + \cdots + B_1 e^{Q_1(z)}f' + \left(A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right) f = H, \tag{3.1}$$

with $H = -\left(f^{(k)} + D_{k-1}f^{(k-1)} + \cdots + D_1 f' + D_0 f\right)$. By the conditions of Theorem 1.1, we have $a_{1,n}a_{2,n} \neq 0$, $a_{1,n} \neq a_{2,n}$ and $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)$, thus

$$\deg(P_1 - P_2) = n,$$

$$\deg(Q_j - P_i) = \deg(Q_j - Q_l) = n$$

with $j \neq l$; $j = 1, ..., k-1$; $l = 1, ..., k-1$; $i = 1, 2$. Then, by Lemma 2.1 and (3.1), we find that the order of growth of the left side of equation (3.1) is n. On the other hand, $\rho(f) < n$, $\rho\left(f^{(j)}\right) < n$, $j = 1, ..., k$, so $\rho(H) < n$, which is a contradiction. Consequently, every meromorphic solution $f\ (\not\equiv 0)$ of equation (1.3) is transcendental with order $\rho(f) \geq n$. Now, we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. By equation (1.3), we get

$$\left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| \leq \left|\frac{f^{(k)}}{f}\right| + \left(|D_{k-1}| + \left|B_{k-1}e^{Q_{k-1}(z)}\right|\right)\left|\frac{f^{(k-1)}}{f}\right|$$

$$+ \cdots + \left(|D_1| + \left|B_1 e^{Q_1(z)}\right|\right)\left|\frac{f'}{f}\right| + |D_0|. \tag{3.2}$$

Set $\rho_1 = \max\limits_{\substack{i=1,2\\ j=1,\ldots,k-1\\ m=0,\ldots,k-1}} \{\rho(A_i), \rho(B_j), \rho(D_m)\} < n$. By Lemma 2.2, for any given ε $(0 < \varepsilon < n - \rho_1)$, there exists a set $E_1 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that when $|z| = r \notin [0, 1] \cup E_1$, $r \to +\infty$, we have

$$|A_i(z)| \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}, |B_j(z)| \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}, |D_m(z)| \leq \exp\left\{r^{\rho_1+\varepsilon}\right\}. \tag{3.3}$$

By Lemma 2.3, there exists a set $E_2 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ of linear measure zero, such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \backslash E_2$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq |z|^{j(\rho-1+\varepsilon)} \,(j = 1, ..., k)\,. \tag{3.4}$$

Set $z = re^{i\theta}$, $a_{1,n} = |a_{1,n}|\, e^{i\theta_1}$, $a_{2,n} = |a_{2,n}|\, e^{i\theta_2}$, $\theta_1, \theta_2 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$. Then

$$\delta(P_1, \theta) = |a_{1,n}| \cos(n\theta + \theta_1)\,, \delta(P_2, \theta) = |a_{2,n}| \cos(n\theta + \theta_2)\,. \tag{3.5}$$

Since $b_{j,n} = c_j a_{1,n}$ $(0 < c_j < 1)\,(j \in I_1)$ and c_j are distinct numbers, then

$$\delta(Q_j, \theta) = c_j \delta(P_1, \theta)\,. \tag{3.6}$$

Case 1. If $\theta_1 = \arg\, a_{1,n} \neq \pi$ which is $\theta_1 \neq \pi$.
(i) Assume that $\theta_1 \neq \theta_2$. By Lemma 2.4, for any given ε with $(0 < \varepsilon < \min\left\{n - \rho_1, \frac{1}{2}\left(\frac{1-c}{1+c}\right)\right\})$, there is a ray $\arg z = \theta$ with $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n})\backslash(E_2 \cup E_5 \cup E_6)$ such that

$$\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$$

or

$$\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0.$$

a) When $\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$, we have $\delta(P_2, \theta) < \delta(P_1, \theta)$. For sufficiently large r, we get by Lemma 2.5

$$\begin{aligned} \left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| &\geq \left|A_1 e^{P_1(z)}\right| - \left|A_2 e^{P_2(z)}\right| \\ &\geq \exp\left\{(1-\varepsilon)\,\delta(P_1, \theta)\, r^n\right\} - \exp\left\{(1-\varepsilon)\,\delta(P_2, \theta)\, r^n\right\} \\ &\geq \frac{1}{2}\exp\left\{(1-\varepsilon)\,\delta(P_1, \theta)\, r^n\right\}. \end{aligned} \tag{3.7}$$

For $j \in I_1$, by (3.6), we have

$$\left|B_j e^{Q_j(z)}\right| \leq \exp\left\{(1+\varepsilon)\, c_j \delta(P_1, \theta)\, r^n\right\} \leq \exp\left\{(1+\varepsilon)\, c\delta(P_1, \theta)\, r^n\right\}, \tag{3.8}$$

and for $j \in I_2$, we have

$$B_j e^{Q_j(z)} = B_j e^{Q_j(z) - b_{j,n} z^n} e^{b_{j,n} z^n} = h_j(z)\, e^{b_{j,n} z^n},$$

where $h_j(z) = B_j e^{Q_j(z) - b_{j,n} z^n}$ with $\rho_2 = \rho(h_j) \leq \max\{n - 1, \rho(B_j)\} < n$. By Lemma 2.2, for any given ε with

$$0 < \varepsilon < \min\left\{n - \rho_1, n - \rho_2, \frac{1}{2}\left(\frac{1-c}{1+c}\right)\right\}$$

when $|z| = r \notin [0, 1] \cup E_1$, $r \to +\infty$, we have

$$|h_j(z)| \leq \exp\left\{r^{\rho_2 + \varepsilon}\right\}.$$

Then

$$\left|B_j e^{Q_j(z)}\right| = \left|h_j(z)\, e^{b_{j,n}z^n}\right| \le \exp\left\{r^{\rho_2+\varepsilon}\right\}\left|e^{b_{j,n}z^n}\right|$$

$$= \exp\left\{r^{\rho_2+\varepsilon}\right\} e^{b_{j,n}r^n \cos n\theta} \le \exp\left\{r^{\rho_2+\varepsilon}\right\} \tag{3.9}$$

because $b_{j,n} \le 0$ and $\cos n\theta > 0$. Substituting (3.3), (3.4), (3.7), (3.8), (3.9) into (3.2), we obtain

$$\begin{aligned}\frac{1}{2}\exp\left\{(1-\varepsilon)\,\delta(P_1,\theta)\,r^n\right\} &\le r^{k(\rho-1+\varepsilon)} \\ &+ \left(\exp\left\{r^{\rho_1+\varepsilon}\right\} + \left|B_{k-1}e^{Q_{k-1}(z)}\right|\right) r^{k(\rho-1+\varepsilon)} \\ &+ \cdots + \left(\exp\left\{r^{\rho_1+\varepsilon}\right\} + \left|B_1 e^{Q_1(z)}\right|\right) r^{k(\rho-1+\varepsilon)} + \exp\left\{r^{\rho_1+\varepsilon}\right\} \\ &\le (k+1)\, r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} \exp\left\{(1+\varepsilon)\, c\delta(P_1,\theta)\, r^n\right\}\end{aligned}$$

which gives

$$\begin{aligned}&\exp\left\{(1-\varepsilon)\,\delta(P_1,\theta)\,r^n\right\} \\ &\le 2(k+1)\, r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} \exp\left\{(1+\varepsilon)\, c\delta(P_1,\theta)\, r^n\right\}.\end{aligned} \tag{3.10}$$

From (3.10) and $0 < \varepsilon < \frac{1}{2}\left(\frac{1-c}{1+c}\right)$, we obtain

$$\exp\left\{\left(\frac{1-c}{2}\right)\delta(P_1,\theta)\, r^n\right\} \le 2(k+1)\, r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} \tag{3.11}$$

which is a contradiction because

$$\delta(P_1,\theta) > 0 \text{ and } 0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}.$$

b) When $\delta(P_1,\theta) < 0, \delta(P_2,\theta) > 0$, we have $\delta(P_1,\theta) < \delta(P_2,\theta)$. For sufficiently large r and the above ε, we get by Lemma 2.5

$$\begin{aligned}\left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| &\ge \left|A_2 e^{P_2(z)}\right| - \left|A_1 e^{P_1(z)}\right| \\ &\ge \exp\left\{(1-\varepsilon)\,\delta(P_2,\theta)\,r^n\right\} - \exp\left\{(1-\varepsilon)\,\delta(P_1,\theta)\,r^n\right\} \\ &\ge \frac{1}{2}\exp\left\{(1-\varepsilon)\,\delta(P_2,\theta)\,r^n\right\}.\end{aligned} \tag{3.12}$$

For $j \in I_1$, by (3.6), we have

$$\left|B_j e^{Q_j(z)}\right| \le \exp\left\{(1-\varepsilon)\, c_j\delta(P_1,\theta)\, r^n\right\} < 1 \tag{3.13}$$

and for $j \in I_2$, (3.9) holds. Substituting (3.3), (3.4), (3.9), (3.12), (3.13) into (3.2), we obtain

$$\exp\left\{(1-\varepsilon)\,\delta(P_2,\theta)\,r^n\right\} \le 2(k+1)\, r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$

which is a contradiction because

$$\delta(P_2,\theta) > 0 \text{ and } 0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}.$$

(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.4, for any given ε with $0 < \varepsilon < \min\left\{n - \rho_1, n - \rho_2, \frac{\delta(P_2,\theta)-\delta(P_1,\theta)}{\delta(P_2,\theta)+\delta(P_1,\theta)}\right\}$, there is a ray $\arg z = \theta$ such that $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n})\setminus(E_2 \cup E_5 \cup E_6)$ and $\delta(P_1,\theta) > 0$. Since $|a_{1,n}| \leq |a_{2,n}|$, $a_{1,n} \neq a_{2,n}$ and $\theta_1 = \theta_2$, then $|a_{1,n}| < |a_{2,n}|$, thus $0 < \delta(P_1,\theta) < \delta(P_2,\theta)$. For sufficiently large r, we have by Lemma 2.5

$$\begin{aligned} \left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| &\geq \left|A_2 e^{P_2(z)}\right| - \left|A_1 e^{P_1(z)}\right| \\ &\geq \exp\{(1-\varepsilon)\delta(P_2,\theta) r^n\} - \exp\{(1+\varepsilon)\delta(P_1,\theta) r^n\} \\ &= (\exp\{\alpha r^n\} - 1)\exp\{(1+\varepsilon)\delta(P_1,\theta) r^n\}, \end{aligned} \tag{3.14}$$

where

$$\alpha = (1-\varepsilon)\delta(P_2,\theta) - (1+\varepsilon)\delta(P_1,\theta).$$

Since $0 < \varepsilon < \frac{\delta(P_2,\theta)-\delta(P_1,\theta)}{\delta(P_2,\theta)+\delta(P_1,\theta)}$, then

$$\alpha = \delta(P_2,\theta) - \delta(P_1,\theta) - \varepsilon(\delta(P_2,\theta) + \delta(P_1,\theta))$$

$$> \delta(P_2,\theta) - \delta(P_1,\theta) - \frac{\delta(P_2,\theta) - \delta(P_1,\theta)}{\delta(P_2,\theta) + \delta(P_1,\theta)}(\delta(P_2,\theta) + \delta(P_1,\theta)) = 0.$$

We get by (3.14)

$$\left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| \geq \frac{1}{2}\exp\{\alpha r^n\}\exp\{(1+\varepsilon)\delta(P_1,\theta) r^n\}. \tag{3.15}$$

On the other hand, by Lemma 2.2 and Lemma 2.5, we get (3.8) and (3.9). Substituting (3.3), (3.4), (3.8), (3.9), (3.15) into (3.2), we obtain

$$\begin{aligned} \frac{1}{2}\exp(\alpha r^n)\exp\{(1+\varepsilon)\delta(P_1,\theta) r^n\} &\leq (k+1) r^{k(\rho-1+\varepsilon)} \\ &\times \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}\exp\{(1+\varepsilon)c\delta(P_1,\theta) r^n\}. \end{aligned} \tag{3.16}$$

By (3.16), we get

$$\begin{aligned} \exp\{(1+\varepsilon)(1-c)\delta(P_1,\theta) r^n + \alpha r^n\} &\leq 2(k+1) r^{k(\rho-1+\varepsilon)} \\ &\times \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}. \end{aligned}$$

which is a contradiction because $\delta(P_1,\theta) > 0$, $\alpha > 0$ and $0 < \varepsilon < \min\{n - \rho_1, n - \rho_2\}$.

Case 2. *If* $a_{1,n} < \frac{b}{1-c}$. which is $\theta_1 = \pi$.

(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.4, for any given ε $(0 < \varepsilon < \min\left\{n - \rho_1, n - \rho_2, \frac{(1-c)}{2(1+c)}\right\})$, there is a ray $\arg z = \theta$ such that $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n})\setminus(E_2 \cup E_5 \cup E_6)$ and $\delta(P_2,\theta) > 0$. On the other hand, we have $\delta(P_1,\theta) = |a_{1,n}|\cos(n\theta+\theta_1) = -|a_{1,n}|\cos(n\theta) < 0$ because $\cos(n\theta) > 0$. For sufficiently large r, we obtain by Lemma 2.5

$$\left|A_1e^{P_1(z)} + A_2e^{P_2(z)}\right| \geq \left|A_2e^{P_2(z)}\right| - \left|A_1e^{P_1(z)}\right|$$
$$\geq \exp\{(1-\varepsilon)\delta(P_2,\theta)r^n\} - \exp\{(1-\varepsilon)\delta(P_1,\theta)r^n\}$$
$$\geq \frac{1}{2}\exp\{(1-\varepsilon)\delta(P_2,\theta)r^n\} \tag{3.17}$$

and $(3.9), (3.13)$ hold. Substituting $(3.3), (3.4), (3.9), (3.13), (3.17)$ into (3.2), we obtain

$$\frac{1}{2}\exp\{(1-\varepsilon)\delta(P_2,\theta)r^n\} \leq (k+1)r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$
$$\times\exp\{(1+\varepsilon)c\delta(P_1,\theta)r^n\}$$
$$\leq (k+1)r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}\exp\{(1+\varepsilon)c\delta(P_2,\theta)r^n\}. \tag{3.18}$$

By $0 < \varepsilon < \frac{(1-c)}{2(1+c)}$ and (3.16), we get

$$\exp\left\{\frac{(1-c)}{2}\delta(P_2,\theta)r^n\right\} \leq 2(k+1)r^{k(\rho-1+\varepsilon)}\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$

which is a contradiction because $\delta(P_2,\theta) > 0$, $1-c > 0$ and $0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}$.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_2 = \pi$. By Lemma 2.4, for any given ε $\left(0 < \varepsilon < \min\left\{n-\rho_1, \frac{\delta(P_2,\theta)-\delta(P_1,\theta)}{\delta(P_2,\theta)+\delta(P_1,\theta)}\right\}\right)$, there is a ray $\arg z = \theta$ such that $\theta \in [\frac{\pi}{2n}, \frac{3\pi}{2n})\setminus(E_2\cup E_5\cup E_6)$, then $\cos n\theta < 0$, $\delta(P_1,\theta) = |a_{1,n}|\cos(n\theta+\theta_1) = -|a_{1,n}|\cos(n\theta) > 0$. Because $|a_{1,n}| \leq |a_{2,n}|$, $a_{1,n} \neq a_{2,n}$, we get $|a_{1,n}| < |a_{2,n}|$, thus $0 < \delta(P_1,\theta) < \delta(P_2,\theta)$. For sufficiently large r, we get (3.8) and (3.15) holds. For $j \in I_2$, we have

$$B_je^{Q_j(z)} = B_je^{Q_j(z)-b_{j,n}z^n}e^{b_{j,n}z^n} = h_j(z)e^{b_{j,n}z^n},$$

where $h_j(z) = B_je^{Q_j(z)-b_{j,n}z^n}$ with $\rho_2 = \rho(h_j) \leq \max(n-1, \rho(B_j)) < n$. By Lemma 2.2, for any given ε $(0 < \varepsilon < n-\rho_2)$, when $|z| = r \notin [0,1]\cup E_1$, $r \to +\infty$, we have

$$|h_j(z)| \leq \exp\left\{r^{\rho_2+\varepsilon}\right\}.$$

Then

$$\left|B_je^{Q_j(z)}\right| = \left|h_j(z)e^{b_{j,n}z^n}\right| \leq \exp\left\{r^{\rho_2+\varepsilon}\right\}\left|e^{b_{j,n}z^n}\right|$$
$$= \exp\left\{r^{\rho_2+\varepsilon}\right\}e^{b_{j,n}r^n\cos n\theta} \tag{3.19}$$

because $b_{j,n} \leq 0$ and $\cos n\theta < 0$, $b = \min\{b_{j,n},\ j \in I_2\}$, from (3.19), we get

$$\left|B_j e^{Q_j(z)}\right| \leq \exp\left\{r^{\rho_2+\varepsilon}\right\} e^{br^n \cos n\theta}. \tag{3.20}$$

Substituting $(3.3), (3.4), (3.8), (3.15), (3.20)$ into (3.2), we obtain

$$\frac{1}{2}\exp\{(1+\varepsilon)\,\delta(P_1,\theta)\,r^n + \alpha r^n\} \leq (k+1)\,r^{k(\rho-1+\varepsilon)}$$
$$\times \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} e^{br^n \cos n\theta} \exp\{(1+\varepsilon)\,c\delta(P_1,\theta)\,r^n\}$$

which gives

$$\exp\{((1-c)(1+\varepsilon)\,\delta(P_1,\theta) + \alpha - b\cos n\theta)\,r^n\} \leq 2(k+1)\,r^{k(\rho-1+\varepsilon)}$$
$$\times \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}.$$

Set $\gamma = (1-c)(1+\varepsilon)\,\delta(P_1,\theta) + \alpha - b\cos n\theta$, we obtain

$$\exp\{\gamma r^n\} \leq 2(k+1)\,r^{k(\rho-1+\varepsilon)} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}. \tag{3.21}$$

Since $\alpha > 0$, $\cos n\theta < 0$, $\delta(P_1,\theta) = -|a_{1,n}|\cos(n\theta)$, $a_{1,n} < \frac{b}{1-c}$ and $b \leq 0$, then

$$\gamma = (1-c)(1+\varepsilon)\,\delta(P_1,\theta) + \alpha - b\cos n$$
$$= -\left[(1-c)(1+\varepsilon)\,|a_{1,n}| + b\right]\cos n\theta + \alpha$$
$$> -\left[(1-c)(1+\varepsilon)\frac{|b|}{(1-c)} + b\right]\cos n\theta + \alpha = \alpha + b\varepsilon\cos n\theta > 0.$$

Since $0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}$ and $\gamma > 0$, then (3.21) is a contradiction. Concluding the above proof, we obtain $\rho(f) = +\infty$.

Second step. We prove that $\rho_2(f) = n$. By

$$\max = \left\{\rho(D_j + B_j e^{Q_j(z)})\ (j = 1, ..., k-1),\right.$$
$$\left.\rho(D_0 + A_1 e^{P_1(z)} + A_2 e^{P_2(z)})\right\} = n$$

and Lemma 2.6, we obtain $\rho_2(f) \leq n$. Remains to show that $\rho_2(f) \geq n$. By Lemma 2.7, there exists a set $E_9 \subset (1,+\infty)$ having finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_9$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,(T(2r,f))^{j+1}. \tag{3.22}$$

Case 1. If $\theta_1 = \arg a_{1,n} \neq \pi$ which is $\theta_1 \neq \pi$.
(i) if $\theta_1 \neq \theta_2$. By Lemma 2.4, for any given ε with

$$0 < \varepsilon < \min\left\{n-\rho_1, n-\rho_2, \frac{1}{2}\left(\frac{1-c}{1+c}\right)\right\},$$

there is a ray $\arg z = \theta$ such that $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \backslash (E_2 \cup E_5 \cup E_6)$ such that

$$\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$$

or

$$\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0.$$

a) When $\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$, we have $\delta(P_2, \theta) < \delta(P_1, \theta)$. For sufficiently large r, using the same reasoning as in Case $1((i), (a))$, we get

$$\left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| \geq \frac{1}{2} \exp\{(1-\varepsilon)\delta(P_1, \theta) r^n\}. \tag{3.23}$$

Substituting $(3.3), (3.8), (3.9), (3.22), (3.23)$ into (3.2), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_9$, $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \backslash (E_2 \cup E_5 \cup E_6)$

$$\begin{aligned} \frac{1}{2} \exp\{(1-\varepsilon)\delta(P_1, \theta) r^n\} &\leq B(T(2r, f))^{k+1} \\ &+ B\left(\exp\left\{r^{\rho_1+\varepsilon}\right\} + \left|B_{k-1} e^{Q_{k-1}(z)}\right|\right)(T(2r, f))^k \\ &+ \cdots + B\left(\exp\left\{r^{\rho_1+\varepsilon}\right\} + \left|B_1 e^{Q_1(z)}\right|\right)(T(2r, f))^2 + \exp\left(r^{\rho_1+\varepsilon}\right) \end{aligned}$$

$$\leq B(k+1)(T(2r, f))^{k+1} \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} \exp\{(1+\varepsilon)c\delta(P_1, \theta) r^n\}. \tag{3.24}$$

By $0 < \varepsilon < \frac{(1-c)}{2(1+c)}$ and (3.24), we get

$$\exp\left\{\left(\frac{1-c}{2}\right)\delta(P_1, \theta) r^n\right\} \leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$

$$\times (T(2r, f))^{k+1}. \tag{3.25}$$

Since $\delta(P_1, \theta) > 0$, $0 < \varepsilon < \min\{n - \rho_1, n - \rho_2\}$, then by using Lemma 2.8 and (3.25), we obtain $\rho_2(f) \geq n$, hence $\rho_2(f) = n$.

b) When $\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0$, we have $\delta(P_1, \theta) < \delta(P_2, \theta)$. For sufficiently large r and the above ε, we get by Lemma 2.4

$$\begin{aligned} \left|A_1 e^{P_1(z)} + A_2 e^{P_2(z)}\right| &\geq \left|A_2 e^{P_2(z)}\right| - \left|A_1 e^{P_1(z)}\right| \\ &\geq \exp\{(1-\varepsilon)\delta(P_2, \theta) r^n\} - \exp\{(1-\varepsilon)\delta(P_1, \theta) r^n\} \end{aligned}$$

$$\geq \frac{1}{2} \exp\{(1-\varepsilon)\delta(P_2, \theta) r^n\}. \tag{3.26}$$

Substituting $(3.3), (3.9), (3.13), (3.22), (3.26)$ into (3.2), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_9$, $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_2 \cup E_5 \cup E_6)$

$$\exp\{(1-\varepsilon)\,\delta(P_2,\theta)\,r^n\} \leq 2B(k+1)$$

$$\times \exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\} (T(2r,f))^{k+1}. \tag{3.27}$$

Since $\delta(P_2,\theta) > 0$, $0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}$, then by using Lemma 2.8 and (3.27), we obtain $\rho_2(f) \geq n$, hence $\rho_2(f) = n$.
(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.4, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_2 \cup E_5 \cup E_6)$ satisfying $\delta(P_1,\theta) > 0$. Since $|a_{1n}| \leq |a_{2n}|, a_{1n} \neq a_{2n}$ and $\theta_1 = \theta_2$, then $|a_{1n}| < |a_{2n}|$, thus $\delta(P_2,\theta) > \delta(P_1,\theta) > 0$. For sufficiently large r, we have by Lemma 2.5, we get (3.15) hold. Substituting $(3.3), (3.8), (3.9), (3.15), (3.22)$ into (3.2), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_9$, $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_2 \cup E_5 \cup E_6)$

$$\exp\{(1+\varepsilon)\,\delta(P_1,\theta)\,r^n + \alpha r^n\} \leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$

$$\times (T(2r,f))^{k+1} \exp\{(1+\varepsilon)\,c\delta(P_1,\theta)\,r^n\}.$$

Then

$$\exp\{((1-c)(1+\varepsilon)\,\delta(P_1,\theta) + \alpha)\,r^n\}$$

$$\leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}(T(2r,f))^{k+1}. \tag{3.28}$$

Since $0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}$, $\delta(P_1,\theta) > 0$, $\alpha > 0$, then by using Lemma 2.8 and (3.28), we obtain $\rho_2(f) \geq n$, hence $\rho_2(f) = n$.
Case 2. *If* $a_{1,n} < \frac{b}{1-c}$.
(i) Assume that $\theta_1 = \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ such that $\theta \in [\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_2 \cup E_5 \cup E_6)$ satisfying $\delta(P_2,\theta) > \delta(P_1,\theta) > 0$. By Lemma 2.5, and for sufficiently large r, we get (3.15) holds. Substituting $(3.3), (3.8), (3.15), (3.20), (3.22)$ into (3.2), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_9$, $\theta \in [\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_2 \cup E_5 \cup E_6)$

$$\exp\{(1+\varepsilon)\,\delta(P_1,\theta)\,r^n + \alpha r^n\} \leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}$$

$$\times (T(2r,f))^{k+1} \exp\{(1+\varepsilon)\,c\delta(P_1,\theta)\,r^n\} \exp\{br^n \cos n\theta\}.$$

Then

$$\exp\{((1-c)(1+\varepsilon)\,\delta(P_1,\theta) + \alpha - b\cos n\theta)\,r^n\}$$

$$\leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}(T(2r,f))^{k+1}. \tag{3.29}$$

Set $\gamma = (1-c)(1+\varepsilon)\delta(P_1,\theta) + \alpha - b\cos n\theta$, we obtain

$$\exp\{\gamma r^n\} \leq 2B(k+1)\exp\left\{r^{\rho_1+\varepsilon} + r^{\rho_2+\varepsilon}\right\}(T(2r,f))^{k+1}. \tag{3.30}$$

Since $0 < \varepsilon < \min\{n-\rho_1, n-\rho_2\}$ and $\gamma > 0$, then by using Lemma 2.8 and (3.30), we obtain $\rho_2(f) \geq n$, hence $\rho_2(f) = n$.
(i) Assume that $\theta_1 \neq \theta_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in [-\frac{\pi}{2n}, \frac{\pi}{2n})\setminus(E_2 \cup E_5 \cup E_6)$ satisfying $\delta(P_2,\theta) > 0, \delta(P_1,\theta) < 0$. By Lemma 2.5, and for sufficiently large r, we get (3.17) holds. Using the same reasoning as in second step (Case 1 (i), (b)), we can get $\rho_2(f) = n$. Concluding the above proof, we obtain that every meromorphic solution $f(\not\equiv 0)$ whose poles are of uniformly bounded multiplicities of equation (1.3) satisfies $\rho_2(f) = n$. The proof of Theorem 1.1 is complete.

4. Proofs of Corollary 1.2 and Corollary 1.3

Using the same reasoning as in the proof of Theorem 1.1, we can obtain Corollary 1.1 and Corollary 1.2.

5. Proof of Theorem 1.2

First step. We prove that

$$\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f) = n.$$

Set $F_0 = D_0 + A_1 e^{P_1(z)} + A_2 e^{P_2(z)}$, $F_j = D_j + B_j e^{Q_j(z)}\,(j = 1, ..., k-1)$. Assume that $f(\not\equiv 0)$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.3), then by Theorem 1.1, we have $\rho(f) = +\infty$. Set $g_0(z) = f(z) - \varphi(z)$ with $\rho(\varphi) < n$. We have $g_0(z)$ is a meromorphic function with $\rho(g_0) = \rho(f) = +\infty$ and $\rho_2(g_0) = \rho_2(f) = n$. Substituting $f(z) = g_0(z) + \varphi(z)$ into (1.3), we obtain

$$g_0^{(k)} + F_{k-1}g_0^{(k-1)} + \cdots + F_1 g_0' + F_0 g_0 = K, \tag{4.1}$$

where $K = -\left[\varphi^{(k)} + F_{k-1}\varphi^{(k-1)} + \cdots + F_1\varphi' + F_0\varphi\right]$. We have $K \not\equiv 0$. In fact, if $K \equiv 0$, then

$$\varphi^{(k)} + F_{k-1}\varphi^{(k-1)} + \cdots + F_1\varphi' + F_0\varphi = 0, \tag{4.2}$$

thus, $\varphi(\not\equiv 0)$ is a solution of equation (1.3) and by Theorem 1.1, φ must be of infinite order, which is a contradiction with $\rho(\varphi) < n$. Hence, $K \not\equiv 0$. By Lemma 2.9, we have

$$\overline{\lambda}(g_0) = \lambda(g_0) = \rho(g_0) = +\infty,\ \overline{\lambda}_2(g_0) = \lambda_2(g_0) = \rho_2(g_0) = n.$$

Then

$$\overline{\lambda}(f-\varphi)=\lambda(f-\varphi)=\rho(f)=+\infty,$$

$$\overline{\lambda}_2(f-\varphi)=\lambda_2(f-\varphi)=\rho_2(f)=n.$$

Second step. Now, we prove that

$$\overline{\lambda}(f'-\varphi)=\lambda(f'-\varphi)=\rho(f)=+\infty,$$

$$\overline{\lambda}_2(f'-\varphi)=\lambda_2(f'-\varphi)=\rho_2(f)=n.$$

Set $g_1(z)=f'(z)-\varphi(z)$ with $\rho(\varphi)<n$. We have $g_1(z)$ is a meromorphic function with $\rho(g_1)=\rho(f')=+\infty$ and $\rho_2(g_1)=\rho_2(f)=n$. Using Lemma 2.10, we get that $f'(z)$ is a solution of the equation

$$(f')^{(k)}+F_{k-1}^1(f')^{(k-1)}+\cdots+F_1^1(f')'+F_0^1f'=0, \tag{4.3}$$

where $F_j^1(z),(j=0,1,...,k-1)$ are defined by (2.4). By (4.3), we obtain

$$f^{(k+1)}+F_{k-1}^1f^{(k)}+\cdots+F_1^1f''+F_0^1f'=0. \tag{4.4}$$

Substituting $f^{(j+1)}(z)=g_1^{(j)}(z)+\varphi^{(j)}(z),(j=0,1,...,k)$ into (4.4), we obtain

$$g_1^{(k)}+F_{k-1}^1g_1^{(k-1)}+\cdots+F_1^1g_1'+F_0^1g_1=h_1, \tag{4.5}$$

where

$$h_1=-\left[\varphi^{(k)}+F_{k-1}^1\varphi^{(k-1)}+\cdots+F_1^1\varphi'+F_0^1\varphi\right]. \tag{4.6}$$

We can get

$$F_j^1=\frac{N_j}{F_0}\ (j=0,1,...,k-1), \tag{4.7}$$

where

$$\begin{gathered}N_0=F_1'F_0+F_0^2-F_1F_0',\\ N_j=(F_{j+1})'F_0+F_jF_0-F_{j+1}(F_0)',\ \ (j=1,...,k-2),\\ N_{k-1}=F_{k-1}F_0-F_0'.\end{gathered} \tag{4.8}$$

Now, we prove that $h_1\not\equiv 0$. In fact, if $h_1\equiv 0$, then $\frac{h_1}{\varphi}\equiv 0$. By (4.6) and (4.7), we have

$$\frac{\varphi^{(k)}}{\varphi}F_0+N_{k-1}\frac{\varphi^{(k-1)}}{\varphi}+\cdots+N_1\frac{\varphi'}{\varphi}+N_0=0, \tag{4.9}$$

with $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, ..., k)$ are meromorphic functions with $\rho\left(\frac{\varphi^{(j)}}{\varphi}\right) < n$ $(j = 1, ..., k)$. Using (2.4) and (4.8), we can rewrite (4.9) in the form

$$2A_1A_2e^{P_1(z)+P_2(z)} + A_1^2e^{2P_1(z)} + A_2^2e^{2P_2(z)} + \sum_{j=1}^{k-1} f_{1,j}e^{P_1(z)+Q_j(z)}$$

$$+\sum_{j=1}^{k-1} f_{2,j}e^{P_2(z)+Q_j(z)} + f_{1,0}e^{P_1(z)} + f_{2,0}e^{P_2(z)} = M, \tag{4.10}$$

where $M = D_0 + D_1 + D_{k-1} + \sum_{j=0}^{k-2} \left(D'_{j+1} + D_j - D_{j+1}\right) \frac{\varphi^{(j)}}{\varphi}$, $f_{1,j}$, $f_{2,j}$ $(j = 0, ..., k-1)$ are meromorphic functions of order less than n. Set

$$J_1 = \{a_{1,n}, a_{2,n}, 2a_{1,n}, 2a_{2,n}, a_{1,n} + a_{2,n},$$
$$a_{1,n} + b_{jn}, a_{2,n} + b_{jn} \ \ (j = 1, ..., k-1)\}.$$

Because

$$\begin{cases} 2a_{1,n} \neq a_{1,n}, \\ 2a_{1,n} \neq a_{1,n} + a_{2,n}, \\ 2a_{1,n} \neq 2a_{2,n}, \end{cases}$$

then by Lemma 2.13, we have $2a_{1,n} \neq a_{1,n} + b_{jn}$ $(j = 1, ..., k-1)$.
(i) If $2a_{1,n} \neq a_{2,n}, 2a_{1,n} \neq a_{2,n} + b_{jn} \,(j = 1, ..., k-1)$, then we can rewrite (4.10) in the form

$$A_1^2e^{2P_1(z)} + \sum_{\beta\in\Gamma}^{k-1} \alpha_\beta e^{R_\beta(z)} = M,$$

where $\Gamma \subseteq J_1 \backslash \{2a_{1,n}\}, \alpha_\beta$ $(\beta \in \Gamma)$, M are meromorphic functions of order less than n and $R_\beta(z)$ are non-constant polynomials with degree n. By Lemma 2.11 and Lemma 2.12, we get $A_1 \equiv 0$, which is a contradiction.
(ii) If $2a_{1,n} = \eta$ such that $\eta \in \{a_{2,n}, a_{2,n} + b_{jn} \,(j = 1, ..., k-1)\}$, then by Lemma 2.14, we obtain $2a_{2,n} \neq \lambda \in J_1 \backslash \{2a_{2,n}\}$. Hence, we can rewrite (4.10) in the form

$$A_2^2e^{2P_2(z)} + \sum_{\beta'\in\Gamma'}^{k-1} \alpha_{\beta'} e^{R_{\beta'}(z)} = M,$$

where $\Gamma' \subseteq J_1 \backslash \{2a_{2,n}\}, \alpha_{\beta'}$ $(\beta' \in \Gamma')$, M are meromorphic functions of order less than n and $R_{\beta'}(z)$ are non-constant polynomials with degree n. By Lemma 2.11 and Lemma 2.12, we get $A_2 \equiv 0$, which is a contradiction. Thus, $h_1 \not\equiv 0$ is proved. By Lemma 2.9, we have

$$\overline{\lambda}(g_1) = \lambda(g_1) = \rho(g_1) = +\infty,\ \overline{\lambda}_2(g_1) = \lambda_2(g_1) = \rho_2(g_1) = n.$$

Then

$$\overline{\lambda}(f' - \varphi) = \lambda(f' - \varphi) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f' - \varphi) = \lambda_2(f' - \varphi) = \rho_2(f) = n.$$

Third step. Now, we prove that

$$\overline{\lambda}(f'' - \varphi) = \lambda(f'' - \varphi) = \rho(f) = +\infty,$$
$$\overline{\lambda}_2(f'' - \varphi) = \lambda_2(f'' - \varphi) = \rho_2(f) = n.$$

Set $g_2(z) = f''(z) - \varphi(z)$ with $\rho(\varphi) < n$. We have $g_2(z)$ is a meromorphic function with $\rho(g_2) = \rho(f'') = +\infty$ and $\rho_2(g_2) = \rho_2(f) = n$. Using Lemma 2.10, we get that $f''(z)$ is a solution of the equation

$$(f'')^{(k)} + F_{k-1}^2 (f'')^{(k-1)} + \cdots + F_1^2 (f'')' + F_0^2 f'' = 0, \tag{4.11}$$

where $F_j^2(z), (j = 0, 1, ..., k-1)$ are defined by (2.4). By (4.11), we obtain

$$f^{(k+2)} + F_{k-1}^2 f^{(k+1)} + \cdots + F_1^2 f^{(3)} + F_0^2 f'' = 0. \tag{4.12}$$

Substituting $f^{(j+2)}(z) = g_2^{(j)}(z) + \varphi^{(j)}(z), (j = 0, 1, ..., k)$ into (4.12), we obtain

$$g_2^{(k)} + F_{k-1}^2 g_2^{(k-1)} + \cdots + F_1^2 g_2' + F_0^2 g_2 = h_2, \tag{4.13}$$

where

$$h_2 = -\left[\varphi^{(k)} + F_{k-1}^2 \varphi^{(k-1)} + \cdots + F_1^2 \varphi' + F_0^2 \varphi\right]. \tag{4.14}$$

Now, we prove that $F_0^1 \not\equiv 0$. Suppose that $F_0^1 \equiv 0$. Then, we have

$$2A_1A_2e^{P_1(z)+P_2(z)} + A_1^2 e^{2P_1(z)} + A_2^2 e^{2P_2(z)} + f_1 e^{P_1(z)+Q_1(z)}$$
$$+ f_2 e^{P_2(z)+Q_1(z)} + f_{1,0} e^{P_1(z)} + f_{2,0} e^{P_2(z)} = D,$$

where $D = D_0^2 + D_1 D_0' + D_0 D_1'$, f_1, f_2 are meromorphic functions of order less than n. By using the same reasoning as above, we can get a contradiction. Hence, $F_0^1 \not\equiv 0$. In this case, we can get

$$F_j^2 = \frac{M_j}{F_0^1}\ \ (j = 0, 1, ..., k-1), \tag{4.15}$$

where

$$\begin{aligned} M_0 &= (F_1^1)' F_0^1 + (F_0^1)^2 - F_1^1 (F_0^1)', \\ M_j &= \left(F_{j+1}^1\right)' F_0^1 + F_j^1 F_0^1 - F_{j+1}^1 (F_0^1)',\ \ (j = 1, ..., k-2), \\ M_{k-1} &= F_{k-1}^1 F_0^1 - (F_0^1)'. \end{aligned} \tag{4.16}$$

We can denote equations (4.13) and (4.14) by the following form

$$F_0^1 g_2^{(k)} + M_{k-1} g_2^{(k-1)} + \cdots + M_1 g_2' + M_0 g_2 = h_2, \tag{4.17}$$

where

$$h_2 = -\left[\varphi^{(k)} F_0^1 + M_{k-1}\varphi^{(k-1)} + \cdots + M_1\varphi' + M_0\varphi\right]. \tag{4.18}$$

Now we prove that $h_2 \not\equiv 0$. In fact, if $h_2 \equiv 0$, then $\frac{h_2}{\varphi} \equiv 0$. By (4.18) , we have

$$F_0^1 \frac{\varphi^{(k)}}{\varphi} + M_{k-1}\frac{\varphi^{(k-1)}}{\varphi} + \cdots + M_1\frac{\varphi'}{\varphi} + M_0 = 0, \tag{4.19}$$

with $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, ..., k)$ are meromorphic functions with $\rho\left(\frac{\varphi^{(j)}}{\varphi}\right) < n$ $(j = 1, ..., k)$. Using (2.4) and (4.16) , we can rewrite (4.19) in the form

$$\begin{aligned} &A_1^3 e^{3P_1(z)} + A_2^3 e^{3P_2(z)} + 3A_1^2 A_2 e^{2P_1(z)+P_2(z)} + 3A_1 A_2^2 e^{P_1(z)+2P_2(z)} \\ &+ f_{1,0} e^{2P_1(z)} + f_{2,0} e^{2P_2(z)} + f_{3,0} e^{P_1(z)+P_2(z)} + \sum_{j=1}^{k-1} f_{1,j} e^{2P_1(z)+Q_j(z)} \\ &+ \sum_{j=1}^{k-1} f_{2,j} e^{2P_2(z)+Q_j(z)} + \sum_{j=1}^{k-1} f_{3,j} e^{P_1(z)+P_2(z)+Q_j(z)} \\ &+ l_{1,0} e^{P_1(z)+Q_1(z)} + l_{2,0} e^{P_2(z)+Q_1(z)} \\ &+ \sum_{j=1}^{k-1} l_{1,j} e^{P_1(z)+Q_1(z)+Q_j(z)} + \sum_{j=1}^{k-1} l_{2,j} e^{P_2(z)+Q_1(z)+Q_j(z)} = H \end{aligned} \tag{4.20}$$

with

$$\begin{aligned} H = {} & H_0 + H_1\left(1 + D_2'' + D_1'\right) + H_2\left(-D_2' - D_1\right) \\ & + \left(H_1\left(\sum_{j=1}^{k-1} H_j\right) + H_2\left(\sum_{j=1}^{k-1} K_j\right)\right)\frac{\varphi^{(j)}}{\varphi}, \end{aligned}$$

where

$$\begin{cases} H_0 = D_2'' + D_1' + H_1 + H_2\left(-D_2' - D_1\right), \\ H_1 = D_1' + D_0, \\ H_2 = D_1'' + D_0', \\ H_j = D_{j+1}'' + D_{j+2}' + D_j, \\ K_j = D_{j+1}' - D_j \end{cases}$$

and $f_{1,j}, f_{2,j}, f_{3,j,} l_{1,j}, l_{2,j}$ $(j = 0, 1, ..., k-1)$ are meromorphic functions of order less than n. Set

$$J_2 = \{3a_{1,n}, 3a_{2,n}, 2a_{1,n} + a_{2,n}, a_{1,n} + 2a_{2,n}, 2a_{1,n}, 2a_{2,n}, a_{1,n} + a_{2,n},$$

$$2a_{1,n}+b_{jn},2a_{2,n}+b_{jn},a_{1,n}+a_{2,n}+b_{jn},a_{1,n}+b_{1n},$$
$$a_{2,n}+b_{1n},a_{1,n}+b_{1,n}+b_{jn},a_{2,n}+b_{1,n}+b_{jn}\,(j=1,...,k-1)\}\,.$$

Because

$$\begin{cases} 3a_{1,n}\neq 2a_{1,n}, \\ 3a_{1,n}\neq 2a_{1,n}+a_{2,n}, \\ 3a_{1,n}\neq 3a_{2,n}, \\ 3a_{1,n}\neq a_{1,n}+2a_{2,n}, \end{cases}$$

then by Lemma 2.13, we have $3a_{1,n}\neq a_{1,n}+b_{1n},2a_{1,n}+b_{jn},a_{1,n}+a_{2,n}+b_{jn}\,(j=1,...,k-1)$.

(i) If $3a_{1,n}\neq 2a_{2,n},a_{1,n}+a_{2,n},a_{2,n}+b_{1,n},2a_{2,n}+b_{jn},a_{1,n}+a_{2,n}+b_{jn},a_{2,n}+b_{1,n}+b_{jn}\,(j=1,...,k-1)$, then we can rewrite (4.20) in the form

$$A_1^3e^{3P_1(z)}+\sum_{\beta\in\Gamma}^{k-1}\alpha_\beta e^{S_\beta(z)}=H,$$

where $\Gamma\subseteq J_2\backslash\{3a_{1,n}\}\,,\alpha_\beta\ (\beta\in\Gamma)\,,\ H$ are meromorphic functions of order less than n and $S_\beta(z)$ are non-constant polynomials with degree n. By Lemma 2.11 and Lemma 2.12, we get $A_1\equiv 0$, which is a contradiction.

(ii) If $3a_{1,n}=\eta$ such that

$$\eta\in\{2a_{2,n},a_{1,n}+a_{2,n},a_{2,n}+b_{1n},2a_{2,n}+b_{jn},a_{1,n}+a_{2,n}+b_{jn},$$
$$a_{2,n}+b_{1,n}+b_{jn}\ \ (j=1,...,k-1)\}\,,$$

then by Lemma 2.14, we obtain $3a_{2,n}\neq\lambda$ for all $\lambda\in J_2\backslash\{3a_{2,n}\}$. Hence, we can rewrite (4.20) in the form

$$A_2^3e^{3P_2(z)}+\sum_{\beta'\in\Gamma'}^{k-1}\alpha_{\beta'}e^{S_{\beta'}(z)}=H,$$

where $\Gamma'\subseteq J_2\backslash\{3a_{2,n}\}\,,\alpha_{\beta'}\ (\beta'\in\Gamma')\,,\ M$ are meromorphic functions of order less than n and $S_{\beta'}(z)$ are non-constant polynomials with degree n. By Lemma 2.11 and Lemma 2.12, we get $A_2\equiv 0$, which is a contradiction. Thus, $h_2\not\equiv 0$ is proved. By Lemma 2.9, we have

$$\overline{\lambda}(g_2)=\lambda(g_2)=\rho(g_2)=+\infty,\ \overline{\lambda}_2(g_2)=\lambda_2(g_2)=\rho_2(g_2)=n.$$

Then

$$\overline{\lambda}(f''-\varphi)=\lambda(f''-\varphi)=\rho(f)=+\infty,$$
$$\overline{\lambda}_2(f''-\varphi)=\lambda_2(f''-\varphi)=\rho_2(f)=n.$$

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