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NEW TYPES OF FUZZY CONTINUITY VIA β-SEMIOPEN SET

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Abstract. This paper deals with a new type of fuzzy open-like sets, viz., fuzzy β-semiopen sets, the class of which is strictly larger than that of fuzzy semiopen sets [1], but strictly smaller than the classes of fuzzy β-open sets [8], respectively of fuzzy e^*-open sets [4]. It is shown that the collection fuzzy β-semiopen sets does not form a fuzzy topology. In Section 4, a new type of continuous-like function, viz., fuzzy (β-semi, r)-continuous function is introduced and studied. In Section 5, some applications of this new type of function are established.

1. Introduction

After introduction of the notion of fuzzy open set by Chang [7], several classes of fuzzy open-like sets have been studied, in connection with generalized form of fuzzy continuity [1, 2, 4, 5, 6]. In this context we have to mention [1, 2, 4, 5, 6, 8]. In [4], fuzzy δ-semiopen, fuzzy e-open, fuzzy e^*-open, fuzzy a-open sets are introduced and studied.

Keywords and phrases: Fuzzy β-semiopen set, fuzzy e^*-open set, fuzzy semiopen set, fuzzy (β-semi, r)-continuous function, fuzzy extremally disconnected space, fuzzy β-semicontinuous function, fuzzy almost β-semicontinuous function.
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Here we introduce fuzzy β-semiopen set the class of which is strictly larger than that of fuzzy semiopen sets, fuzzy δ-semiopen sets, fuzzy a-open sets, fuzzy δ-open sets [9], fuzzy α-open sets [6], fuzzy preopen sets [12], fuzzy γ-open sets [5] and strictly smaller than the class of fuzzy e^*-open sets and fuzzy β-open sets [8]. It is also shown that the concept fuzzy β-semiopen set is independent concept of fuzzy δ-preopen set [2], fuzzy e-open set. Next we introduce fuzzy (β-semi, r)-continuity, fuzzy β-continuity, fuzzy almost β-continuity and establish the mutual relationships of the newly defined functions with the functions defined in [4]. Lastly a new type of separation axiom is introduced and studied in connection with fuzzy (β-semi, r)-continuity.

2. Preliminaries

Throughout the paper, (X,τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [7]. In [15], Zadeh introduced fuzzy set as follows : A fuzzy set A in an fts X is a mapping from a non-empty set X into the closed interval $I=[0,1]$, i.e., $A \in I^X$. The support [15] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [15] of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$ [15]. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [15] while AqB means A is quasi-coincident (q-coincident, for short) [13] with B, i.e., there exists $x \in X$ such that $A(x)+B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy set A, clA and $intA$ stand for fuzzy closure and fuzzy interior of A in X [7]. $A \in I^X$ is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [12], fuzzy α-open [6], fuzzy β-open [8], fuzzy γ-open [5]) if $A = int(clA)$ (resp., $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$, $A \leq (clintA) \bigvee (intclA))$. The complement of fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open) set is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy preclosed [12], fuzzy α-closed [6], fuzzy β-closed [8], fuzzy γ-closed [5]) set. The fuzzy δ-closure and fuzzy δ-interior [9] of a fuzzy set A in X are defined as : $\delta clA = \{x_\alpha \in X : Aq(int(clU))$, for all $U \in \tau$ with $x_\alpha qU\}$, $\delta intA = \bigvee\{W : W$ is fuzzy regular open in $X, W \leq A\}$. $A \in I^X$ is called fuzzy δ-preopen if $A \leq int(\delta clA)$ [2].

The complement of a fuzzy δ-preopen set is called fuzzy δ-preclosed [2]. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed, fuzzy δ-preclosed) set containing a fuzzy set A in X is called fuzzy semiclosure [1] (resp., fuzzy prelosure [12], fuzzy α-closure [6], fuzzy β-closure [8], fuzzy γ-closure [5], fuzzy δ-preclosure [2]) of A, denoted by $sclA$ (resp., $pclA$, αclA, βclA, γclA, $\delta pclA$). $A \in I^X$ is fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed, fuzzy δ-closed, fuzzy δ-preclosed) if $A = sclA$ (resp., $A = pclA$, $A = \alpha clA$, $A = \beta clA$, $A = \gamma clA$, $A = \delta clA$, $A = \delta pclA$). The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open, fuzzy δ-open, fuzzy δ-preopen) sets in X is denoted by $FRO(X)$ (rssp., $FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $F\beta O(X)$, $F\gamma O(X)$, $F\delta O(X)$, $F\delta PO(X)$) and the collection of all fuzzy regular closed (resp., fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed, fuzzy δ-closed, fuzzy δ-preclosed) sets in X is denoted by $FRC(X)$ (rssp., $FSC(X)$, $FPC(X)$, $F\alpha C(X)$, $F\beta C(X)$, $F\gamma C(X)$, $F\delta C(X)$, $F\delta PC(X)$). For a fuzzy open set A in X, $sclA = int(clA)$ [3].

3. Fuzzy β-Semiopen Set : Some Characterizations

In this section we first recall some definition from [4]

Definition 3.1 [4]. Let (X, τ) be an fts and $A \in I^X$. A fuzzy point x_α in X is said to be fuzzy θ-semicluster point of A if $clUqA$ for all $U \in FSO(X)$ with $x_\alpha qU$. The union of all fuzzy θ-semicluster points of A is called fuzzy θ-semiclosure of A and is denoted by θ-$sclA$.
$A(\in I^X)$ is fuzzy θ-semiclosed if $A = \theta$-$sclA$. The complement of a fuzzy θ-semiclosed set is called fuzzy θ-semiopen.

Definition 3.2 [4]. Let (X, τ) be an fts and $A \in I^X$. Then r-kernel of A, denoted by r-Ker$A$, is defined as follows :

$$r\text{-Ker}A = \bigwedge\{U : U \in FRO(X), A \leq U\}.$$

Definition 3.3 [4]. Let (X, τ) be an fts and $A \in I^X$. Then A is said to be fuzzy
(i) δ-semiopen if $A \leq cl(\delta intA)$,
(ii) e-open if $A \leq cl(\delta intA) \bigvee int(\delta clA)$,
(iii) e^*-open if $A \leq cl(int(\delta clA))$,
(iv) a-open if $A \leq int(cl(\delta intA))$.
The complements of the above mentioned fuzzy sets are called their

respective closed sets.

Let us now introduce the following concept.

Definition 3.4. A fuzzy set A in an fts (X, τ) is called fuzzy β-semiopen if $A \leq cl(int(sclA))$.
The complement of a fuzzy β-semiopen set is called fuzzy β-semiclosed.

Note 3.5. The collection of all fuzzy δ-semiopen (resp., fuzzy e-open, fuzzy e^*-open, fuzzy a-open, fuzzy β-semiopen) sets in X is denoted by $F\delta SO(X)$ (resp., $FeO(X)$, $Fe^*O(X)$, $FaO(X)$, $F\beta SO(X)$). The collection of all fuzzy δ-semiclosed (resp., fuzzy e-closed, fuzzy e^*-closed, fuzzy a-closed, fuzzy β-semiclosed) sets in X is denoted by $F\delta SC(X)$ (resp., $FeC(X)$, $Fe^*C(X)$, $FaC(X)$, $F\beta SC(X)$).

Remark 3.6. The union of any two fuzzy β-semiopen sets is also so. But the intersection of any two fuzzy β-semiopen sets may not be so, as it seen from the following example.

Example 3.7. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus A, V \geq B$ and $FSC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $A \leq 1_X \setminus U \leq 1_X \setminus A, 1_X \setminus V \leq 1_X \setminus B$. Consider two fuzzy sets C and D defined by $C(a) = 0.5, C(b) = 0.3, D(a) = 0.3, D(b) = 0.6$. Then $cl(int(sclC)) = cl(int(sclD)) = 1_X \setminus A$ and $1_X \setminus A \geq C, 1_X \setminus A \geq D$. So C and D are fuzzy β-semiopen sets in (X, τ). Let $E = C \bigwedge D$. Then $E(a) = E(b) = 0.3$. But $cl(int(sclE)) = 0_X \ngeq E$ implies that $E \notin F\beta SO(X)$.

Remark 3.8. It is clear from definitions that
(i) fuzzy open set implies fuzzy semiopen set implies fuzzy β-semiopen set implies fuzzy e^*-open set,
(ii) $FPO(X), F\alpha O(X), F\gamma O(X), F\delta O(X), F\delta SO(X), FaO(X) \subseteq F\beta SO(X)$,
(iii) as for any fuzzy set $A \in I^X, sclA \leq clA, F\beta SO(X) \subseteq F\beta O(X)$,
(iv) fuzzy β-semiopen is an independent concept of fuzzy e-open and fuzzy δ-preopen sets, as follow from the following examples.

Example 3.9. $F\beta SO(X) \supset \tau$ where τ is the topology of an fts (X, τ)
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.4$. Then $B \notin \tau$. But as $cl(int(sclB)) = A \geq B$, $B \in F\beta SO()$.

Example 3.10. $F\beta SO(X) \supset FSO(X), F\alpha O(X)$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then $cl(int(sclB)) = 1_X \setminus A \geq B$ implies that $B \in F\beta SO(X)$. But $cl(intB) = 0_X \ngeq B$ implies that $B \notin FSO(X)$. Also $int(cl(intB)) = 0_X \ngeq B$ and so $B \notin F\alpha O(X)$.

Example 3.11. $F\beta SO(X) \supset F\gamma O(X)$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7.B(b) = 0.5$. Then (X, τ) is an fts. Consider the fuzzy set C defined by $C(a) = 0.4, C(b) = 0.5$. Then $(intclC) \bigvee (clintC) = A \ngeq C$ implies that $C \notin F\gamma O(X)$. But $cl(int(sclC)) = 1_X \setminus A \geq C$ and so $C \in F\beta SO(X)$.

Example 3.12. $F\beta SO(X) \supset FPO(X)$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0$. Then (X, τ) is an fts. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Then $int(clC) = B \ngeq C$ implies that $C \notin FPO(X)$. But $cl(int(sclC)) = 1_X \setminus A \geq C$. Hence $C \in F\beta SO(X)$.

Example 3.13. $Fe^*O(X) \supset F\beta SO(X)$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then clearly $B \in Fe^*O(X)$. But as $cl(int(sclB)) = 0_X \ngeq B$, $B \notin F\beta SO(X)$.

Example 3.14. $F\beta SO(X) \supset F\delta O(X)$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Clearly $B \notin F\delta O(X)$. But $cl(int(sclB)) = 1_X \setminus A \geq B$ and so $B \in F\beta SO(X)$.

Example 3.15. $F\delta PO(X) \supset F\beta SO(X)$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.45$. Then (X,τ) is an fts. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.59$. Then $cl(int(sclC)) = 1_X \setminus B \not\geq C$ implies that $C \notin F\beta SO(X)$. But $int(\delta clC) = 1_X > C$. Hence $C \in F\delta PO(X)$.

Example 3.16. $F\beta SO(X) \supset F\delta PO(X)$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5$. Then (X,τ) is an fts. Consider the fuzzy set D defined by $D(a) = 0.46, D(b) = 0.6$. Then $cl(int(sclD)) = 1_X \setminus B \geq D$and so $D \in F\beta SO(X)$. But $int(\delta clD) = C \not\geq D$ and consequently, $D \notin F\delta PO(X)$.

Example 3.17. $FeO(X) \supset F\beta SO(X)$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.7$. Then (X,τ) is an fts. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Then $cl(int(sclC)) = 0_X \not\geq C$ implies that $C \notin F\beta SO(X)$. But as $int(\delta clC) = A \geq C$, $(cl\delta intC) \bigvee (int\delta clC) \geq C$. Hence $C \in FeO(X)$.

Example 3.18. $F\beta SO(X) \supset FeO(X), FaO(X), F\delta SO(X)$
Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B, C\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5$. Then (X,τ) is a fts. Consider the fuzzy set D defined by $D(a) = 0.56, D(b) = 0.6$. Then $cl(int(sclD)) = 1_X \setminus B \geq D$ implies that $D \in F\beta SO(X)$. Now $(cl\delta intD) \bigvee (int\delta clD) = C \not\geq D$ and so $D \notin FeO(X)$. Also $D \notin F\delta SO(X)$. Again $int(cl(\delta intD)) = B \not\geq D$. Hence $D \notin FaO(X)$.

Note 3.19. For a fuzzy semiclosed set A, $A \in F\beta SO(X)$ implies that $A \in FSO(X)$.

Theorem 3.20. Let (X,τ) be an fts. Then the union of any collection of fuzzy β-semiopen sets in X is fuzzy β-semiopen in X.

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy β-semiopen sets in X. Then for any $\alpha \in \Lambda$, $G_\alpha \leq cl(int(sclG_\alpha))$. Also, $G_\alpha \leq \bigvee_{\alpha\in\Lambda} G_\alpha$. Then $sclG_\alpha \leq scl(\bigvee_{\alpha\in\Lambda} G_\alpha)$ implies that

$G_\alpha \leq cl(int(sclG_\alpha)) \leq cl(int(scl(\bigvee_{\alpha\in\Lambda} G_\alpha)))$, and this is true for all $\alpha \in \Lambda$. Taking union on both sides, $\bigvee_{\alpha\in\Lambda} G_\alpha \leq cl(int(scl(\bigvee_{\alpha\in\Lambda} G_\alpha)))$. Hence $\bigvee_{\alpha\in\Lambda} G_\alpha$ is a fuzzy β-semiopen in X.

Let us now introduce a new type of closure-like operator.

Definition 3.21. Let (X, τ) be an fts and $A \in I^X$. Then fuzzy β-semiclosure of A, denoted by $\beta sclA$, is defined by $\beta sclA = \bigwedge\{U \in I^X : A \leq U, U \in F\beta SC(X)\}$.

Lemma 3.22. Let (X, τ) be an fts. Then the following statements are true :
(i) for any fuzzy point x_α in X and any $U \in I^X$, $x_\alpha \in \beta sclU$ and so for any $V \in F\beta SO(X)$ with $x_\alpha qV$, VqU,
(ii) for any two fuzzy sets U, V where $V \in F\beta SO(X)$, $U \not{q} V$. Hence $\beta sclU \not{q} V$.

Proof (i). Let $x_\alpha \in \beta sclU$ and $V \in F\beta SO(X)$ with $x_\alpha qV$. Then $x_\alpha \notin 1_X \setminus V \in F\beta SC(X)$. Then $U \not\leq 1_X \setminus V$ implies that UqV.
(ii). If possible, let $\beta sclUqV$, but $U \not{q} V$. Then there exists $x \in X$ such that $(\beta sclU)(x) + V(x) > 1$ and so $V(x) + t > 1$ where $t = (\beta sclU)(x)$. Then $x_t \in \beta sclU$ where $x_t qV, V \in F\beta SO(X)$. By definition, VqU, a contradiction.

Let us now recall the following Lemma from [4] for ready references.

Lemma 3.23 [4]. Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold.:
(i) for any $A \in FRO(X)$, θ-$sclA = A$,
(ii) for any $A \in F\beta O(X), clA = \alpha clA$,
(iii) for any $A \in FSO(X)$, $clA = pclA$,
(iv) for any $A \in \tau, sclA = \theta$-$sclA$.

4. Fuzzy (β-semi, r)-Continuous Function : Some Characterizations

In this section we introduce and characterize three different types of fuzzy continuous functions and establish the mutual relationships of these newly defined functions with the functions defined in [4].

We first recall the following definitions from [4] for ready references.

Definition 4.1 [4]. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is called fuzzy
(i) (δ, r)-continuous if $f^{-1}(A) \in F\delta C(X)$ for all $A \in FRO(Y)$,
(ii) (δ-semi, r)-continuous if $f^{-1}(A) \in F\delta SC(X)$ for all $A \in FRO(Y)$,
(iii) (δ-pre, r)-continuous if $f^{-1}(A) \in F\delta PC(X)$ for all $A \in FRO(Y)$,
(iv) (e^*, r)-continuous if $f^{-1}(A) \in Fe^*C(X)$ for all $A \in FRO(Y)$,
(v) (e, r)-continuous if $f^{-1}(A) \in FeC(X)$ for all $A \in FRO(Y)$,
(vi) (a, r)-continuous if $f^{-1}(A) \in FaC(X)$ for all $A \in FRO(Y)$.

Let us now introduce the following concept.

Definition 4.2. Let (X, τ) and (Y, τ_1) be two fts's. Then $f : X \to Y$ is called fuzzy (β-semi, r)-continuous function if $f^{-1}(A) \in F\beta SC(X)$, for all $A \in FRO(Y)$.

Remark 4.3. (i) Fuzzy (β-semi, r)-continuity implies fuzzy (e^*, r)-continuity,
(ii) fuzzy (a, r)-continuity, fuzzy (δ-semi, r)-continuity imply fuzzy (β-semi, r)-continuity.
But the reverse implications are not true, in general, follow from the next examples.
(iii) Fuzzy (β-semi, r)-continuity is independent concept of fuzzy (e, r)-continuity and fuzzy (δ-pre, r)-continuity follow from the following examples.

Example 4.4. Fuzzy (e^*, r)-continuity, fuzzy (e, r)-continuity does not imply fuzzy (β-semi, r)-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B, C\}$, $\tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.4, C(a) = 0.45, C(b) = 0.4, D(a) = 0.4, D(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $D \in FRO(X, \tau_2)$. $i^{-1}(D) = D$. Now $FSO(X, \tau_1) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus C, B \leq V \leq 1_X \setminus A$. Then $int_{\tau_1}(cl_{\tau_1}(sint_{\tau_1} D)) = C \not\leq D$ implies that $D \notin F\beta SC(X, \tau_1)$ and so i is not fuzzy (β-semi, r)-continuous function. But $int_{\tau_1}(cl_{\tau_1}(\delta int_{\tau_1} D)) = 0_X \leq D$ implies that $D \in Fe^*C(X, \tau_1)$. So i is fuzzy (e^*, r)-continuous function. Again $(cl_{\tau_1} \delta int_{\tau_1} D) \bigwedge (int_{\tau_1} \delta cl_{\tau_1} D) = 0_X \leq D$. Hence $D \in FeC(X, \tau_1)$

implies that i is fuzzy (e,r)-continuous function.

Example 4.5. Fuzzy (β-semi, r)-continuity does not imply fuzzy (e,r)-continuity, fuzzy (δ-pre, r)-continuity
Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X, A, B, C\}$, $\tau_2 = \{0_X, 1_X, B, D\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.54, D(b) = 0.4$. Then (X,τ_1) and (X,τ_2) are fts's. Consider the identity function $i : (X,\tau_1) \to (X,\tau_2)$. Now $FRO(X,\tau_2) = \tau_2$. Then $i^{-1}(D) = D$, $int_{\tau_1}(cl_{\tau_1}(sint_{\tau_1}D)) = B \leq D$ implies that $D \in F\beta SC(X,\tau_1)$. Also $i^{-1}(B) = B$, $int_{\tau_1}(cl_{\tau_1}(sint_{\tau_1}B)) = B \leq B$ and so $B \in F\beta SC(X,\tau_1)$. Hence i is fuzzy (β-semi, r)-continuous function. But $(int_{\tau_1}\delta cl_{\tau_1}D)\bigwedge(cl_{\tau_1}\delta int_{\tau_1}D) = 1_X \setminus C \nleq D$. Then $D \notin FeC(X,\tau_1)$ which shows that i is not fuzzy (e,r)-continuous function. Again $cl_{\tau_1}(\delta int_{\tau_1}D) = 1_X \setminus C \nleq D$. So$D \notin F\delta PC(X,\tau_1)$. Hence i is not fuzzy (δ-pre, r)-continuous function.

Example 4.6. Fuzzy (δ-pre, r)-continuity does not imply fuzzy (β-semi, r)-continuity
Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.45, C(a) = 0.5, C(b) = 0.41$. Then (X,τ_1) and (X,τ_2) are fts's. Consider the identity function $i : (X,\tau_1) \to (X,\tau_2)$. Here $C \in FRO(X,\tau_2)$. $i^{-1}(C) = C$. Since $cl_{\tau_1}(\delta int_{\tau_1}C) = 0_X < C$. So $C \in F\delta PC(X,\tau_1)$. Then i is fuzzy (δ-pre, r)-continuous function. But $int_{\tau_1}(cl_{\tau_1}(sint_{\tau_1}C)) = B \nleq C$ implies that $C \notin F\beta SC(X,\tau_1)$. Hence i is not fuzzy (β-semi, r)-continuous function.

Example 4.7. Fuzzy (β-semi, r)-continuity does not imply fuzzy (a,r)-continuity, fuzzy (δ-semi, r)-continuity, fuzzy (δ,r)-continuity
Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X, A, B, C\}$, $\tau_2 = \{0_X, 1_X, D\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.44, D(b) = 0.4$. Then (X,τ_1) and (X,τ_2) are fts's. Consider the identity function $i : (X,\tau_1) \to (X,\tau_2)$. Now $D \in FRO(X,\tau_2), i^{-1}(D) = D$. Now $int_{\tau_1}(cl_{\tau_1}(sint_{\tau_1}D)) = B \leq D$ implies $D \in F\beta SC(X,\tau_1)$. So i is fuzzy (β-semi, r)-continuous function. But $cl_{\tau_1}(int_{\tau_1}(\delta cl_{\tau_1}D)) = 1_X \setminus B \nleq D$ and so $D \notin FaC(X,\tau_1)$. Hence i is not fuzzy (a,r)-continuous function. Again $int_{\tau_1}(\delta cl_{\tau_1}D) = C \nleq D$. Then $D \notin F\delta SC(X,\tau_1)$. So i is not fuzzy (δ-semi, r)-continuous function. Also $\delta cl_{\tau_1}D \neq D$ and so $D \notin F\delta C(X,\tau_1)$ which shows that

i is not fuzzy $(\delta,\, r)$-continuous function.

Let us now recall the following definition and theorem from [4] for ready references.

Definition 4.8 [4]. An fts (X, τ) is called fuzzy e^*-$T_{\frac{1}{2}}$-space if every fuzzy e^*-closed set in X is fuzzy δ-closed in X.

Theorem 4.9 [4]. Let (X, τ) and (Y, τ_1) be two fts's. where (X, τ) is fuzzy e^*-$T_{\frac{1}{2}}$-space and $f : X \to Y$ be a function. Then the following statements are equivalent :
(i) f is fuzzy (e^*, r)-continuous,
(ii) f is fuzzy (e, r)-continuous,
(iii) f is fuzzy (δ-semi, r)-continuous,
(iv) f is fuzzy (δ-pre, r)-continuous,
(v) f is fuzzy (a, r)-continuous,
(vi) f is fuzzy (δ, r)-continuous.

Theorem 4.10. Let (X, τ) and (Y, τ_1) be two fts's. where (X, τ) is fuzzy e^*-$T_{\frac{1}{2}}$-space and $f : X \to Y$ be a function. Then the following statements are equivalent :
(i) f is fuzzy (β-semi, r)-continuous,
(ii) f is fuzzy (e^*, r)-continuous,
(iii) f is fuzzy (e, r)-continuous,
(iv) f is fuzzy (δ-semi, r)-continuous,
(v) f is fuzzy (δ-pre, r)-continuous,
(vi) f is fuzzy (a, r)-continuous,
(vii) f is fuzzy (δ, r)-continuous.

Proof (i)$\Rightarrow$ (ii). Let $V \in FRO(Y)$. Then by (i), $f^{-1}(V) \in F\beta SC(X)$. By Remark 3.8 (i), $f^{-1}(V) \in Fe^*C(X)$ and hence f is fuzzy (e^*, r)-continuous function.
(ii)$\Rightarrow$ (vi). Follows from Theorem 4.9 (i)$\Rightarrow$ (v).
(vi)$\Rightarrow$ (i). Let $V \in FRO(Y)$. By (vi), $f^{-1}(V) \in FaC(X)$. By Remark 3.8 (ii), $f^{-1}(V) \in F\beta SC(X)$ and so f is fuzzy (β-semi, r)-continuous function.
The rest follows from Theorem 4.9.

Theorem 4.11. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then the following statements are equivalent :
(i) f is fuzzy (β-semi, r)-continuous,

(ii) $f^{-1}(A) \in F\beta SO(X)$, for all $A \in FRC(Y)$,
(iii) $f(\beta scl_\tau U) \leq r\text{-ker}(f(U))$, for all $U \in I^X$,
(iv) $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A))$, for all $A \in I^Y$,
(v) for each fuzzy point x_α in X and each $A \in FSO(Y)$ with $f(x_\alpha)qA$, there exists $U \in F\beta SO(X)$ with $x_\alpha qU$, $f(U) \leq cl_{\tau_1}A$,
(vi) $f(\beta scl_\tau P)) \leq \theta\text{-}scl_{\tau_1}(f(P))$, for all $P \in I^X$,
(vii) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in I^Y$.
(viii) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in \tau_1$,
(ix) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(scl_{\tau_1}R))$, for all $R \in \tau_1$.
(x) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1}R))$, for all $R \in \tau_1$.
(xi) for each fuzzy point x_α in X and each $A \in FSO(Y)$ with $f(x_\alpha) \in A$, there exists $U \in F\beta SO(X)$ such that $x_\alpha \in U$ and $f(U) \leq cl_{\tau_1}A$,
(xii) $f^{-1}(A) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$, for all $A \in FSO(Y)$,
(xiii) $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta SC(X)$, for all $A \in \tau_1$,
(xiv) $f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta SO(X)$, for all $F \in \tau_1^c$,
(xv) $f^{-1}(cl_{\tau_1}U) \in F\beta SO(X)$, for all $U \in F\beta O(Y)$,
(xvi) $f^{-1}(cl_{\tau_1}U) \in F\beta SO(X)$, for all $U \in FSO(Y)$,
(xvii) $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta SC(X)$, for all $U \in FPO(Y)$,
(xviii) $f^{-1}(\alpha cl_{\tau_1}U) \in F\beta SO(X)$, for all $U \in F\beta O(Y)$,
(xix) $f^{-1}(pcl_{\tau_1}U) \in F\beta SO(X)$, for all $U \in FSO(Y)$,
(xx) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in FSO(Y)$,
(xxi) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in FPO(Y)$,
(xxii) $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$, for all $R \in F\beta O(Y)$.

Proof (i) $\Rightarrow$ (ii). Let $W \in FRC(Y)$. Then $1_Y \setminus W \in FRO(Y)$. By (i), $f^{-1}(1_Y \setminus W) = 1_X \setminus f^{-1}(W) \in F\beta SC(X)$. Hence $f^{-1}(W) \in F\beta SO(X)$.

(ii) $\Rightarrow$ (i). Let $W \in FRO(Y)$. Then $1_Y \setminus W \in FRC(Y)$. By (ii), $f^{-1}(1_Y \setminus W) = 1_X \setminus f^{-1}(W) \in F\beta SO(X)$. Hence $f^{-1}(W) \in F\beta SC(X)$.

(ii) $\Rightarrow$ (iii). Let $U \in I^X$ and suppose that y_α be a fuzzy point in Y with $y_\alpha \notin r\text{-ker}(f(U))$. Then there exists $V \in FRO(Y)$ such that $f(U) \leq V$ and $y_\alpha \notin V$, which implies $V(y) < \alpha$ and so $y_\alpha q(1_Y \setminus V) \in FRC(Y)$ and $1_Y \setminus f(U) \geq 1_Y \setminus V$. So $f(U)\not{q}(1_Y \setminus V)$ implies that $U\not{q}f^{-1}(1_Y \setminus V)$. By (ii), $f^{-1}(1_Y \setminus V) = 1_X \setminus f^{-1}(V) \in F\beta SO(X)$. By Lemma 3.22(ii), $\beta scl_\tau U\not{q}(1_X \setminus f^{-1}(V))$. Then $\beta scl_\tau U \leq f^{-1}(V)$. So $f(\beta scl_\tau U) \leq V$ implies that $1_Y \setminus f(\beta scl_\tau U) \geq 1_Y \setminus V$. So $1 - f(\beta scl_\tau U)(y) \geq 1 - V(y) > 1 - \alpha$ Then $\alpha > f(\beta scl_{\tau U})(y)$. Then $y_\alpha \notin f(\beta scl_\tau U)$. Therefore, $f(\beta scl_\tau U) \leq r\text{-ker}(f(U))$.

(iii) $\Rightarrow$ (iv). Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. By (iii), $f(\beta scl_\tau f^{-1}(A)) \leq r\text{-ker}(f(f^{-1}(A))) \leq r\text{-ker}(A)$ implies that $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A))$.
(iv) $\Rightarrow$ (i). Let $A \in FRO(Y)$. By (iv), $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A)) = f^{-1}(A)$. But $f^{-1}(A) \leq \beta scl_\tau(f^{-1}(A))$ and so $f^{-1}(A) = \beta scl_\tau(f^{-1}(A))$ and so $f^{-1}(A) \in F\beta SC(X)$. Hence f is fuzzy (β-semi, r)-continuous function.
(v) $\Rightarrow$ (vi). Let $P \in I^X$ and x_α be any fuzzy point in X such that $x_\alpha \in \beta scl_\tau P$ and let $G \in FSO(Y)$ with $f(x_\alpha)qG$. By (v), there exists $U \in F\beta SO(X)$ with $x_\alpha qU$, $f(U) \leq cl_{\tau_1}G$. As $x_\alpha \in \beta scl_\tau P$, by Lemma 3.22(i), UqP and so $f(U)qf(P)$. Then $f(P)qcl_{\tau_1}G$. Then $f(x_\alpha) \in \theta\text{-}scl_{\tau_1}(f(P))$. Hence $f(\beta scl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$.
(vi) $\Rightarrow$ (vii). Let $R \in I^Y$. Then $f^{-1}(R) \in I^X$. By (vi), $f(\beta scl_\tau(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}(f(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}R$ and so $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$.
(vii) $\Rightarrow$ (v). Let x_α be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_\alpha)qA$. Since, $cl_{\tau_1}A\not{q}(1_Y \setminus cl_{\tau_1}A)$, by definition $f(x_\alpha) \notin \theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A)$ and so $x_\alpha \notin f^{-1}(\theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A))$. By (vii), $x_\alpha \notin \beta scl_\tau(f^{-1}(1_Y \setminus cl_{\tau_1}A))$. So there exists $U \in F\beta SO(X)$ with $x_\alpha qU$, $U\not{q}f^{-1}(1_Y \setminus cl_{\tau_1}A)$ implies that $f(U)\not{q}(1_Y \setminus cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.
(vii) $\Rightarrow$ (viii). Let $A \in \tau_1$. By (vii), $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}A)$.
(viii) $\Rightarrow$ (ix). Follows from Lemma 3.23 (iv).
(ix) $\Rightarrow$ (x). Obvious.
(x) $\Rightarrow$ (i). Let $A \in FRO(Y)$. By (x), $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(A)$ implies that $f^{-1}(A) \in F\beta SC(X)$. Hence f is fuzzy (β-semi, r)-continuous function.
(i) $\Rightarrow$ (x). Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$. By (i), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta SC(X)$ and so $\beta scl_\tau(f^{-1}(A)) \leq \beta scl_\tau(f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(scl_{\tau_1}A)$.
(x) $\Rightarrow$ (ix). Obvious.
(ix) $\Rightarrow$ (viii). Follows from Lemma 3.23 (iv).
(vii) $\Rightarrow$ (i). Let $R \in FRO(Y)$. By (vii), $\beta scl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R) = f^{-1}(R)$ implies that $f^{-1}(R) \in F\beta SC(X)$. Hence f is fuzzy (β-semi, r)-continuous function.
(v) $\Rightarrow$ (xii). Let $A \in FSO(Y)$ and x_α be any fuzzy point in X such that $x_\alpha qf^{-1}(A)$. Then $f(x_\alpha)qA$. By (v), there exists $U \in F\beta SO(X)$ such that $x_\alpha qU$, $f(U) \leq cl_{\tau_1}A \Rightarrow x_\alpha qU \leq f^{-1}(cl_{\tau_1}A)$ and so $s_\alpha qU = \beta sint_\tau U \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$ implies that $x_\alpha q\beta sint_\tau(f^{-1}(cl_{\tau_1}A))$ as $\beta sint_\tau(f^{-1}(cl_{\tau_1}A))$ is the union of all

fuzzy β-semiopen sets in X contained in $f^{-1}(cl_{\tau_1}A)$ and hence $f^{-1}(A) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$.

(xii) $\Rightarrow$ (v). Let x_α be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_\alpha)qA$. Then $x_\alpha q f^{-1}(A) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$ (by (xii)) implies that there exists $U \in F\beta SO(X)$ with $x_\alpha qU$, $U \leq f^{-1}(cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.

(xi) $\Rightarrow$ (xii). Let $A \in FSO(Y)$ and x_α be any fuzzy point in X such that $x_\alpha \in f^{-1}(A)$. Then $f(x_\alpha) \in A$. By (xi), there exists $U \in F\beta SO(X)$ with $x_\alpha \in U$ and $f(U) \leq cl_{\tau_1}A$ implies that $U \leq f^{-1}(cl_{\tau_1}A)$ and so $x_\alpha \in U = \beta sint_\tau U \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$. Hence $f^{-1}(A) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$.

(xii) $\Rightarrow$ (xi). Let x_α be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_\alpha) \in A$. Then $x_\alpha \in f^{-1}(A) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}A))$ (by (xii)) implies that there exists $U \in F\beta SO(X)$ with $x_\alpha \in U$ and $U \leq f^{-1}(cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.

(i) $\Rightarrow$ (xiii). Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$ and so by (i), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\beta SC(X)$.

(xiii) $\Rightarrow$ (i). Let $A \in FRO(Y)$ implies that $A \in \tau_1$ and so by (xiii), $f^{-1}(A) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A) \in F\beta SC(X)$.

(xii) $\Rightarrow$ (ii). Let $F \in FRC(Y)$ implies that $F \in FSO(Y)$. By (xii), $f^{-1}(F) \leq \beta sint_\tau(f^{-1}(cl_{\tau_1}F)) = \beta sint_\tau(f^{-1}(F))$.

(ii) $\Rightarrow$ (xiv). Let $F \in \tau_1^c$. Then $cl_{\tau_1}int_{\tau_1}F \in FRC(Y)$. By (ii), $f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta SO(X)$.

(xiv) $\Rightarrow$ (ii). Let $F \in FRC(Y)$. By (xiv), $f^{-1}(F) = f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\beta SO(X)$.

(ii) $\Rightarrow$ (xv). Let $U \in F\beta O(Y)$. Then $U \leq cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}U$ implies that $cl_{\tau_1}U \leq cl_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}(cl_{\tau_1}U) = cl_{\tau_1}U$. So $cl_{\tau_1}U = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))$. Then $cl_{\tau_1}U \in FRC(Y)$ and so by (ii), $f^{-1}(cl_{\tau_1}U) \in F\beta SO(X)$.

(xv) $\Rightarrow$ (xvi). Since $FSO(Y) \subseteq F\beta O(Y)$, by (xv), $f^{-1}(cl_{\tau_1}U) \in F\beta SO(X)$, for all $U \in FSO(Y)$.

(xvi) $\Rightarrow$ (xvii). Let $U \in FPO(Y)$. Then $U \leq int_{\tau_1}(cl_{\tau_1}U)$. We claim that $int_{\tau_1}(cl_{\tau_1}U) \in FRO(Y)$. Indeed, $int_{\tau_1}(cl_{\tau_1}U) \leq int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) \leq int_{\tau_1}(cl_{\tau_1}U)$ implies that $int_{\tau_1}(cl_{\tau_1}U) = int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)))$. So $\Rightarrow 1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FRC(Y)$. So $1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FSO(Y)$. By (xvi), $f^{-1}(cl_{\tau_1}(1_Y \setminus int_{\tau_1}(cl_{\tau_1}U))) \in F\beta SO(X)$. Then $1_X \setminus f^{-1}(int_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = 1_X \setminus f^{-1}((int_{\tau_1}(cl_{\tau_1}U)) \in F\beta SO(X)$. Hence $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta SC(X)$.

(xvii) $\Rightarrow$ (i). Let $U \in FRO(Y)$. Then $U \in FPO(Y)$. By (xvii),

$f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta SC(X)$. Hence $f^{-1}(U) = f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\beta SC(X)$. Then (i) follows.
(xv) $\Leftrightarrow$ **(xviii)**. The proof follows from Lemma 3.23(ii).
(xv) $\Leftrightarrow$ **(xix)**. The proof follow from Lemma 3.23(iii).
(vii) $\Rightarrow$ **(xx)**. Obvious.
(xx) $\Rightarrow$ **(viii)**. Let $A \in \tau_1$. Since $FSO(Y) \supseteq \tau_1$, by (xx), $\beta scl_\tau(f^{-1}(A)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}A)$.
(vii) $\Rightarrow$ **(xxii)**. Obvious.
(xxii) $\Rightarrow$ **(xx)**. Since $FSO(Y) \subseteq F\beta O(Y)$, the result follows.
(vii) $\Rightarrow$ **(xxi)**. Obvious.
(xxi) $\Rightarrow$ **(viii)**. Since $\tau_1 \subseteq FPO(Y)$, the result follows.

Let us now recall the following definition from [4] for ready references.

Definition 4.12 [4]. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be fuzzy
(i) e^*-continuous if $f^{-1}(A) \in Fe^*O(X)$, for all $A \in \tau_1$,
(ii) almost e^*-continuous if $f^{-1}(A) \in Fe^*O(X)$, for all $A \in FRO(Y)$,
(iii) almost e-continuous if $f^{-1}(A) \in FeO(X)$, for all $A \in FRO(Y)$,
(iv) almost a-continuous if $f^{-1}(A) \in FaO(X)$, for all $A \in FRO(Y)$.

Let us now introduce the following concept.

Definition 4.13. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be fuzzy
(i) β-semicontinuous if $f^{-1}(A) \in F\beta SO(X)$, for all $A \in \tau_1$,
(ii) almost β-semicontinuous if $f^{-1}(A) \in F\beta SO(X)$, for all $A \in FRO(Y)$.

Remark 4.14. (i) Fuzzy β-semicontinuity implies fuzzy almost β-semicontinuity.
(ii) Fuzzy almost a-continuity implies fuzzy β-semicontinuity as well as fuzzy almost β-semicontinuity,
(iii) Fuzzy β-semicontinuity implies fuzzy e^*-continuity and hence fuzzy almost e^*-continuity. Also fuzzy almost β-semicontinuity implies fuzzy almost e^*-continuity.
But the reverse implications are not necessarily true, follow from the following examples.
(iv) Fuzzy almost e-continuity is an independent concept of fuzzy

β-semicontinuity as well as fuzzy almost β-semicontinuity, follow from the next examples.

Example 4.15. Fuzzy almost β-semicontinuity, fuzzy e^*-continuity, fuzzy almost e^*-continuity, fuzzy almost e-continuity, fuzzy almost a-continuity do not imply fuzzy β-semicontinuity

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = A(b) = 0.4, B(a) = B(b) = 0.5, C(a) = 0.5, C(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in \tau_2, i^{-1}(C) = C \nleq cl_{\tau_1}(int_{\tau_1}(scl_{\tau_1}C)) = B$ implies that $C \notin F\beta SO(X, \tau_1)$ and so i is not fuzzy β-semicontinuous function. But clearly i is fuzzy almost β-semicontinuous, fuzzy almost a-continuous, fuzzy almost e-continuous, fuzzy almost e^*-continuous function. Now $cl_{\tau_1}(int_{\tau_1}(\delta cl_{\tau_1}C)) = 1_X > C$ and so $C \in Fe^*O(X, \tau_1)$. Hence i is fuzzy e^*-continuous function.

Example 4.16. Fuzzy β-semicontinuity, fuzzy almost β-semicontinuity do not imply fuzzy almost a-continuity

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in FRO(X, \tau_2)$. $i^{-1}(B) = B \nleq int_{\tau_1}(cl_{\tau_1}(\delta int_{\tau_1}B)) = 0_X$ implies that $B \notin FaO(X, \tau_1)$. Hence i is not fuzzy almost a-continuous function. But $cl_{\tau_1}(int_{\tau_1}(scl_{\tau_1}B)) = A \geq B$ implies that $B \in F\beta SO(X, \tau_1)$. So i is fuzzy β-semicontinuous as well as fuzzy almost β-semicontinuous function.

Example 4.17. Fuzzy β-semicontinuity, fuzzy almost β-semicontinuity do not imply fuzzy almost e-continuity

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B.C\}$, $\tau_2 = \{0_X, 1_X, D, E\}$ where $A(a) = 0.3, A(b) = 0.4, B(a) = B(b) = 0.4, C(a) = 0.6, C(b) = 0.5, D(a) = 0.56, D(b) = 0.6, E(a) = E(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $D \in FRO(X, \tau_2)$. $i^{-1}(D) = D$. Now $(cl_{\tau_1}\delta int_{\tau_1}D) \bigvee (int_{\tau_1}\delta cl_{\tau_1}D) = C \ngeq D$ and so $D \notin FeO(X, \tau_1)$. Then i is not fuzzy almost e-continuous function. But $cl_{\tau_1}(int_{\tau_1}(scl_{\tau_1}D)) = 1_X \setminus B \geq D$. Then $D \in F\beta SO(X, \tau_1)$. Also $E \in FRO(X, \tau_2)$. $i^{-1}(E) = E < 1_X \setminus C = cl_{\tau_1}(int_{\tau_1}(scl_{\tau_1}E))$, So $E \in F\beta SO(X, \tau_1)$. Hence i is fuzzy β-semicontinuous as well as fuzzy almost β-semicontinuous function.

Example 4.18. Fuzzy almost e-continuity does not imply fuzzy β-semicontinuity, fuzzy almost β-semicontinuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.7, C(a) = 0.5, C(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in FRO(X, \tau_2)$. $i^{-1}(C) = C$. Now $cl_{\tau_1}(int_{\tau_1}(scl_{\tau_1}C)) = 0_X \not\geq C$ and so $C \notin F\beta SO(X, \tau_1)$. Hence i is not fuzzy β-semicontinuous as well as fuzzy almost β-semicontinuous function. But $(cl_{\tau_1}\delta int_{\tau_1}C) \bigvee (int_{\tau_1}\delta cl_{\tau_1}C) = A \geq C$ implies that $C \in FeO(X, \tau_1)$. Hence i is fuzzy almost e-continuous function.

Definition 4.19 [11]. An fts (X, τ) is said to be fuzzy extremally disconnected if the closure of ever fuzzy open set in X is fuzzy open in X.

Theorem 4.20. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. If (Y, τ_1) is fuzzy extremally disconnected, then f is fuzzy (β-semi, r)-continuous if and only if f is fuzzy almost β-semicontinuous function.

Proof. First suppose that f is fuzzy (β-semi, r)-continuous function. Let $U \in FRO(Y)$. Then $U = int_{\tau_1}(cl_{\tau_1}U)$. As Y is fuzzy extremally disconnected, $cl_{\tau_1}U \in \tau_1$ and so $U = int_{\tau_1}cl_{\tau_1}U = cl_{\tau_1}U = cl_{\tau_1}int_{\tau_1}U$ implies that $U \in FRC(Y)$. By hypothesis, $f^{-1}(U) \in F\beta SO(X)$ and so f is fuzzy almost β-semicontinuous function.

Conversely, let $U \in FRC(Y)$. As Y is fuzzy extremally disconnected, $U \in FRO(Y)$. By hypothesis, $f^{-1}(U) \in F\beta SO(X)$. Hence f is fuzzy (β-semi, r)-continuous function.

5. Applications of Fuzzy (β-semi, r)-Continuous and Fuzzy β-Semicontinuous Functions

In this section we first introduce a new type of compactness in an fts and then introduce a new type of separation axiom. Afterwards, the applications of the functions defined in Section 4 are established.

First we recall some definitions from [7, 10] for ready references.

Definition 5.1. Let A be a fuzzy set in X. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A if $sup\{U(x) : U \in \mathcal{U}\} = 1$, for each $x \in suppA$ [10]. In particular, if $A = 1_X$, we get the definition of fuzzy cover of X [7].

Definition 5.2. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A [10]. In particular, if $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$ [7].

Definition 5.3. A fuzzy set A in an fts (X, τ) is said to be fuzzy compact [10] if every fuzzy covering $\mathcal{U}$ of A by fuzzy open sets in X has a finite subcovering $\mathcal{U}_0$ of $\mathcal{U}$. In particular, if $A = 1_X$, we get the definition of fuzzy compact [7] space.

Definition 5.4. An fts (X, τ) is said to be fuzzy s-closed [14] (resp. fuzzy nearly compact [11]) if every fuzzy covering of X by fuzzy regular closed (resp., fuzzy regular open) sets of X contains a finite subcovering.

Let us now introduce the following concept.

Definition 5.5. A fuzzy set A in an fts (X, τ) is called fuzzy β-semicompact if every fuzzy covering of A by fuzzy β-semiopen sets of X has a finite subcovering. In particular, if $A = 1_X$, we get the definition of fuzzy β-semicompact space.

Result 5.6. It is clear from above discussion that fuzzy β-semicompact space is fuzzy compact. But the converse is not necessarily true follows from the next example.

Example 5.7. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. The clearly (X, τ) is a fuzzy compact space. Here every fuzzy set is fuzzy β-semiopen set in X. Consider the fuzzy cover $\mathcal{U} = \{U_n : n \in \mathbf{N}\}$ where $U_n(a) = \{\frac{n}{n+1} : n \in \mathbf{N}\}$. Then $\mathcal{U}$ is a fuzzy β-semiopen cover of X. But it does not have any subcovering of X. Hence X is not fuzzy β-semicompact space.

Theorem 5.8. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be surjective, fuzzy (β-semi, r)-continuous function. If X is fuzzy β-semicompact space, then Y is fuzzy s-closed space.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of Y

by fuzzy regular closed sets of Y. As f is fuzzy (β-semi, r)-continuous, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ covers X by fuzzy β-semiopen sets of X. As X is fuzzy β-semicompact, there exists a finite subset Λ_0 of Λ such that $1_X = \bigvee_{\alpha\in\Lambda_0} f^{-1}(U_\alpha)$ implies that $1_Y = f(\bigvee_{\alpha\in\Lambda_0} f^{-1}(U_\alpha)) = \bigvee_{\alpha\in\Lambda_0} f(f^{-1}(U_\alpha)) \leq \bigvee_{\alpha\in\Lambda_0} U_\alpha$. Hence Y is fuzzy s-closed space.

Theorem 5.9. Every fuzzy β-semiclosed set A in a fuzzy β-semicompact space X is fuzzy β-semicompact.
Proof. Let A be a fuzzy β-semiclosed set in a fuzzy β-semicompact space X. Let $\mathcal{U}$ be a fuzzy covering of A by fuzzy β-semiopen sets in X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy β-semiopen covering of X. By hypothesis, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcovering of A. Consequently, A is fuzzy β-semicompact.

Theorem 5.10. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be fuzzy β-semicontinuous function. If A is fuzzy β-semicompact set relative to X, then the image $f(A)$ is fuzzy compact relative to Y.
Proof. Let A be fuzzy β-semicompact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of $f(A)$ by fuzzy open sets of Y, i.e, $f(A) \leq \bigvee_{\alpha\in\Lambda} U_\alpha$ implies that $A \leq f^{-1}(\bigvee_{\alpha\in\Lambda} U_\alpha) = \bigvee_{\alpha\in\Lambda} f^{-1}(U_\alpha)$. So $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy β-semiopen sets in X. As A is fuzzy β-semicompact set relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i}))$ $\leq \bigvee_{i=1}^{n} U_{\alpha_i}$ implies that $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Hence the proof.

Theorem 5.11. Let (X, τ) and (Y, τ) be two fts's and $f : X \to Y$ be fuzzy almost β-semicontinuous function. If A is fuzzy β-semicompact relative to X, then the image $f(A)$ is fuzzy nearly compact relative to Y.

Proof. The proof is similar to that of Theorem 5.10.

Let us now introduce the following concept.

Definition 5.12. Let (X,τ) be an fts. Then X is said to be fuzzy β-semi-T_2-space if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy β-semiopen sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy β-semiopen sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Definition 5.13 [4]. An fts (X,τ) is said to be fuzzy s-Urysohn if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy semiopen sets U_1, U_2, V_1, V_2 in X such that $x_\alpha \in U_1, y_\beta q V_1$ and $clU_1 \not{q} clV_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $clU_2 \not{q} clV_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy semiopen sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $clU \not{q} clV$.

Theorem 5.14. Let (X,τ) and (Y,τ_1) be two fts's and $f : X \to Y$ be injective, fuzzy (β-semi, r)-continuous function and Y is fuzzy s-Urysohn space. Then X is fuzzy β-semi-T_2-space.

Proof. Let x_α and y_β be two distinct fuzzy points in X where $x \neq y$. Then as f is injective, $f(x_\alpha) \neq f(y_\beta)$. As Y is fuzzy s-Urysohn, there exist fuzzy semiopen sets U_1, U_2, V_1, V_2 in Y such that $f(x_\alpha) \in U_1, f(y_\beta) q V_1$ and $cl_{\tau_1} U_1 \not{q} cl_{\tau_1} V_1$ and $f(x_\alpha) q U_2$, $f(y_\beta) \in V_2$ and $cl_{\tau_1} U_2 \not{q} cl_{\tau_1} V_2$. By Theorem 4.11, there exist $W_1, W_2 \in F\beta SO(X)$ such that $x_\alpha \in W_1, W_1 \leq f^{-1}(cl_{\tau_1} U_1), y_\beta q W_2, W_2 \leq f^{-1}(cl_{\tau_1} V_1)$ or $x_\alpha q W_2, W_2 \leq f^{-1}(cl_{\tau_1} U_2), y_\beta \in W_1, W_1 \leq f^{-1}(cl_{\tau_1} V_2)$.We claim that $W_1 \not{q} W_2$. Indeed, $cl_{\tau_1} U_1 \not{q} cl_{\tau_1} V_1$ and $cl_{\tau_1} U_2 \not{q} cl_{\tau_1} V_2$ implies that $f^{-1}(cl_{\tau_1} U_1) \not{q} f^{-1}(cl_{\tau_1} V_1)$ and $f^{-1}(cl_{\tau_1} U_2) \not{q} f^{-1}(cl_{\tau_1} V_2)$.
Similarly, when $x = y$, $\alpha < \beta$ (say), there exist $U_1, U_2 \in FSO(Y)$ such that $f(x_\alpha) \in U_1, f(y_\beta) q U_2$ and $cl_{\tau_1} U_1 \not{q} cl_{\tau_1} U_2$. By Theorem 4.11 (i) $\Leftrightarrow$ (xi), there exist $W_1, W_2 \in F\beta SO(X)$ such that $x_\alpha \in W_1, W_1 \leq f^{-1}(cl_{\tau_1} U_1), y_\beta q W_2, W_2 \leq f^{-1}(cl_{\tau_1} U_2)$. Then as above, $W_1 \not{q} W_2$. Hence X is fuzzy β-semi-T_2-space.

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