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A NEW TYPE OF REGULAR SPACE VIA FUZZY PREOPEN SETS

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Abstract. In this paper we introduce a new closure-like operator in fuzzy topological spaces, via fuzzy preopen sets. Then mutual relationships of this operator with several closure operators in fuzzy topological spaces, studied in [2, 3, 4, 6, 7, 8, 11, 12] are established. The newly introduced operator is idempotent in fuzzy spaces satisfying some regularity property with respect to this operator, but it is not idempotent in general. Some characterizations of the new operator via nets are given in the last section.

1. Introduction and Preliminaries

After the introduction of the notion of fuzzy closure operator by Chang in 1968 [9], various types of fuzzy closure-like operators have been introduced and studied. In this context we have to mention [2, 3, 4, 6, 7, 8, 11, 12, 13]. In this paper, a new type of closure-like operator is introduced and studied which is not an idempotent operator.

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This new operator commutes with union but not with intersection. Throughout the paper, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [9]. A fuzzy set A is a function from a non-empty set X into a closed interval $I = [0, 1]$, i.e., $A \in I^X$ [15]. The support of a fuzzy set A in X will be denoted by $suppA$ [15] and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. A fuzzy point [14] with the singleton support $x \in X$ and the value t $(0 < t \leq 1)$ at x will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 in X respectively. The complement of a fuzzy set A in X will be denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for all $x \in X$ [15]. For two fuzzy sets A and B in X, we write $A \leq B$ if and only if $A(x) \leq B(x)$, for each $x \in X$, and AqB means A is quasi-coincident (q-coincident, for short) with B if $A(x) + B(x) > 1$, for some $x \in X$ [14]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. clA and $intA$ of a fuzzy set A in X respectively stand for the fuzzy closure [9] and fuzzy interior [9] of A in X. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [13], fuzzy β-open [10], fuzzy γ-open [5]) if $A = intclA$ (resp., $A \leq clintA$, $A \leq intclA$, $A \leq clintclA$, $A \leq (intclA) \bigcup (clintA)$). The complement of a fuzzy semiopen (resp., fuzzy preopen, fuzzy β-open, γ-open) set is called a fuzzy semiclosed [1] (resp., fuzzy preclosed [13], fuzzy β-closed [10], γ-closed [5]) set. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) set containing a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy preclosure [13], fuzzy β-closure [10], γ-closure [5]) of A and is denoted by $sclA$ (resp., $pclA$, βclA, γclA). The collection of all fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy β-open, fuzzy γ-open) sets in an fts X is denoted by $FRO(X)$ (resp., $FSO(X)$, $FPO(X)$, $F\beta O(X)$, $F\gamma O(X)$) and that of fuzzy semiclosed (resp., fuzzy preclosed, fuzzy β-closed, fuzzy γ-closed) sets is denoted by $FSC(X)$ (resp., $FPC(X)$, $F\beta C(X)$, $F\gamma C(X)$).

2. Fuzzy p^c-Closure Operator: Some Properties

In this section we introduce fuzzy p^c-closure operator using fuzzy preopen set as a basic tool. It is not an idempotent operator and this operator is characterized via fuzzy open sets.

Definition 2.1. A fuzzy point x_t in an fts (X, τ) is called a fuzzy p^c-cluster point of a fuzzy set A in an fts X if $clUqA$ for every $U \in FPO(X)$ with $x_t qU$.

The union of all fuzzy p^c-cluster points of A is called fuzzy p^c-closure of A, to be denoted by $[A]^c_p$. A is called fuzzy p^c-closed set if $A = [A]^c_p$. The complement of a fuzzy p^c-closed set in an fts X is called fuzzy p^c-open set in X.

Note 2.2. It is clear from definition that for any $A \in I^X$, $pclA \leq [A]^c_p$. But the converse is not necessarily true, as the following example shows.

Example 2.3. $x_t \in [A]^c_p$, but $x_t \notin pclA$, for any $A \in I^X$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $U \leq A$, $V \not\leq 1_X \setminus A$ and $FPC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $1_X \setminus U \geq 1_X \setminus A, 1_X \setminus V \not\geq A$. Now consider the fuzzy set B defined by $B(a) = B(b) = 0.45$ and the fuzzy point $a_{0.6}$. Here $pclB = B \not\ni a_{0.6}$. Now $a_{0.6} q U_1 \in FPO(X)$ where $0.4 < U_1(a) \leq 0.5, U_1(b) \leq 0.4$ or $a_{0.6} q V_1 \in FPO(X)$ where $V_1 \not\leq 1_X \setminus A$, $0.4 < V_1(a) \leq 0.5, V_1(b) > 0.6$ or $a_{0.6} q V_2 \in FPO(X)$ where $V_2(a) > 0.5, V_2(b) \geq 0$. Then $clU_1 = (1_X \setminus A) q B$, $clV_1 = clV_2 = 1_X q B$. Then $a_{0.6} \in [B]^c_p$.

The following theorem shows that under which condition fuzzy preclosure and fuzzy p^c-closure operators coincide on the family of fuzzy open sets.

Theorem 2.4. For a fuzzy open set A in an fts X, $pclA = [A]^c_p$.

Proof. By Note 2.2, it suffices to show that $[A]^c_p \leq pclA$ for every fuzzy open set A in X. Let $A \in \tau$. Then $A \in FPO(X)$. If possible, let $x_t \notin pclA$. Then there exists $V \in FPO(X)$, $x_t q V, V \not{q} A \Rightarrow V \leq 1_X \setminus A$ where $1_X \setminus A$ is fuzzy closed set in X. Therefore, $clV \leq cl(1_X \setminus A) = 1_X \setminus A \Rightarrow clV \not{q} A \Rightarrow x_t \notin [A]^c_p$. Hence the proof.

The next theorem characterizes fuzzy p^c-closure operator of a fuzzy set in an fts X.

Theorem 2.5. For any fuzzy set A in an fts (X, τ),
$[A]^c_p = \bigcap\{[U]^c_p : U$ is fuzzy open set in X with $A \leq U\}$.

Proof. Clearly left hand side is $\leq$ right hand side.
Assume that there exists a fuzzy point x_t that belongs to the right hand side but does not belong to the left hand side. Then there

exists $V \in FPO(X)$ with $x_t qV$ and $clV\not{q}A \Rightarrow A \leq 1_X \setminus clV(\in \tau)$. By hypothesis, $x_t \in [1_X \setminus clV]^c_p$. But as $clV\not{q}(1_X \setminus clV)$, $x_t \notin [1_X \setminus clV]^c_p$, a contradiction.

Remark 2.6. From Theorem 2.4 and Theorem 2.5, we conclude that $[A]^c_p$ is fuzzy preclosed set in X for any $A \in I^X$.

Theorem 2.7. In an fts (X, τ), the following statements are equivalent :
(a) 0_X and 1_X are fuzzy p^c-closed sets in X,
(b) For any two fuzzy sets A, B in X with $A \leq B$ we have $[A]^c_p \leq [B]^c_p$,
(c) For any two $A, B \in I^X$, $[A \bigcup B]^c_p = [A]^c_p \bigcup [B]^c_p$,
(d) For any two $A, B \in I^X$, $[A \bigcap B]^c_p \leq [A]^c_p \bigcap [B]^c_p$; here the equality does not hold in general, as follows from the next example,
(e) The union of any two fuzzy p^c-closed sets in X is also so,
(f) The intersection of any two fuzzy p^c-closed sets in X is also so.

Proof. (a) and (b) are obvious.
(c) By (b), we can write, $[A]^c_p \bigcup [B]^c_p \leq [A \bigcup B]^c_p$.
To prove the converse, let $x_t \in [A \bigcup B]^c_p$. Then for any $U \in FPO(X)$ with $x_t qU$, $clUq(A \bigcup B)$. Then there exists $y \in X$ such that $(clU)(y) + max\{A(y), B(y)\} > 1 \Rightarrow$ either $(clU)(y) + A(y) > 1$ or $(clU)(y) + B(y) > 1 \Rightarrow$ either $clUqA$ or $clUqB \Rightarrow$ either $x_t \in [A]^c_p$ or $x_t \in [B]^c_p \Rightarrow x_t \in [A]^c \bigcup [B]^c_p$.
(d) It follows from (b).
(e) It follows from (c).
(f) From (d), we have $[A \bigcap B]^c_p \leq [A]^c_p \bigcap [B]^c_p$ for any two fuzzy sets $A, B \in X$.
Conversely, let A, B be two fuzzy p^c-closed sets in X. Then $[A]^c_p = A, [B]^c_p = B$. Let $x_t \in [A]^c_p \bigcap [B]^c_p = A \bigcap B \leq [A \bigcap B]^c_p \Rightarrow [A]^c_p \bigcap [B]^c_p \leq [A \bigcap B]^c_p$.

Example 2.8. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $0.3 \not< U(a)$ $\not< 0.7, U(b) = 0.5$, $V \not\leq 1_X \setminus A$. Consider two fuzzy sets C and D defined by $C(a) = 0.4, C(b) = 0.1, D(a) = 0.1, D(b) = 0.6$ and the fuzzy point $a_{0.8}$. We claim that $a_{0.8} \in [C]^c_p \bigcap [D]^c_p$, but $a_{0.8} \notin [C \bigcap D]^c_p$. Now $a_{0.8}qT \in FPO(X)$ where $0.2 < T(a) \leq 0.5, T(b) > 0.5$. Then $clS = 1_X \setminus B$ or $clS = 1_X$ according as $S(b) = 0.5$ or $S(b) > 0.5$ and $clT = 1_X$. So $clSqC, clSqD, clTqC, clTqD \Rightarrow a_{0.8} \in [C]^c_p, a_{0.8} \in$

$[D]_p^c \Rightarrow a_{0.8} \in [C]_p^c \bigcap [D]_p^c$. Let $E = C \bigcap D$. Then $E(a) = E(b) = 0.1$. Then $clS \not{q} E$ where $S(a) = 0.3, S(b) = 0.5 \Rightarrow a_{0.8} \notin [E]_p^c = [C \bigcap D]_p^c$.

Note 2.9. In fact, intersection of any collection of fuzzy p^c-closed sets in an fts (X, τ) is also fuzzy p^c-closed in X. So we can conclude that fuzzy p^c-open sets in an fts (X, τ) form a fuzzy topology τ_{p^c} (say) which is coarser than fuzzy topology τ of (X, τ).

Note 2.10. The next example shows that fuzzy p^c-closure operator is not an idempotent operator.

Example 2.11. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $U \leq A, V \not\leq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$ and the fuzzy points $a_{0.5}$ and $a_{0.7}$. We claim that $a_{0.7} \in [a_{0.5}]_p^c$, $a_{0.5} \in [B]_p^c$, i.e., $[a_{0.7}]_p^c \subseteq [[a_{0.5}]_p^c]_p^c$, but $a_{0.7} \notin [B]_p^c$.
Now $a_{0.7} q U_1 \in FPO(X)$ where $0.3 < U_1(a) \leq 0.4, U_1(b) \leq 0.5$. Then $clU_1 = (1_X \setminus A) q a_{0.5}$. Also $a_{0.7} q V_1 \in FPO(X)$ where $0.3 < V_1(a) \leq 0.6, V_1(b) > 0.5$ and $a_{0.7} q V_2 \in FPO(X)$ where $V_2(a) > 0.6, V_2(b) \geq 0$. Then $clV_1 = clV_2 = 1_X q a_{0.5}$. So $a_{0.7} \in [a_{0.5}]_p^c$.
Now $a_{0.5} q V_3 \in FPO(X)$ where $0.5 < V_3(a) \leq 0.6, V_3(b) > 0.5$, also $a_{0.5} q V_4 \in FPO(X)$ where $V_4(a) > 0.6, V_4(b) \geq 0$. Then $clV_3 = clV_4 = 1_X q B \Rightarrow a_{0.5} \in [B]_p^c$. Now $a_{0.7} q U_2 \in FPO(X)$ where $U_2(a) = 0.31, U_2(b) = 0$. But $clU_2 = (1_X \setminus A) \not{q} B \Rightarrow a_{0.7} \notin [B]_p^c$, i.e., $a_{0.7} \in [a_{0.5}]_p^c \subseteq [[B]_p^c]_p^c$, but $a_{0.7} \notin [B]_p^c \Rightarrow [[B]_p^c]_p^c \not\subset [B]_p^c$.

Result 2.12. We conclude that $x_t \in [y_{t'}]_p^c$ does not imply $y_{t'} \in [x_t]_p^c$ where $x_t, y_{t'}$ $(0 < t, t' \leq 1)$ are fuzzy points in X as shown from the following example.

Example 2.13. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.7, B(b) = 0.5$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $0.3 < U(a) \leq 0.5, U(b) \leq 0.4, V \not\leq 1_X \setminus A$. Now consider two fuzzy points $a_{0.1}$ and $b_{0.61}$. We claim that $a_{0.1} \in [b_{0.61}]_p^c$, but $b_{0.61} \notin [a_{0.1}]_p^c$. Now $a_{0.1} q V_1 \in FPO(X)$ where $V_1(a) > 0.9, V_1(b) \geq 0$. Then $clV_1 = 1_X q b_{0.61} \Rightarrow a_{0.1} \in [b_{0.61}]_p^c$. Now $b_{0.61} q U_1 \in FPO(X)$ where $U_1(a) = U_1(b) = 0.4$. But $clU_1 = (1_X \setminus A) / q a_{0.1} \Rightarrow b_{0.61} \notin [a_{0.1}]_p^c$.

3. Mutual Relationships

In this section we establish the mutual relationship of this newly defined operator with the operators defined in [2, 3, 4, 6, 7, 8, 11, 12].

Definition 3.1. A fuzzy point x_t in an fts (X, τ) is called fuzzy θ-cluster point [12] (resp., fuzzy δ-cluster point [11], fuzzy s^*-cluster point [3], fuzzy p^*-cluster point [4], fuzzy β^*-cluster point [2], fuzzy γ^*-cluster point [6], fuzzy s^c-cluster point [7], fuzzy β^c-cluster point [8]) of a fuzzy set A in X if for every $U \in \tau$ (resp., $U \in FRO(X)$, $U \in FSO(X)$, $U \in FPO(X)$, $U \in F\beta O(X)$, $F\gamma O(X)$, $U \in FSO(X)$, $U \in F\beta O(X)$) with $x_t qU$, $clUqA$ (resp., UqA, $sclUqA$, $pclUqA$, $\beta clUqA$, $\gamma clUqA$, $clUqA$, $clUqA$).
The union of all fuzzy θ-cluster (resp., fuzzy δ-cluster, fuzzy s^*-cluster, fuzzy p^*-cluster, fuzzy β^*-cluster, fuzzy γ^*-cluster, fuzzy s^c-cluster, fuzzy β^c-cluster) points of a fuzzy set A is called fuzzy θ-closure [12] (resp., fuzzy δ-closure [11], fuzzy s^*-closure [3], fuzzy p^*-closure [4], fuzzy β^*-closure [2], fuzzy γ^*-closure [6], fuzzy s^c-closure [7], fuzzy β^c-closure [8]) of A, denoted by $[A]_\theta$ (resp., $[A]_\delta$, $[A]_s$, $[A]_p$, $[A]_\beta$, $[A]_\gamma$, $[A]^c_s$, $[A]^c_\beta$).

Note 3.2. It is clear from definitions that for any fuzzy set A in an fts (X, τ),
(i) $[A]_p \subseteq [A]^c_p \subseteq [A]_\theta$, again $[A]_\beta, [A]^c_\beta, [A]_\gamma \subseteq [A]^c_p$, but the reverse implications are not necessarily true, as the following examples show,
(ii) $[A]_\delta, [A]_s, [A]^c_s$ are independent concepts of $[A]^c_p$.

Example 3.3. $x_t \in [A]^c_p$, but $x_t \notin [A]_\gamma, [A]_p$
Consider Example 2.3. Here $a_{0.6} \in [B]^c_p$. Now $a_{0.6} qU_1 \in FPO(X)$ where $U_1(a) = 0.41, U_1(b) = 0$. But $pclU_1 = U_1 \not{q} B \Rightarrow a_{0.6} \notin [B]_p$. Again $a_{0.6} qA \in F\gamma O(X)$, but $\gamma clA = A \not{q} B \Rightarrow a_{0.6} \notin [B]_\gamma$.

Example 3.4. $x_t \in [A]^c_p$, but $x_t \notin [A]_s, [A]^c_s$
Consider Example 2.3 and the fuzzy set D defined by $D(a) = D(b) = 0.3$ and the fuzzy point $b_{0.5}$. Then $b_{0.5} qV \in FPO(X)$ where $V(a) > 0.5, V(b) > 0.5$ or $V(a) < 0.5, V(b) > 0.6$. Then $clV = 1_X qD \Rightarrow b_{0.5} \in [D]^c_p$. Now $FSO(X) = \{0_X, 1_X, W\}$ where $A \leq W \leq 1_X \setminus A$ and $FSC(X) = \{0_X, 1_X, 1_X \setminus W\}$ where $A \leq 1_X \setminus W \leq 1_X \setminus A$. Now $b_{0.5} qW_1 \in FSO(X)$ where $W_1(a) = 0.5, 0.5 < W_1(b) \leq 0.6$. Then $clW_1 = 1_X \setminus A, sclW_1 = W_1$.

But $(1_X \setminus A) \not{q} D$ and $W_1 \not{q} D \Rightarrow b_{0.5} \notin [D]^c_s, b_{0.5} \notin [D]_s$.

Example 3.5. $x_t \in [A]^c_p$, but $x_t \notin [A]_\delta$
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $FPO(X) = \{0_X, 1_X, U, V\}$ where $U \leq A$, $V \not\leq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$ and the fuzzy point $a_{0.7}$. Then $a_{0.7}qA \in FRO(X)$, but $A \ /qB \Rightarrow a_{0.7} \notin [B]_\delta$. But $a_{0.7}qU_1$ and $a_{0.7}qV_1 \in FPO(X)$ where $0.3 < U_1(a) \leq 0.5, U_1(b) \leq 0.4$ and $0.3 < V_1(a) \leq 0.5, V_1(b) > 0.6$ or $V_1(a) > 0.5, V_1(b) \geq 0$. Then $clU_1 = (1_X \setminus A)qB$, $clV_1 = 1_X qB \Rightarrow a_{0.7} \in [B]^c_p$.

Example 3.6. $x_t \in [A]^c_p$, but $x_t \notin [A]^c_\beta$
Consider Example 3.5 and the fuzzy set D defined by $D(a) = D(b) = 0.2$ and the fuzzy point $b_{0.41}$. Then $b_{0.41}qV_1 \in FPO(X)$ where $V_1(a) > 0.5, V_1(b) > 0.59$ or $V_1(a) \leq 0.5, V_1(b) > 0.6$. Then $clV_1 = 1_X qD \Rightarrow b_{0.41} \in [D]^c_p$. But $b_{0.41}qS \in F\beta O(X)$ where $S(a) = 0, S(b) = 0.6$. Then $clS = (1_X \setminus A) \not{q} D \Rightarrow b_{0.41} \notin [D]^c_\beta$.

Example 3.7. $x_t \in [A]^c_p$, but $x_t \notin [A]_\beta$
Consider Example 2.11 and the fuzzy point $a_{0.5}$. Here $a_{0.5} \in [B]^c_p$. Now every fuzzy set in (X, τ) is fuzzy β-open as well as fuzzy β-closed. Now consider the fuzzy set S defined by $S(a) = 0.51, S(b) = 0.5$. Then $a_{0.5}qS \in F\beta O(X)$, but $\beta clS = S \not{q} B \Rightarrow a_{0.5} \notin [B]_\beta$.

4. Fuzzy p^c-Regular Space : Some Characterizations

In this section a new type of fuzzy separation axiom is introduced and characterized by fuzzy p^c-closure operator. Also it is shown that p^c-closure operator is idempotent in every space satisfying the new separation axiom.

Definition 4.1. An fts (X, τ) is called fuzzy p^c-regular space if for each fuzzy point x_t and each fuzzy preopen set U in X with $x_t qU$, there exists $V \in \tau$ such that $x_t qV \leq clV \leq U$.

Theorem 4.2. For an fts (X, τ), the following statements are equivalent :
(a) X is fuzzy p^c-regular space,
(b) for any $A \in I^X$, $pclA = [A]^c_p$,
(c) for each fuzzy point x_t and each $U \in FPC(X)$ with $x_t \notin U$, there

exists $V \in \tau$ such that $x_t \notin clV$ and $U \leq V$,
(d) for each fuzzy point x_t and each $U \in FPC(X)$ with $x_t \notin U$, there exist $V, W \in \tau$ such that $x_t qV$, $U \leq W$ and $V \not{q} W$,
(e) for any $A \in I^X$ and any $U \in FPC(X)$ with $A \not\leq U$, there exist $V, W \in \tau$ such that $AqV, U \leq W$ and $V \not{q} W$,
(f) for any $A \in I^X$ and any $U \in FPO(X)$ with AqU, there exists $V \in \tau$ such that $AqV \leq clV \leq U$.

Proof. (a) $\Rightarrow$ (b) By Note 3.2, it suffices to show that $[A]_p^c \subseteq pclA$, for any $A \in I^X$. Let $x_t \in [A]_p^c$ be arbitrary and $V \in FPO(X)$ with $x_t qV$. By (a), there exists $U \in \tau$ such that $x_t qU \leq clU \leq V$. Since $U \in \tau \Rightarrow U \in FPO(X)$, by hypothesis, $clUqA \Rightarrow VqA \Rightarrow x_t \in pclA \Rightarrow [A]_p^c \subseteq pclA$.
(b) $\Rightarrow$ (a) Let x_t be a fuzzy point in X and $U \in FPO(X)$ with $x_t qU$. Then $U(x)+t > 1 \Rightarrow x_t \notin 1_X \backslash U (\in FPC(X)) = pcl(1_X \backslash U) = [1_X \backslash U]_p^c$ (by (b)). Then there exists $V \in FPO(X)$ with $x_t qV$, $clV \not{q}(1_X \setminus U) \Rightarrow clV \leq U$. Therefore, $x_t qV \leq clV \leq U \Rightarrow X$ is fuzzy p^c-regular space.
(a) $\Rightarrow$ (c) Let x_t be a fuzzy point in X and $U \in FPC(X)$ with $x_t \notin U$. Then $x_t q(1_X \setminus U) \in FPO(X)$. By (a), there exists $V \in \tau$ such that $x_t qV \leq clV \leq 1_X \setminus U$. Therefore, $U \leq 1_X \setminus clV$ ($= W$, say). Then $W \in \tau$. Now $x_t qV = intV \Rightarrow x_t qintV \leq V \leq intclV \Rightarrow x_t q(intclV) \Rightarrow (intclV)(x) + t > 1 \Rightarrow 1 - (intclV)(x) < t \Rightarrow x_t \notin 1_X \setminus intclV = cl(1_X \setminus clV) = clW$.
(c) $\Rightarrow$ (d) Let x_t be a fuzzy point in X and $U \in FPC(X)$ with $x_t \notin U$. By (c), there exists $V \in \tau$ such that $U \leq V$ and $x_t \notin clV \Rightarrow$ there exists $W \in \tau$ such that $x_t qW$, $W \not{q} V$. Now $V \not{q}(1_X \setminus clV)$. Now $x_t q(1_X \setminus clV)$ ($= W$, say). Then $W \in \tau$ and $V \not{q} W$. So we get, $V, W \in \tau$ with $x_t qW$, $U \leq V$ and $V \not{q} W$.
(d) $\Rightarrow$ (e) Let $A \in I^X$ and $U \in FPC(X)$ with $A \not\leq U$. Then there exists $x \in X$ such that $A(x) > U(x)$. Let $A(x) = t$. Then $x_t \notin U$. By (d), there exist $V, W \in \tau$ such that $x_t qV, U \leq W$ and $V \not{q} W$. Again $V(x) + t > 1 \Rightarrow V(x) + A(x) > 1 - t + t = 1 \Rightarrow AqV$.
(e) $\Rightarrow$ (f) Let $A \in I^X$ and $U \in FPO(X)$ with AqU. Then $A \not\leq 1_X \setminus U \in FPC(X)$. By (e), there exist $V, W \in \tau$ such that $A \leq V, 1_X \setminus U \leq W$ and $V \not{q} W \Rightarrow V \leq 1_X \setminus W \in \tau^c \Rightarrow clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$. Therefore, $A \leq V \leq clV \leq U$.
(f) $\Rightarrow$ (a) Obvious.

Corollary 4.3. An fts (X, τ) is fuzzy p^c-regular if and only if every fuzzy preclosed set in X is fuzzy p^c-closed set in X.

Proof. Let (X,τ) be fuzzy p^c-regular space and $A \in FPC(X)$. Then by Theorem 5.2 (a)$\Rightarrow$ (b), $A = pclA = [A]^c_p \Rightarrow A$ is fuzzy p^c-closed set in X.
Conversely, let $A = [A]^c_p$ for any $A \in FPC(X)$. Let $B \in I^X$. Then $pclB \in FPC(X)$ and so by hypothesis, $pclB = [pclB]^c_p$. Then $[B]^c_p \leq [pclB]^c_p = pclB$. By Note 2.2, $pclB \leq [B]^c_p$. Combining these two, we get $[B]^c_p = pclB$ for any $B \in I^X$. Then by Theorem 5.2 (b)$\Rightarrow$ (a), X is fuzzy p^c-regular space.

Remark 4.4. In a fuzzy p^c-regular space, $[[A]^c_p]^c_p = [A]^c_p$, for any $A \in I^X$.

5. Characterizations of The p^c-Closure Operator

In this section we first introduce fuzzy p^c-cluster point and fuzzy p^c-convergence of a fuzzy net and then fuzzy p^c-closure operator of a fuzzy set is characterized in terms of these concepts.

Definition 5.1. A fuzzy point x_t in an fts (X,τ) is called a fuzzy p^c-cluster point of a fuzzy net $\{S_n : n \in (D,\geq)\}$ if for every fuzzy preopen set U in X with x_tqU and for any $n \in D$, there exists $m \in D$ with $m \geq n$ such that S_mqclU.

Definition 5.2. A fuzzy net $\{S_n : n \in (D,\geq)\}$ in an fts (X,τ) is said to p^c-converge to a fuzzy point x_t if for every fuzzy preopen set U in X, x_tqU, there exists $m \in D$ such that S_nqclU, for all $n \geq m$ $(n \in D)$. This is denoted by $S_n \overrightarrow{p^c} x_t$.

Theorem 5.3. A fuzzy point x_t is a fuzzy p^c-cluster point of a fuzzy net $\{S_n : n \in (D,\geq)\}$ in an fts (X,τ) if and only if there exists a fuzzy subset of $\{S_n : n \in (D,\geq)\}$ which p^c-converges to x_t.

Proof. Let x_t be a fuzzy p^c-cluster point of the fuzzy net $\{S_n : n \in (D,\geq)\}$. Let $C(Q_{x_t})$ denote the set of fuzzy closures of all fuzzy preopen sets of X q-coincident with x_t. Then for any $A \in C(Q_{x_t})$, there exits $n \in D$ such that S_nqA. Let E denote the set of all ordered pairs (n,A) such that $n \in D$, $A \in C(Q_{x_t})$ and S_nqA. Then $(E,\gg)$ is a directed set, where $(m,A) \gg (n,B)$ $((m,A),(n,B) \in E)$ if and only if $m \geq n$ in D and $A \leq B$. Then $T : (E,\gg) \to (X,\tau)$ given by $T(m,A) = S_m$ is clearly a fuzzy subnet of $\{S_n : n \in (D,\geq)\}$. We claim that $T \overrightarrow{p^c} x_t$. Let V be any fuzzy preopen set in X with $x_t qV$. Then there exists $n \in D$ such that $(n, clV) \in E$ and so $S_n qclV$. Now for any $(m, A) \gg (n, clV)$, $T(m, A) = S_m qA \leq clV \Rightarrow T(m, A)qclV$. Consequently, $T\overrightarrow{p^c}x_t$.

Conversely, assume that x_t is not a fuzzy p^c-cluster point of the fuzzy net $\{S_n : n \in (D, \geq)\}$. Then there exists $U \in FPO(X)$ with $x_t qU$ and an $n \in D$ such that $S_m \not{q} clU$, for all $m \geq n$. Then clearly no fuzzy subnet of the net $\{S_n : n \in (D, \geq)\}$ can p^c-converge to x_t.

Theorem 5.4. Let A be a fuzzy set in an fts (X, τ). A fuzzy point x_t belongs to $[A]^c_p$ if and only if there exists a fuzzy net $\{S_n : n \in (D, \geq)\}$ in A, which p^c-converges to x_t.

Proof. Let $x_t \in [A]^c_p$. Then for any $U \in FPO(X)$ with $x_t qU$, we have $clUqA$, i.e., there exists $y^U \in suppA$ and a real number s_U with $0 < s_U \leq A(y^U)$ such that the fuzzy point $y^U_{s_U}$ with support y^U and the value s_U belong to A and $y^U_{s_U} qclU$. We choose and fix one such $y^U_{s_U}$ for each U. Let $\mathcal{D}$ denote the set of all fuzzy preopen set in X q-coincident with x_t. Then $(\mathcal{D}, \succeq)$ is a directed set under inclusion relation, i.e., $B, C \in \mathcal{D}$, $B \succeq C$ if and only if $B \leq C$. Then $\{y^U_{s_U} \in A : y^U_{s_U} qclU, U \in \mathcal{D}\}$ is a fuzzy net in A such that it p^c-converges to x_t. Indeed, for any fuzzy preopen set U in X with $x_t qU$, if $V \in \mathcal{D}$ and $V \succeq U$ (i.e., $V \leq U$) then $y^V_{s_V} qclV \leq clU \Rightarrow y^V_{s_V} qclU$.

Conversely, let $\{S_n : n \in (D, \geq)\}$ be a fuzzy net in A such that $S_n\overrightarrow{p^c}x_t$. Then for any $U \in FPO(X)$ with $x_t qU$, there exists $m \in D$ such that $n \geq m \Rightarrow S_n qclU \Rightarrow AqclU$ (since $S_n \in A$). Hence $x_t \in [A]^c_p$.

Remark 5.5. It is clear that an improved version of the sufficiency part of the last theorem can be written as "$x_t \in [A]^c_p$ if there exists a fuzzy net in A with x_t as a fuzzy p^c-cluster point".

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