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A NEW MENON-TYPE IDENTITY DERIVED FROM GROUP ACTIONS

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Abstract. In this short note, we give a new Menon-type identity involving the sum of element orders and the sum of cyclic subgroup orders of a finite group. It is based on applying the weighted form of Burnside's lemma to a natural group action.

1. Introduction

One of the most interesting arithmetical identities is due to P.K. Menon [5].

Menon's identity. *For every positive integer n we have*

$$\sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)=\varphi(n)\,\tau(n),$$

where $\mathbb{Z}_n^*$ is the group of units of the ring $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$, $\gcd(,)$ *represents the greatest common divisor, φ is the Euler's totient function and $\tau(n)$ is the number of divisors of n.*

There are several approaches to Menon's identity and many generalisations. One of the methods used to prove Menon-type identities is based on the Burnside's Lemma concerning group actions (see e.g. [5–10]). In what follows, we will use a generalization of this result, called the Weighted Form of Burnside's Lemma (see e.g. [2]).

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Weighted Form of Burnside's Lemma. *Given a finite group G acting on a finite set X, we denote*

$$Fix(g) = \{x \in X : g \circ x = x\}, \forall\, g \in G.$$

Let R be a commutative ring containing the rationals and $w : X \longrightarrow R$ be a weight function that is constant on the distinct orbits O_{x_1}, ..., O_{x_k} of X. For every $i = 1, ..., k$, let $w(O_{x_i}) = w(x)$, where $x \in O_{x_i}$. Then

$$\sum_{i=1}^{k} w(O_{x_i}) = \frac{1}{|G|} \sum_{g\in G} \sum_{x\in Fix(g)} w(x). \tag{1}$$

Note that the Burnside's Lemma is obtained from (1) by taking the weight function $w(x) = 1$, $\forall\, x \in X$.

Next we will consider a finite group G of order n and the functions

$$\psi(G) = \sum_{g\in G} o(g) \ \text{ and } \ \sigma(G) = \sum_{H\in C(G)} |H|,$$

where $o(g)$ is the order of $g \in G$ and $C(G)$ is the set of cyclic subgroups of G.[1] Also, for every divisor m of n, we will denote $G_m = \{g \in G : g^m = 1\}$.

Our main result is stated as follows.

Theorem 1. *Under the above notations, we have*

$$\sum_{a\in\mathbb{Z}_n^*} \psi(G_{\gcd(a-1,n)}) = \varphi(n)\, \sigma(G). \tag{2}$$

Clearly, (2) gives a new connection between the above functions $\psi(G)$ and $\sigma(G)$. We remark that an alternative way of writing (2) is

$$\sum_{a\in\mathbb{Z}_n^*} \ \sum_{d\,|\,\gcd(a-1,n)} d\varphi(d) n_d(G) = \varphi(n)\, \sigma(G), \tag{3}$$

where $n_d(G)$ denotes the number of cyclic subgroups of order d in G, for all d dividing n.

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[1] For more details concerning these functions, we refer the reader to [1,3] and [4], respectively.

For $G = \mathbb{Z}_n$, Theorem 1 leads to the following corollary.

Corollary 2. *We have*

$$\sum_{a\in\mathbb{Z}_n^*} \psi(\mathbb{Z}_{\gcd(a-1,n)}) = \varphi(n)\,\sigma(n), \tag{4}$$

where $\sigma(n)$ is the sum of divisors of n.

Finally, since $\psi(\mathbb{Z}_n) \geq \frac{q}{p+1}\, n^2$, where q and p are the smallest and the largest prime divisor of $n \geq 2$ (see the proof of Lemma 2.9(2) in [3]), from (4) we infer the following inequalities.

Corollary 3. *We have*

$$\frac{q}{p+1}\,\frac{1}{\varphi(n)} \sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)^2 \leq \sigma(n) \leq \frac{1}{\varphi(n)} \sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)^2. \tag{5}$$

2. Proof of Theorem 1

Let $\mathbb{Z}_n^* = \{a \in \mathbb{N} : 1 \leq a \leq n, \gcd(a,n) = 1\}$ be the group of units (mod n). The natural action of $\mathbb{Z}_n^*$ on G is defined by

$$a \circ g = g^a,\ \forall\,(a,g) \in \mathbb{Z}_n^* \times G.$$

Then two elements of G belong to the same orbit if and only if they generate the same cyclic subgroup. This shows that the weight function $w : G \longrightarrow \mathbb{R}$, $w(g) = o(g)$, $\forall\, g \in G$, is constant on the distinct orbits O_{g_1}, ..., O_{g_k} of G. Thus we can apply the Weighted Form of Burnside's Lemma.

First of all, we observe that $w(O_{g_i}) = o(g_i) = |\langle g_i\rangle|$, $\forall\, i = 1, ..., k$, and therefore the left side of (1) is $\sigma(G)$.

Next we will prove that $Fix(a) = G_{\gcd(a-1,n)}$, for any $a \in \mathbb{Z}_n^*$. Indeed, if $g \in Fix(a)$ then $g^a = g$, that is $g^{a-1} = 1$. Since $|G| = n$, we also have $g^n = 1$. Consequently, $g^{\gcd(a-1,n)} = 1$, i.e. $g \in G_{\gcd(a-1,n)}$. The converse inclusion is obvious.

Now (1) becomes

$$\sigma(G) = \frac{1}{\varphi(n)} \sum_{a\in\mathbb{Z}_n^*}\ \sum_{g\in G_{\gcd(a-1,n)}} o(g) = \frac{1}{\varphi(n)} \sum_{a\in\mathbb{Z}_n^*} \psi(G_{\gcd(a-1,n)}),$$

as desired. ∎

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