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ON ω^c-SETS AND ω-SETS IN TOPOLOGICAL SPACES

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Abstract. In this paper we introduce and study the notion of ω^c-set. We also study ω-sets, as complements of ω^c-sets. Finally, using the ω^c-sets and ω-sets of a topological space, we introduce and study the notion of ω-extremally disconnected topological spaces. We obtain a result akin to Urysohn's Lemma in the setting of ω-extremally disconnected topological spaces.

1. Introduction

Unless otherwise mentioned, X stands for the topological space $(X, \mathscr{P})$. For a subset A of a topological space X, $Int(A)$ (resp. $Cl(A)$) denotes the interior (resp. closure) of A with respect to the topological space $(X, \mathscr{P})$. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers.

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Generalization of existing topological notions is an important concern for topologists. Levine [7] generalized the concept of open sets and introduced the notion of semi-open sets: a subset A of a topological space X is called semi-open if there exists an open set G such that $G \subset A \subset Cl(G)$. Equivalently, a subset A of a topological space X is semi-open in X if and only if $A \subset Cl(Int(A))$ [7]. A new type of set, namely pre-open set (see Mashhour et al. [9]) introduced by Corson and Michael [3] under the name locally dense sets is originated in the topological space X, if we change the order of application of interior and closure operators in semi-open sets, i.e., a subset A of a topological space X is pre-open if $A \subset Int(Cl(A))$. Generalizing both ideas of semi-open and pre-open sets, Andrijević [1] introduced and studied the notion of semi-pre-open sets: a subset A of a topological space X is semi-pre-open if there exists a pre-open set U such that $U \subset A \subset Cl(U)$. It can be seen that a subset A of a topological space X is semi-pre-open if and only if $A \subset (Cl(Int(Cl(A)))$. Njåstad [8] and El-Monsef et al. [5] independently introduced and studied β-sets and β-open sets respectively which are same as semi-pre-open sets. Interchanging the role of interior and closure operators in semi-pre-open sets, we get α-sets, i.e. a subset A of X is α-sets [8] if and only if $A \subset Int(Cl(Int(A)))$. Here we agree to call α-sets as α-open sets and β-sets or semi-pre-open sets as β-open sets. Note that both notions of semi-open and pre-open sets are generalization of α-open sets. Interestingly, concepts of semi-open and pre-open sets are independent. Obviously, open sets are included in either of above four categories of sets. We depict the implication relations among above type of sets in the following diagram. The implications are not reversible.

$$\begin{tikzcd}
 & & \text{semi-open} \arrow[dr] & \\
\text{open} \arrow[r] & \alpha\text{-open} \arrow[ur] \arrow[dr] & & \beta\text{-open} \\
 & & \text{pre-open} \arrow[ur] &
\end{tikzcd}$$

Note: '$\longrightarrow$' in above diagram stands to mean 'implies that'.

As usual, a subset of X is called semi-closed [4], pre-closed [9], α-closed and β-closed if its complement is semi-open, pre-open, α-open and β-open respectively. So a subset B of a topological space X is

(1) [(i)]
(2) semi-closed if and only if $Int(Cl(B)) \subset B$ [4],
(3) pre-closed if and only if $Cl(Int(B)) \subset B$ [9],

(4) α-closed if and only if $Cl(Int(Cl(B))) \subset B$,
(5) β-closed if and only if $Int(Cl(Int(B))) \subset B$.

The introduction and study of new types of sets is an active field of research in topological spaces e.g. Modak and Islam [6] studied two types of sets in ideal topological spaces akin to semi-open sets and α-sets due to Levine [7] and Njåstad [8] respectively. We also see that Ekici [2] introduced and studied a-open sets, A^*-sets in topological spaces.

Section 2 and Section 3 are main parts of the paper. In Section 2, we introduce the notions of ω^c-sets and ω-sets in topological spaces and then obtain some of their properties. Some examples are also framed according to the necessity of contents. As we have in the literature of topological spaces the term like α-sets [8], β-sets [8] and β-open sets [5], we coin the term ω-sets for the new type of set introduced here following the tradition of naming some sets in topological spaces and its complement is then denoted by ω^c-sets. Since ω^c-sets are associated with the closure of an open set of topological spaces, they are named so and the complement of ω^c-sets are simply denoted by ω-sets. In Section 3, $\bigwedge_{\omega^c}$ and $\bigvee_{\omega}$ operations are defined using the concepts of ω^c-sets and ω-sets respectively. Finally, we obtain a result analogous to Urysohn's Lemma on ω-extremally disconnected spaces.

2. ω^c-SETS AND ω-SETS

Gazing at semi-open, pre-open, α-open and β-open sets, we notice that for each of these sets A, there exists an open set G such that $A \subset Cl(G)$. This observation instigates us to introduce the following notion.

Definition 1. *A subset A of a topological space X with $Cl(A) \neq X$ is said to be an ω^c-set if there exists an open set $G(\neq X)$ such that $A \subset Cl(G)$.*

It is easy to see that all semi-open and α-open sets except X are ω^c-sets. A pre-open or β-open set A with $Cl(A) \neq X$ is also an ω^c-set.

Example 2 (Mukharjee [10])**.** *For $a \in \mathbb{R}$, we define*

$$\mathscr{T} = \{\emptyset, \mathbb{R}, \{a\}, (-\infty, a), (-\infty, a], [a, \infty)\}.$$

In the topological space $(\mathbb{R}, \mathscr{T})$, we choose two subsets A and B such that $A \subset (-\infty, a)$ and $B \subset (a, \infty)$. We see that $A \cup B$ is an ω^c-set but it is not β-open. It means that an ω^c-set may not be a β-open set. Since semi-open, pre-open and α-open sets are also β-open, an ω^c-set may not be a semi-open or pre-open or α-open set.

The complement of an ω^c-set is said to be an ω-set. So a subset A of a topological space X with $Int(A) \neq \emptyset$ is an ω-set if and only if there exists a closed set $E(\neq \emptyset)$ such that $Int(E) \subset A$.

It is easy to see that all semi-closed and α-closed sets except $\emptyset$ are ω-sets. A pre-closed or β-closed set A with $Int(A) \neq \emptyset$ is also an ω-set.

In the topological space of Example 2, there exists ω-sets which are not β-closed sets. It means that an ω-set may not be a β-closed set. Since semi-closed, pre-closed and α-closed sets are also β-closed sets, an ω-set may not be a semi-closed or pre-closed or α-closed set.

Slightly changing the topological space of Example 2, we have the following space.

Example 3. *For $a \in \mathbb{R}$, we define*

$$\mathscr{T} = \{\emptyset, \mathbb{R}, (-\infty, a), [a, \infty)\}.$$

In the topological space $(\mathbb{R}, \mathscr{T})$, we choose $c, d \in \mathbb{R}$ such that $-\infty < c < a < d < \infty$. Then $(-\infty, c)$ and (d, ∞) are ω^c-sets but their union is not an ω^c-set. Also (c, ∞) and $(-\infty, d)$ are ω-sets but their intersection is not an ω-set.

So it follows that the union of even finitely many ω^c-sets may not be an ω^c-set and intersection of even finitely many ω-sets may not be an ω-set.

Theorem 4. *If $\{A_\alpha \mid \alpha \in \Delta\}$ is a collection of ω^c-sets in X, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is an ω^c-set in X.*

Proof. For each $\alpha \in \Delta$, we have an open set $G\alpha \neq X$ such that $A\alpha \subset Cl(G\alpha)$. Then $\bigcap_{\alpha\in\Delta} A_\alpha \subset \bigcap_{\alpha\in\Delta} Cl(G_\alpha) \subset Cl(G_\alpha)$ for any $\alpha \in \Delta$. So $\bigcap_{\alpha\in\Delta} A_\alpha$ is an ω^c-set. ∎

Theorem 5. *If $\{A_\alpha \mid \alpha \in \Delta\}$ is a collection of ω-sets in X, then $\bigcup_{\alpha\in\Delta} A_\alpha$ is an ω-set in X.*

Proof. Similar to that of Theorem 4.

Alternatively, $\{X - A_\alpha \mid \alpha \in \Delta\}$ is a collection of ω^c-sets in X. By Theorem 4, $\bigcap_{\alpha\in\Delta}(X - A_\alpha) = X - \bigcup_{\alpha\in\Delta}(A_\alpha)$ is an ω^c-set and so $\bigcup_{\alpha\in\Delta} A_\alpha$ is an ω-set. ∎

Theorem 6. *A subset A of a topological space X with $Cl(A) \neq X$ is an ω^c-set if there exists a dense open set in X.*

Proof. Straightforward. ∎

Theorem 7. *If A is an ω^c-set in X and $B \subset A$, then B is also an ω^c-set in X.*

Proof. We have $Cl(B) \subset Cl(A)$. A being an ω^c-set, $Cl(A) \neq X$ which means $Cl(B) \neq X$. By the ω^c-setness of A, we obtain an open set $G \neq X$ such that $A \subset Cl(G)$ which in turn implies that $B \subset Cl(G)$. ∎

Theorem 8. *If A is an ω^c-set in X and B is a subset of X such that $Cl(B) = Cl(A)$, then B is also an ω^c-set.*

Proof. As A is an ω^c-set, $Cl(A) \neq X$ and hence $Cl(B) \neq X$. Due to the ω^c-setness of A, we get an open set $G \neq X$ such that $A \subset Cl(G)$. So we have $B \subset Cl(B) = Cl(A) \subset Cl(G)$. ∎

Theorem 9. *For any ω^c-set A in X, there exists an open set $G \neq X$ such that $Cl(A) - G$ is a nowhere dense set in X.*

Proof. By the ω^c-setness of A, we obtain an open set $G \neq X$ such that $A \subset Cl(G)$. Since $Cl(G) - G$ is a nowhere dense set in X and $Cl(A) - G \subset Cl(G) - G$, $Cl(A) - G$ is a nowhere dense set in X. ∎

Corollary 10. *For any ω^c-set A in X, there exists an open set $G \neq X$ such that $A - G$ is a nowhere dense set in X.*

Proof. By Theorem 9, we have an open set $G \neq X$ such that $Cl(A) - G$ is a nowhere dense set in X. Since $A - G \subset Cl(A) - G$, $A - G$ is also a nowhere dense set in X. ∎

Theorem 11. *If A is an ω^c-set in X, then $Cl(A)$ is also an ω^c-set in X provided $Cl(A) \neq X$.*

Proof. By the ω^c-setness of A, we obtain an open set $G \neq X$ such that $A \subset Cl(G)$ which implies $Cl(A) \subset Cl(G)$. So $Cl(A)$ is also an ω^c-set if $Cl(A) \neq X$. ∎

Theorem 12. *For an ω^c-set A in X, the following assertions hold good in X:*

(1) [(i)]
(2) $A \cap B$ is an ω^c-set in X for each subset B of X.
(3) $A \cup G$ is also an ω^c-set in X for some open set G in X provided $Cl(A \cup G) \neq X$.

Proof. (i) Since A is an ω^c-set in X and $A \cap B \subset A$, by Theorem 7, $A \cap B$ is an ω^c-set in X for each subset B of X.

(ii) There exists an open set $G \neq X$ in X such that $A \subset Cl(G)$. So $A \cup G \subset Cl(G)$ which means that $A \cup G$ is an ω^c-set in X if $Cl(A \cup G) \neq X$. ∎

Theorem 13. *If A and B are semi-open and pre-open sets respectively in X such that $Cl(A \cup B) \neq X$, then $A \cup B$ is an ω^c-set in X.*

Proof. For A, we have an open set G such that $G \subset A \subset Cl(G)$ which in turns imply that $Cl(A) = Cl(G)$. As B is pre-open, $B \subset Int(Cl(B)) \subset Cl(B)$. We put $H = Int(Cl(B))$. So $Cl(H) = Cl(B)$. Now $Cl(G \cup H) = Cl(G) \cup Cl(H) = Cl(A) \cup Cl(B) = Cl(A \cup B) \neq X$ which also implies that $G \cup H \neq X$. Since $A \cup B \subset Cl(G) \cup Cl(H) = Cl(G \cup H)$, $A \cup B$ is an ω^c-set in X. ∎

Dualizing results from Theorem 6 to Theorem 13, we have results from Theorem 14 to Theorem 21. The proofs of these results are omitted as they are similar to the proofs of corresponding results already established.

Theorem 14. *A subset A of a topological space X with $Int(A) \neq \emptyset$ is an ω-set if there exists a closed set E in X such that $Int(E) = \emptyset$.*

Theorem 15. *If A is an ω-set in X and $A \subset B$, then B is also an ω-set in X.*

Theorem 16. *If A is an ω-set in X and B is a subset of X such that $Int(B) = Int(A)$, then B is also an ω-set.*

Theorem 17. *For any ω-set A in X, there exists a closed set $E \neq \emptyset$ such that $E - Int(A)$ is a nowhere dense set in X.*

Corollary 18. *For any ω-set A in X, there exists a closed set $E \neq \emptyset$ such that $E - A$ is a nowhere dense set in X.*

Theorem 19. *If A is an ω-set in X, then $Int(A)$ is also an ω-set in X provided $Int(A) \neq \emptyset$.*

Theorem 20. *For an ω-set A in X, the following assertions hold good in X:*

(1) *[(i)]*
(2) *$A \cup B$ is an ω-set in X for each subset B of X.*
(3) *$A \cap E$ is also an ω-set in X for some closed set E in X provided $Int(A \cap E) \neq \emptyset$.*

Theorem 21. *If A and B are semi-closed and pre-closed sets respectively in X such that $Int(A \cap B) \neq \emptyset$, then $A \cap B$ is an ω-set in X.*

In the following theorem, we write $Cl_X(A)$ to denote the closure of $A \subset X$ with respect to the topological space X.

Theorem 22. *Let Y be an open subset of the topological space X. If A is an ω^c-set in Y, then A is an ω^c-set in X also.*

Proof. We have an open set $G \neq Y$ in Y such that $A \subset Cl_Y(G)$. We see that $Cl_Y(G) = Y \cap Cl_X(G) \subset Cl_X(G)$. So $A \subset Cl_X(G)$. Y being open in X, $G \neq X$ also open in X. Due to the ω^c-setness of A in Y, $Cl_Y(A) \neq Y$. If $Cl_X(A) = X$, then we get $Cl_Y(A) = Y \cap Cl_X(A) = Y$, a contradiction. So $Cl_X(A) \neq X$. ∎

3. $\bigwedge_{\omega^c}$ AND $\bigvee_{\omega}$ OPERATIONS

For a subset A of a topological space X, we define the following two operations:

(1) [i.]

(2) $$\bigwedge_{\omega^c}(A) = \begin{cases} \bigcap\{W \mid A \subset W, W \text{ is an } \omega^c\text{-set}\}, & \text{if there exists an } \omega^c\text{-set } W \text{ such that } A \subset W; \\ \emptyset, & \text{otherwise.} \end{cases}$$

(3) $$\bigvee_{\omega}(A) = \begin{cases} \bigcup\{W \mid W \subset A, W \text{ is an } \omega\text{-set}\}, & \text{if there exists an } \omega\text{-set } W \text{ such that } W \subset A; \\ X, & \text{otherwise.} \end{cases}$$

Since the complement of an ω^c-set is an ω-set, we conclude that $X - \bigwedge_{\omega^c}(A) = \bigvee_{\omega}(X - A)$. For any subset A of X, we also have $\bigvee_{\omega}(A) \subset A \subset \bigwedge \omega^c(A)$.

By Theorem 4 and Theorem 5, it follows respectively that $\bigwedge_{\omega^c}(A)$ is an ω^c-set and $\bigvee \omega(A)$ is an ω-set in X.

Definition 23. *A topological space X is said to be ω-extremally disconnected if $\bigwedge_{\omega^c}(A)$ is an ω-set for each ω-set A in X.*

Now we give following two lemmas without proofs as they are easy to follow.

Lemma 24. *If A is any subset of X and W is an ω-set in X with $A \cap W = \emptyset$, then $\bigwedge_{\omega^c}(A) \cap W = \emptyset$.*

Lemma 25. *If A is a subset of X and W is an ω^c-set such that $A \subset W$, then $\bigwedge_{\omega^c}(A) \subset W$.*

Theorem 26. *The following properties are equivalent:*

(1) [(i)]
(2) X is ω-extremally disconnected.
(3) For any two ω-sets A and B with $A \cap B = \emptyset$, there exist ω^c-sets U, V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

(4) If A and B are two disjoint ω-sets, then $\bigwedge_{\omega^c}(A) \cap \bigwedge_{\omega^c}(B) = \emptyset$.

Proof. (i)⇒(ii): Let A and B be two ω-sets such that $A \cap B = \emptyset$. By Lemma 24, $\bigwedge_{\omega^c}(A) \cap B = \emptyset$. We put $U = \bigwedge_{\omega^c}(A)$ and $V = X - U$. Obviously, $U \cap V = \emptyset$ and $A \subset U$. We see that U is an ω^c-set. By the ω-extremal disconnectedness of X, V also is an ω^c-set. From $\bigwedge_{\omega^c}(A) \cap B = \emptyset$, we get $B \subset X - U = V$.

(ii)⇒(iii): Let A and B be two ω-sets such that $A \cap B = \emptyset$. By (ii), there exists two disjoint ω^c-sets U, V such that $A \subset U$ and $B \subset V$. Since U, V are ω^c-sets, we have $\bigwedge_{\omega^c}(A) \subset U$ and $\bigwedge_{\omega^c}(B) \subset V$ by Lemma 25. As $U \cap V = \emptyset$, we see that $\bigwedge_{\omega^c}(A) \cap \bigwedge_{\omega^c}(B) = \emptyset$.

(iii)⇒(i): Let W be an ω-set in X. Then $(X - \bigwedge_{\omega^c}(W))$ is an ω-set with $W \cap (X - \bigwedge_{\omega^c}(W)) = \emptyset$. By (iii), $\bigwedge_{\omega^c}(W) \cap \bigwedge_{\omega^c}(X - \bigwedge_{\omega^c}(W)) = \emptyset$ which implies that $\bigwedge_{\omega^c}(W) \cap (X - \bigvee \omega(\bigwedge_{\omega^c}(W))) = \emptyset$. Since $\bigwedge_{\omega^c}(W) \cup (X - \bigwedge_{\omega^c}(W)) = X$ and $\bigwedge_{\omega^c}(W) \subset \bigvee \omega(\bigwedge_{\omega^c}(W))$, we also have $\bigwedge_{\omega^c}(W) \cup (X - \bigvee \omega(\bigwedge_{\omega^c}(W))) = X$. Now $\bigwedge_{\omega^c}(W) \cap (X - \bigvee \omega(\bigwedge_{\omega^c}(W))) = \emptyset$ and $\bigwedge_{\omega^c}(W) \cup (X - \bigvee \omega(\bigwedge_{\omega^c}(W))) = X$ together imply that $\bigwedge_{\omega^c}(W) = X - (X - \bigvee \omega(\bigwedge_{\omega^c}(W))) = \bigvee \omega(\bigwedge_{\omega^c}(W))$. Hence $\bigwedge_{\omega^c}(W)$ is an ω-set. ∎

Theorem 27. *A topological space X is ω-extremally disconnected if and only if for each ω-set W and each ω^c-set V with $W \subset V$, there exist an ω^c-set H and an ω-set G such that $W \subset H \subset G \subset V$.*

Proof. Firstly, suppose that X is ω-extremally disconnected. Let W be an ω-set and V be an ω^c-set such that $W \subset V$. We see that $X - V$ is an ω-set and $W \cap (X - V) = \emptyset$. Then by Theorem 26, $\bigwedge_{\omega^c}(W) \cap \bigwedge_{\omega^c}(X - V) = \emptyset$ which implies $\bigwedge_{\omega^c}(W) \subset X - \bigwedge_{\omega^c}(X - V) = \bigvee_{\omega}(V)$. Putting $H = \bigwedge_{\omega^c}(W)$ and $G = \bigvee_{\omega}(V)$, we obtain $W \subset H \subset G \subset V$.

Conversely, let V and W be ω-sets in X such that $V \cap W = \emptyset$. Then we have $W \subset X - V$ and $X - V$ is an ω^c-set. So we obtain an ω^c-set H and an ω-set G such that $W \subset H \subset G \subset X - V$. Since H is an ω^c-set and $W \subset H$, we have $\bigwedge_{\omega^c}(W) \subset H$ by Lemma 25. We also see that $X - G$ is an ω^c-set and $V \subset X - G$. So by Lemma 25, we get $\bigwedge_{\omega^c}(V) \subset X - G$. Now $H \subset G$ implies that $H \cap (X - G) = \emptyset$ which implies $\bigwedge_{\omega^c}(W) \cap \bigwedge_{\omega^c}(V) = \emptyset$. Hence by Theorem 26, it follows that X is ω-extremally disconnected. ∎

Definition 28. *Let X be a topological space and $\mathbb{R}$ be the real line with usual topology. A function $f : X \to \mathbb{R}$ is said to be ω-upper semi-continuous (resp. ω-lower semi-continuous) if for each $a \in \mathbb{R}$, $\{x \in X : f(x) < a\}$ (resp. $\{x \in X : f(x) > a\}$ is an ω-set in X.*

Theorem 29. *Suppose that X is an ω-extremally disconnected topological space and V, W are two ω-sets such that $V \cap W = \emptyset$. Then there exist an ω-upper semi-continuous and ω-lower semi-continuous function $f : X \to \mathbb{R}$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ with $f(V) = \{0\}$ and $f(W) = \{1\}$.*

Proof. Given that $V \cap W = \emptyset$. It means that $V \subset X - W$. By Theorem 27, there exist an ω^c-set $H(\frac{1}{2})$ and an ω-set $G(\frac{1}{2})$ such that $V \subset H(\frac{1}{2}) \subset G(\frac{1}{2}) \subset X - W$. Since $H(\frac{1}{2})$ is an ω^c-set and $V \subset H(\frac{1}{2})$, we obtain an ω^c-set $H(\frac{1}{4})$ and an ω-set $G(\frac{1}{4})$ such that $V \subset H(\frac{1}{4}) \subset G(\frac{1}{4}) \subset H(\frac{1}{2})$. Similarly, for $G(\frac{1}{2}) \subset X - W$, we obtain an ω^c-set $H(\frac{3}{4})$ and an ω-set $G(\frac{3}{4})$ such that $G(\frac{1}{2}) \subset H(\frac{3}{4}) \subset G(\frac{3}{4}) \subset X - W$. Combining these two results, we get $V \subset H(\frac{1}{4}) \subset G(\frac{1}{4}) \subset H(\frac{1}{2}) \subset G(\frac{1}{2}) \subset H(\frac{3}{4}) \subset G(\frac{3}{4}) \subset X - W$.

Continuing the process, we obtain ω^c-sets $H(\frac{m}{2^n})$ and ω-sets $G(\frac{m}{2^n})$ $(m = 1, 2, 3, \ldots, 2^n - 1)$ such that $V \subset H(\frac{1}{2^n}) \subset G(\frac{1}{2^n}) \cdots G(\frac{m-1}{2^{n-1}}) \subset H(\frac{m}{2^{n-1}}) \subset G(\frac{m}{2^{n-1}}) \cdots G(\frac{2^{n-1}-1}{2^{n-1}}) \subset X - W$. So we have ω^c-sets $H(\frac{m}{2^n})$ and ω-sets $G(\frac{m}{2^n})$ $(m = 1, 2, 3, \ldots, 2^{n-1})$ such that $V \subset H(\frac{m}{2^n}) \subset G(\frac{m}{2^n}) \subset X - W$.

We put $t = \frac{m}{2^n}$ where $(m = 1, 2, 3, \ldots, 2^n - 1)$. Then for $t = t_1, t_2$ with $t_1 < t_2$, there exist ω^c-sets H_{t_1}, H_{t_2} and ω-sets G_{t_1}, G_{t_2} such that $V \subset H_{t_1} \subset G_{t_1} \subset H_{t_2} \subset G_{t_2} \subset X - W$.

We define a mapping $f : X \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \in G_t \text{ for some } t; \\ \sup\{t : t \notin G_t\}, & \text{otherwise.} \end{cases}$$

It is easy to follow that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(V) = \{0\}, f(W) = \{1\}$.

We may now have the following two cases.

Case i: For $0 < a < 1$, let $x \in f^{-1}([0, a))$ which implies $0 \leq f(x) < a$. Hence there exists a dyadic rational number $t < a$ of the form $t = \frac{m}{2^n}$ $(m = 1, 2, 3, \ldots, 2^{n-1})$ such that $x \in G_t$. So $f^{-1}([0, a)) \subset \bigcup_{t<a} G_t$. If $x \in \bigcup_{t<a} G_t$, then $x \in G_{t_0}$ for some $t_0 < a$. So $x \in f^{-1}([0, a))$ which implies $\bigcup_{t<a} G_t \subset f^{-1}([0, a))$. Thus we conclude that $\bigcup_{t<a} G_t = f^{-1}([0, a))$. Since $\bigcup_{t<a} G_t$ is an ω-set, the mapping f is ω-upper semi-continuous.

Case ii: For $0 < a < 1$, let $x \in f^{-1}((a, 1])$ which implies $a < f(x) \leq 1$. Hence there exists a dyadic rational number $t > a$ of the form $t = \frac{m}{2^n}$ $(m = 1, 2, 3, \ldots, 2^{n-1})$ such that $x \notin H_t$ where H_t is an ω^c-set. So $f^{-1}((a, 1]) \subset \bigcup_{t>a}(X - H_t)$. If $x \in \bigcup_{t>a}(X - H_t)$, then $x \in X - H_t$

for some $t > a$ which in turn implies that $x \notin G_{t_0}$ for some t_0 with $t > t_0 > a$. So $f(x) > a$ which implies $\bigcup_{t>a} G_t \subset f^{-1}((0,1])$. Thus we conclude that $\bigcup_{t>a}(X - H_t) = f^{-1}((a,1])$. Since $\bigcup_{t>a}(X - H_t)$ is an ω-set, the mapping f is ω-lower semi-continuous. ∎

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