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ISOMETRY GROUPS OF TRUNCATED TETRAKIS HEXAHEDRON AND TRUNCATED TRIAKIS OCTAHEDRON SPACES

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Abstract. One of the class of non-Euclidean geometries for a finite dimensions is Minkowski geometry. As well as known, only difference between Euclidean geometry and Minkowski geometry is the used distance function. For this reason, its unit ball of Minkowski geometry is a closed, certain symmetric, convex set which is different from sphere in Euclidean geometry. The truncated tetrakis hexahedron and the truncated triakis octahedron are convex solids in the class Truncated Catalan solids. The aim of this work is to develop two new Minkowski geometries by $d_{TTH}-$metric and $d_{TTO}-$metric which unit spheres are truncated tetrakis hexahedron and truncated triakis octahedron, respectively and to find their isometry groups. After we derive these metrics we also give some properties of them. Furthermore, we give that the group of isometries of the 3−dimensional analytical space furnished by $d_{TTH}-$metric or $d_{TTO}-$metric is the semi-direct product of octahedral group O_h and translation group T(3).

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1. Introduction

Polyhedra are always in nature and art. Observing galaxies, viruses, molecules or crystals it has seen that they have polyhedra-like structures. Designers and architects uses polyheral shapes for centuries as it has seen by archaeological discoveries and modern architectural works. The theory with respect to convex sets which has rich applications is one of the oldest, most interesting field of modern matematics. Furthermore, with introducing and developing primarily polyhedra, investigations with respect to geometric properties of convex sets was started. Although it is so ancient it is not that easy to describe what a polyhedron is. A polyhedron would simply be defined as a finite, connected set of planar polygons. The polygons are called faces, and their sides edges. For more detail see [4]. As stated in [3] there are many different ways to obtain a polyhedron. Polyhedra, like polygons, may be convex or non-convex. A polyhedron which is especially convex is one of the most particular solid in $\mathbb{R}^n$. A polyhedron which has as its faces just one type of regular polygon, and all its vertices are congurent is named regular. As well known the number of regular convex polyhedra is only five. Also, these structures are called Platonic bodies for well-known traditional reasons. For more detail see [3]. Archimedean solids which are called semi-regular convex polyhedra have their faces consist of two or more different types of regular polygons meeting in identical vertices, and the total number of their are thirteen. Also dual polyhedra of the Archimedean solids are called Catalan solids, naturally, they are all convex and exactly thirteen just like Archimedean solids. But faces of the Catalan solids are not regular polygons on the contrary Platonic and Archimedean solids.

One of the class of non-Euclidean geometries for a finite dimensions is Minkowski geometry. As well as known, only difference between Euclidean geometry and Minkowski geometry is the distance function used. For this reason, unit ball of a Minkowski geometry is a certain symmetric, closed, convex set which is different from the sphere in Euclidean geometry. Except for the distance function, the linear structure of the Minkowski geometry is the same as the Euclidean case. That is, the planes, lines and points are the same, and also the same methodology with Euclidean case for meausure of angles is used. (See [1] and [2])

By the studies on metric space geometry it has seen that metrics and convex polyhedra are closely related. Unit spheres of Minkowski geometries which are obtained by covering 3-dimensional analytical space with maximum and taxicab metrics are cube and octahedron, respectively. These polyhedra clearly are two of Platonic Solids, moreover unit sphere of 3-dimensional analytical space covered with Chinese checkers metric is a deltoidal icositetrahedron which is a Catalan Solid.

The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as

$$\|(x,y,z)\|_1 = |x| + |y| + |z|$$

and

$$\|(x,y,z)\|_\infty = \max\{|x|, |y|, |z|\},$$

respectively and they are special cases of l_p-norm;

$$\|(x,y,z)\|_p = (|x|^p + |y|^p + |z|^p)^{1/p},$$

where $(x,y,z) \in \mathbb{R}^3$ and CC-metric is defined as

$$d_{CC}(P_1,P_2) = d_L(P_1,P_2) + \left(\sqrt{2}-1\right) d_S(P_1,P_2),$$

where

$$d_L(P_1,P_2) = \max\{|x_1-x_2|, |y_1-y_2|, |z_1-z_2|\},$$

$$d_S(P_1,P_2) = \min\{|x_1-x_2| + |y_1-y_2|, |x_1-x_2| + |z_1-z_2|, |y_1-y_2| + |z_1-z_2|\},$$

$P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$. So convex polyhedra are associated with some metrics. (See [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [17], [18], [19], [20], [21], [22]). By these motivations, firstly we present two novel metrics, and show that the spheres of Minkowski geometries constituted with these metrics are truncated tetrakis hexahedron and truncated triakis octahedron, after that we give some useful properties of these novel metrics. Furthermore, we search the answer of the question that "What is the isometry group of 3-dimensional analytical space furnished by $d_{TTH}-$metric or $d_{TTO}-$metric?". We give the answer to above question as "the relevant isometry group is semi-direct product of octahedral group and translation group."

2. Truncated Tetrakis Hexahedron Metric and Some Properties

As stated in [5] and [14] there are various methods to obtain a new polyhedron. Truncation is one of the methods that comes to the mind first. Truncated tetrakis hexahedron is a convex solid obtained by truncating tetrakis hexahedron which is a Catalan solid, with 24 mirror-symmetric pentagonal and 8 regular hexagonal faces, 54 vertices and 84 edges.

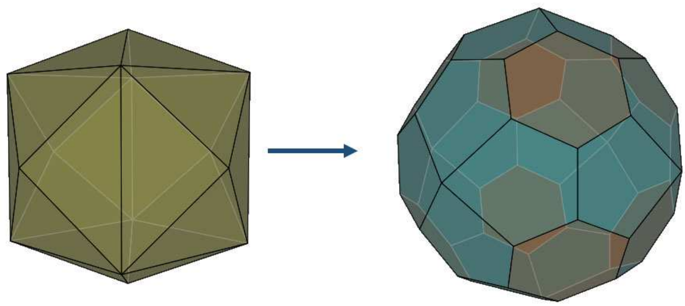

Figure 1: Tetrakis Hexahedron and Truncated Tetrakis Hexahedron

First we give some notions that will be used in the descriptions of distance functions we define. For $P_1{=}(x_1, y_1, z_1)$, $P_2{=}(x_2, y_2, z_2) \in \mathbb{R}^3$, M and S denotes

$$\|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|_\infty \text{ and } \|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|_1 .$$

Moreover orientations $X - Y - Z - X$ and $Z - Y - X - Z$ are called positive (+) direction and negative (-) direction, respectively. M^+ and M^- expresses the next term in the respective direction according to M. For example, if $M = |x_1 - x_2|$, then $M^+ = |y_1 - y_2|$ and $M^- = |z_1 - z_2|$. The $d_{TTH}-$ metric which unit sphere is the truncated tetrakis hexahedron is defined as following:

Definition 1. *The distance function $d_{TTH} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0, \infty)$ which is defined by*

$$d_{TTH}(P_1, P_2) = \max\left\{\frac{9-\sqrt{6}}{10}S, M + \frac{1}{2}M^+, M + \frac{1}{2}M^-\right\}$$

is called the truncated tetrakis hexahedron distance between P_1 and P_2, where $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$.

There are two separate paths from P_1 to P_2 with the same length with respect to the truncated tetrakis hexahedron distance. These paths are

i) consisting three line segments which each one is parallel to a coordinate axis,

ii) consisting two line segments one of which is parallel to a coordinate axis and the other line segment makes $arctan\left(\frac{5}{4}\right)$ angle with another coordinate axis.

Thus the truncated tetrakis hexahedron distance between P_1 and P_2 is for (i) $\frac{9-\sqrt{6}}{10}$ times of the sum of Euclidean lengths of the three line segments or for (ii) the sum of Euclidean lengths of the two line segments. Figure 2 illustrates the truncated tetrakis hexahedron paths from P_1 to P_2 if maximum value is $\frac{9-\sqrt{6}}{10}\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right)$ or $|x_1-x_2|+\frac{1}{2}|y_1-y_2|$.

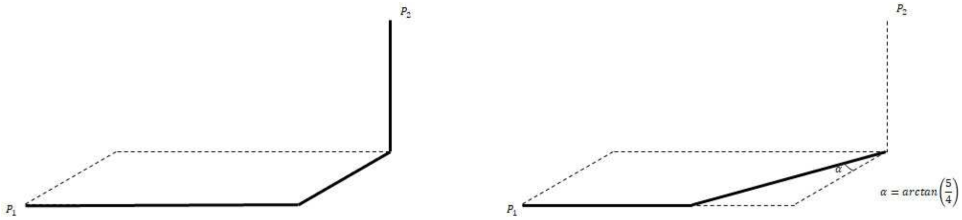

Figure 2: Some TTH way from P_1 to P_2

Lemma 2. *Let* $P_1 \neq P_2$, $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2) \in \mathbb{R}^3$, $M{=}\left\|(x_1-x_2,y_1-y_2,z_1-z_2)\right\|_\infty$ *and* $S{=}\left\|(x_1-x_2,y_1-y_2,z_1-z_2)\right\|_1$. *Then*

$$\begin{array}{l} d_{TTH}\left(P_1,P_2\right) \geq \frac{9-\sqrt{6}}{10}S \\ d_{TTH}\left(P_1,P_2\right) \geq M+\frac{1}{2}M^{+} \\ d_{TTH}\left(P_1,P_2\right) \geq M+\frac{1}{2}M^{-} \end{array}$$

Proof. Proof would be obtained clearly from the properties of maximum function. □

Theorem 3. *The distance function d_{TTH} is a metric. Furthermore unit sphere of d_{TTH} is a truncated tetrakis hexahedron in $\mathbb{R}^3$.*

Proof. Let $d_{TTH}:\mathbb{R}^3\times\mathbb{R}^3 \longrightarrow [0,\infty)$ denotes the truncated tetrakis hexahedron distance function and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ be different points in $\mathbb{R}^3$. To show that d_{TTH} is a metric in $\mathbb{R}^3$ it would be seen that the metric axioms satisfies for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

Since absolute values never have negative value $d_{TTH}\left(P_1,P_2\right) \geq 0$. If $d_{TTH}\left(P_1,P_2\right)=0$ then

$$d_{TTH}\left(P_1,P_2\right)=\max\left\{\frac{9-\sqrt{6}}{10}S_{12}, M_{12}+\frac{1}{2}M_{12}^{+}, M_{12}+\frac{1}{2}M_{12}^{-}\right\}=0,$$

where

$M_{12}=\left\|(x_1-x_2,y_1-y_2,z_1-z_2)\right\|_\infty$ and $S_{12}=\left\|(x_1-x_2,y_1-y_2,z_1-z_2)\right\|_1$

Thus $\frac{9-\sqrt{6}}{10}S_{12}=0$, $M_{12}+\frac{1}{2}M_{12}^{+}=0$ and $M_{12}+\frac{1}{2}M_{12}^{-}=0$. So obviously it is obtained that $x_1=x_2$, $y_1=y_2$, $z_1=z_2$. That is, $P_1=P_2$. Thus the first of the metric axioms; $d_{TTH}(P_1, P_2) \geq 0$ and $d_{TTH}(P_1, P_2) = 0$ if and only if $P_1 = P_2$, holds.

Since $|x - y| = |y - x|$ for all $x, y \in \mathbb{R}$, it is clear that $d_{TTH}(P_1, P_2) = d_{TTH}(P_2, P_1)$, that is d_{TTH} is symmetric.

M_{13} and M_{23} denotes

$$\|(x_1 - x_3, y_1 - y_3, z_1 - z_3)\|_\infty \text{ and } \|(x_2 - x_3, y_2 - y_3, z_2 - z_3)\|_\infty,$$

respectively and S_{13} and S_{23} denotes

$$\|(x_1 - x_3, y_1 - y_3, z_1 - z_3)\|_1 \text{ and } \|((x_2 - x_3, y_2 - y_3, z_2 - z_3)\|_1,$$

respectively.

$$\begin{aligned}
&d_{TTH}(P_1, P_3) = \max\left\{\tfrac{9-\sqrt{6}}{10}S_{13}, M_{13} + \tfrac{1}{2}M_{13}^+, M_{13} + \tfrac{1}{2}M_{13}^-\right\} \\
&\leq \max\left\{\begin{array}{l} \tfrac{9-\sqrt{6}}{10}S_{12} + S_{23}, (M_{12} + M_{23}) + \tfrac{1}{2}\left(M_{12}^+ + M_{23}^+\right), \\ (M_{12} + U_{23}) + \tfrac{1}{2}\left(M_{12}^- + M_{23}^-\right) \end{array}\right\} \\
&= I
\end{aligned}$$

Thus by Lemma2, it is obtained that $d_{TTH}(P_1, P_2) + d_{TTH}(P_2, P_3) \geq I$. Thus triangle inequality $d_{TTH}(P_1, P_2) + d_{TTH}(P_2, P_3) \geq d_{TTH}(P_1, P_3)$ holds. Consequently, truncated tetrakis hexahedron distance is a metric in $\mathbb{R}^3$.

Eventually, the unit sphere of the $d_{TTH}-$metric is

$$S_{TTH} = \left\{(x, y, z) : \max\left\{\frac{9-\sqrt{6}}{10}S, M + \frac{1}{2}M^+, M + \frac{1}{2}M^-\right\} = 1\right\}.$$

Thus the graph of S_{TTH} is a truncated tetrakis hexahedron as in the Figure 3:

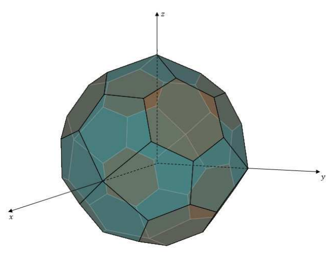

Figure 3: The S_{TTH}: Truncated tetrakis hexahedron

□

Corollary 4. *Let (x_0, y_0, z_0) and r be center and radius of a sphere in the trunceted tetrakis hexahedron space. Then the equation of the* *sphere is*

$$\max\left\{\frac{9-\sqrt{6}}{10}S_0, M_0+\frac{1}{2}M_0^+, M_0+\frac{1}{2}M_0^-\right\}=r.$$

This sphere is a polyhedron with 32 *faces,* 54 *vertices and and* 84 *edges, where* $M_0 = \|(x-x_0, y-y_0, z-z_0)\|_\infty$ *and* $S_0 = \|(x-x_0, y-y_0, z-z_0)\|_1$. *Coordinates of the vertices are translation to* (x_0, y_0, z_0) *all circular shift of the three axis components and all possible* $+/-$ *sign changes of each axis component of* $(0, 0, C_3r)$, (C_0r, C_0r, C_2r) *and* (C_1r, C_4r, C_1r), *where* $C_1 = \frac{1+\sqrt{6}}{2}$, $C_2 = \frac{3+4\sqrt{6}}{6}$, $C_3 = \frac{3(1+\sqrt{6})}{4}$ *and* $C_4 = \frac{1}{2}$.

Lemma 5. *Denote by* l *the Euclidean line passing through* $P_1 = (x_1, y_1, z_1)$ *and* $P_2 = (x_2, y_2, z_2)$ *in* $\mathbb{R}^3$ *and by* d_E *the Euclidean metric. Let* l *has the direction vector* (p, q, r). *Then*

$$d_{TTH}(P_1, P_2) = \mu(P_1P_2).d_E(P_1, P_2)$$

where

$$\mu(P_1P_2) = \frac{\max\left\{\frac{9-\sqrt{6}}{10}S_d, M_d+\frac{1}{2}M_d^+, M_d+\frac{1}{2}M_d^-\right\}}{\sqrt{p^2+q^2+r^2}},$$

M_d *is the* $\|(p, q, r)\|_\infty$ *and* S_d *is the* $\|(p, q, r)\|_1$.

Proof. From equation of l one can get that $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Therefore,

$$d_{TTH}(P_1, P_2) = |\lambda|\max\left\{\frac{9-\sqrt{6}}{10}S_d, M_d+\frac{1}{2}M_d^+, M_d+\frac{1}{2}M_d^-\right\}$$

where M_d is the $\|(p, q, r)\|_\infty$ and S_d is the $\|(p, q, r)\|_1$, and $d_E(P_1, P_2) = |\lambda|\sqrt{p^2+q^2+r^2}$. By proportioning the resulting equations the required consequence is obtained. □

The next corollaries are direct consequences of the lemma 5:

Corollary 6. *If* P, Q *and* X *are any collinear points in* $3-$*dimensional analytical space, then* $d_E(P, X) = d_E(Q, X)$ *iff* $d_{TTH}(P, X) = d_{TTH}(Q, X)$.

Corollary 7. *If* P, Q *and* X *are any three collinear points in* $3-$*dimensional analytical space, then*

$$d_{TTH}(X, P)/d_{TTH}(X, Q) = d_E(X, P)/d_E(X, Q)$$

Namely, the ratios of the d_{TTH} and Euclidean distances along a line are the same.

3. Truncated Triakis Octahedron Metric and Some Properties

Truncated triakis octahedron is a Truncated Catalan solid obtained by trunction operation from triakis octahedron. The truncated triakis octahedron has 24 mirror-symmetric pentagonal and 6 regular octagonal faces, 56 vertices and 84 edges.

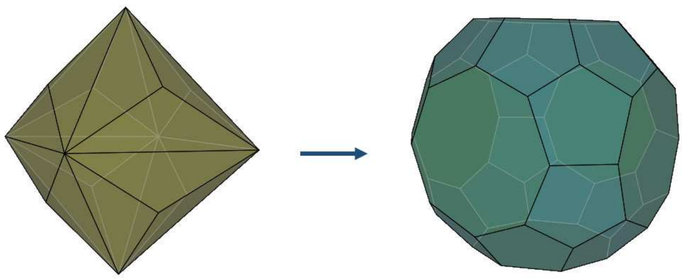

Figure 4: Triakis octahedron, Truncated triakis octahedron

The concepts M, M^{+} and M^{-} are in the same meaning as defined in the previous section. The metric for which the unit sphere is the truncated triakis octahedron is defined as follows:

Definition 8. *The distance function $d_{TTO}: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0,\infty)$ which is defined by*

$$d_{TTO}(P_1, P_2) = \max\left\{M, aM + b\left(M^{-} + M^{+}\right)\right\}$$

where $a = \sqrt{2+\sqrt{2}}\left(\frac{11\sqrt{2}-15}{17}\right) + \frac{3\sqrt{2}-1}{17}$ and $b = \sqrt{2+\sqrt{2}}\left(\frac{7-4\sqrt{2}}{17}\right) + \frac{5+2\sqrt{2}}{17}$ is called the truncated triakis octahedron distance between P_1 and P_2 that $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in $\mathbb{R}^3$.

According to truncated triakis octahedron distance, there are two different paths from P_1 to P_2 with the same length. These paths are

$i)$ a line segment which is parallel to a coordinate axis,

$ii)$ consisting of three line segments one of which is parallel to a coordinate axis and the other two line segments makes $\frac{\pi}{4}$ angle with the according coordinate axes.

Thus truncated tetrakis hexahedron distance between P_1 and P_2 is for (i) Euclidean length of a line segment or for (ii) $\sqrt{2+\sqrt{2}}\left(\frac{11\sqrt{2}-15}{17}\right)+\frac{3\sqrt{2}-1}{17}$ times of sum of Euclidean lengths of relevant three line segments. Figure 5 illustrates truncated triakis octahedron path from P_1 to P_2 in case of maximum value is $|y_1-y_2|$ or $a\,|y_1-y_2|+b\,(|x_1-x_2|+|z_1-z_2|)$ where $a=\sqrt{2+\sqrt{2}}\left(\frac{11\sqrt{2}-15}{17}\right)+\frac{3\sqrt{2}-1}{17}$ and $b=\sqrt{2+\sqrt{2}}\left(\frac{7-4\sqrt{2}}{17}\right)+\frac{5+2\sqrt{2}}{17}$.

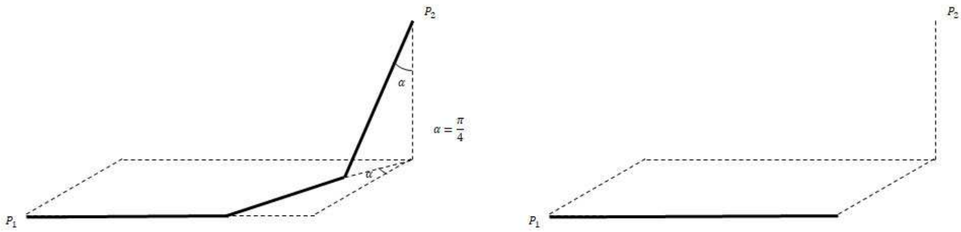

Figure 5: Some TTO ways from P_1 to P_2

Lemma 9. *Let* $P_1\neq P_2$, $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)\in\mathbb{R}^3$, $M{=}\|(x_1-x_2,y_1-y_2,z_1-z_2)\|_\infty$, $a=\sqrt{2+\sqrt{2}}\left(\frac{11\sqrt{2}-15}{17}\right)+\frac{3\sqrt{2}-1}{17}$, *and* $b=\sqrt{2+\sqrt{2}}\left(\frac{7-4\sqrt{2}}{17}\right)+\frac{5+2\sqrt{2}}{17}$. *Then*

$$\begin{aligned}&d_{TTO}\left(P_1,P_2\right)\geq aM+b\left(M^-+M^+\right)\\&d_{TTO}\left(P_1,P_2\right)\geq M\end{aligned}$$

Proof. Proof would be obtained clearly from the properties of maximum function. □

Theorem 10. *The distance function* d_{TTO} *is a metric. Furthermore according to* d_{TTO}*, the unit sphere is a truncated triakis octahedron in* $\mathbb{R}^3$.

Proof. The proof would be done by similar way used for d_{TTH}. □

Finally, the set of points for which the truncated triakis octahedron distance from the origin is 1 (the unit sphere with respect to the truncated triakis octahedron distance) is

$$S_{TTO}=\left\{(x,y,z):\max\left\{M,aM+b\left(M^-+M^+\right)\right\}=1\right\}$$

where $a=\sqrt{2+\sqrt{2}}\left(\frac{11\sqrt{2}-15}{17}\right)+\frac{3\sqrt{2}-1}{17}$ and $b=\sqrt{2+\sqrt{2}}\left(\frac{7-4\sqrt{2}}{17}\right)+\frac{5+2\sqrt{2}}{17}$. Thus the graph of S_{TTO}, the unit sphere in terms of d_{TTO} is as in the Figure 6:

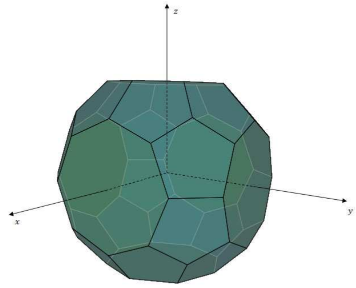

Figure 6: The S_{TTO}: Truncated triakis octahedron

A sphere of the truncated triakis octahedron space with center (x_0, y_0, z_0) and radius r is

$$\max\left\{M_0, aM_0 + b\left(M_0^- + M_0^+\right)\right\} = r$$

which is a polyhedron with 30 faces, 56 vertices and 84 edges, where M_0 is the $\left\|\left(\left|x - x_0\right|, \left|y - y_0\right|, \left|z - z_0\right|\right)\right\|_\infty$. Coordinates of the vertices are translation to (x_0, y_0, z_0) all circular shift of the three axis components and all possible $+/-$ sign changes of each axis component of $(C_1 r, 0, C_3 r)$, $(C_0 r, C_0 r, C_3 r)$ and $(C_2 r, C_2 r, C_2 r)$, where $C_0 = \frac{\sqrt{2+\sqrt{2}}}{2}$, $C_1 = \frac{\sqrt{2\left(2+\sqrt{2}\right)}}{2}$, $C_2 = \frac{2-\sqrt{2}+2\sqrt{2\left(2-\sqrt{2}\right)}}{2}$ and $C_3 = \frac{\sqrt{2}+\sqrt{2\left(2+\sqrt{2}\right)}}{2}$.

Lemma 11. *Denote by l the Euclidean line passing through $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ and by d_E the Euclidean metric. If direction vector of l is (p, q, r), then*

$$d_{TTO}\left(P_1, P_2\right) = \mu\left(P_1 P_2\right).d_E\left(P_1, P_2\right)$$

where

$$\mu\left(P_1 P_2\right) = \frac{\max\left\{M_d, aM_d + b\left(M_d^- + M_d^+\right)\right\}}{\sqrt{p^2 + q^2 + r^2}},$$

and M_d is the $\left\|(p, q, r)\right\|_\infty$.

Proof. From equation of l one can get that $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $\lambda \in \mathbb{R}$. Thus,

$$d_{TTO}\left(P_1, P_2\right) = \left|\lambda\right| \max\left\{M_d, aM_d + b\left(M_d^- + M_d^+\right)\right\},$$

where M_d is the $\left\|(p, q, r)\right\|_\infty$ and $d_E\left(P_1, P_2\right) = \left|\lambda\right| \sqrt{p^2 + q^2 + r^2}$ which implies the required result. □

The lemma 11 states that $d_{TTO}-$distance along a line is a particular positive constant multiple of Euclidean distance along the same line, thus the next corollaries are directly consequences of this statement:

Corollary 12. *If P, Q and X are any collinear points in $3-$dimensonal analytical space, then $d_E(P, X) = d_E(Q, X)$ iff $d_{TTO}(P, X) = d_{TTO}(Q, X)$.*

Corollary 13. *If P, Q and X are any three collinear points in $3-$dimensonal analytical space, then*

$$d_{TTO}(X, P) / d_{TTO}(X, Q) = d_E(X, P) / d_E(X, Q).$$

Namely, the ratios of the d_{TTO} and Euclidean distances along a line are the same.

4. Isometry Groups of Truncated Tetrakis Hexahedron and Truncated Triakis Octahedron

Geometric investigations can be classified into three main methods as metric, group approach and synthetic. The method of group approach is interested in isometry groups and also convex sets play a serious role in dedecting of the group of isometries. There are lots of variety studies with respect to group of isometries of a space (See [10], [11] and [16]). In [2] the author gives the following theorem:

Theorem 14. *If the unit ball B of $(V, \|\|)$ does not intersect a two-plane in an ellipse, then the group of isometries of $(V, \|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant ± 1.*

By Theorem 14, there only left to determine what the pertinent subgroup is.

To find that the isometries of the $3-$dimensional analytical space furnished by $d_{TTH}-$metric or $d_{TTO}-$metric is the semi-direct product of O_h and $T(3)$ we first give the following definition. In the rest of the article we take $\Delta_1 = TTH$ and $\Delta_2 = TTO$. That is, $\Delta_i \in \{\Delta_1, \Delta_2\}$, $i = 1, 2$.

Definition 15. *Let P, Q be two points in $\mathbb{R}^3_{\Delta_i}$, $i = 1, 2$. The least distance set of the points Q, P is defined as follows:*

$$\{X : d_{\Delta_i}(X, P) + d_{\Delta_i}(X, Q) = d_{\Delta_i}(Q, P), i = 1, 2\}$$

and denoted by $[QP]_{\Delta_i}$, $i = 1, 2$.

$[PQ]_{TTH}$ stands for a hexagonal dipyramid in $\mathbb{R}^3_{TTH}$ and $[PQ]_{TTO}$ stands for a octagonal dipyramid in $\mathbb{R}^3_{TTO}$ as shown in Figure(7a) and Figure(7b).

Figure 7(a)

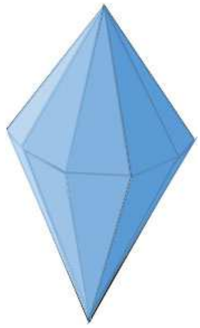

Figure 7(b)

Proposition 16. *Let $\psi : \mathbb{R}^3_{\Delta_i} \longrightarrow \mathbb{R}^3_{\Delta_i}$, be an isometry and let $[PQ]_{\Delta_i}$ be the least distance set of the points Q, P where $i = 1, 2$. Then $\psi\left([PQ]_{\Delta_i}\right) = [\psi(P)\,\psi(Q)]_{\Delta_i}$, $i = 1, 2$.*

Proof. Let $Y \in \psi\left([PQ]_{\Delta_i}\right)$, $i = 1, 2$. Then, there exists $X \in [PQ]_{\Delta_i}$such that $Y = \psi(X)$. $d_{\Delta_i}(P, X) + d_{\Delta_i}(Q, X) = d_{\Delta_i}(P, Q)$, $i = 1, 2$, since $X \in [PQ]_{\Delta_i}$. Thus $d_{\Delta_i}(\psi(P), \psi(X)) + d_{\Delta_i}(\psi(Q), \psi(X)) = d_{\Delta_i}(\psi(P), \psi(Q))$, which means $Y = \psi(X) \in [\psi(P)\,\psi(Q)]_{\Delta_i}$, where $i = 1, 2$. By similar way one can easily prove that $[\psi(P)\,\psi(Q)]_{\Delta_i} \subset \psi\left([PQ]_{\Delta_i}\right)$, $i = 1, 2$. So $\psi\left([PQ]_{\Delta_i}\right) = [\psi(P)\,\psi(Q)]_{\Delta_i}$, $i = 1, 2$, is obtained. $\square$

Corollary 17. *Let $\psi : \mathbb{R}^3_{\Delta_i} \longrightarrow \mathbb{R}^3_{\Delta_i}$, be an isometry and let $[PQ]_{\Delta_i}$ be the least distance set of points Q, P where $i = 1, 2$. Then ψ leaves invariant the lengths of the edges of $[PQ]_{\Delta_i}$, $i = 1, 2$ and maps vertices to vertices.*

Proposition 18. *Let $\psi : \mathbb{R}^3_{\Delta_i} \longrightarrow \mathbb{R}^3_{\Delta_i}$ be an isometry such that $\psi(O) = O$, where $i = 1, 2$. Then $\psi \in O_h$.*

Proof. There are two possibilities for Δ_i, $i = 1, 2$. For $\Delta_1 = TTH$, $C_0 = \frac{6+4\sqrt{6}}{45}$, $C_1 = \frac{2}{3}$, $C_2 = \frac{42-2\sqrt{6}}{45}$, $C_3 = \frac{2\sqrt{6}-2}{15}$ and let $P_1 = (C_0, C_0, C_2)$, $P_2 = (C_2, C_0, C_0)$, $P_3 = (C_0, C_2, C_0)$, $P_4 = (C_1, C_3, C_1)$, $P_5 = (C_1, C_1, C_3)$, $P_6 = (C_3, C_1, C_1)$ and $R = \left(\frac{36+4\sqrt{6}}{45}, \frac{36+4\sqrt{6}}{45}, \frac{36+4\sqrt{6}}{45}\right)$be seven points in $\mathbb{R}^3_{TTH}$. Consider $[OR]_{TTH}$ which is the hexagonal dipyramid (Figure 8(a)).

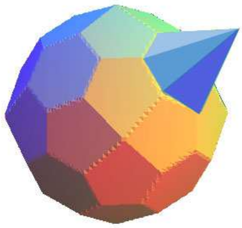

Figure 8(a)

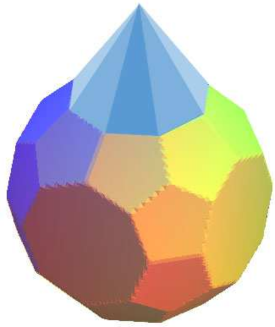

Figure 8(b)

Morever P_1, P_2, P_3, P_4, P_5, P_6 lie on unit sphere centered at origin and minimum distance set $[OR]_{TTH}$. Furthermore these points P_i for $i = 1, ..., 6$ are the vertices of a truncated tetrakis hexahedron's hexagonal face. ψ maps points P_i $(i = 1, 2, ..., 6)$ to the vertices of a truncated tetrakis hexahedron by Corollary 17. Since ψ preserves the lenghts of the edges and truncated tetrakis hexahedron has 8 hexagonal faces and for each face there are 6 possibilities to the points P_i $(i = 1, 2, ..., 6)$ which they can map to, the total number of possibilities is 48. By dealing with each possibility it would seem that all of the elements of pertinent subgroup are found.

If $\Delta_2 = TTO$, $C_0 = \sqrt{2+\sqrt{2}}\left(\frac{\sqrt{2}}{2} - 1\right) + 1$, $C_1 = \sqrt{2+\sqrt{2}}\left(1-\sqrt{2}\right) + \sqrt{2}$, $C_2 = \sqrt{2+\sqrt{2}}\left(2\sqrt{2}-3\right) + 1$, $C_3 = 1$ and let $P_1 = (C_1, 0, C_3)$, $P_2 = (-C_1, 0, C_3)$, $P_3 = (0, C_1, C_3)$, $P_4 = (0, -C_1, C_3)$, $P_5 = (C_0, C_0, C_3)$, $P_6 = (C_0, -C_0, C_3)$, $P_7 = (-C_0, C_0, C_3)$, $P_8 = (-C_0, -C_0, C_3)$ and $R = (0, 0, 2)$ be nine points in $\mathbb{R}^3_{TTO}$. Consider $[OR]_{TTO}$ which is the octagonal dipyramid (Figure 8(b)).Also points P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8 lie on least distance set $[OR]_{TTO}$ and the unit sphere centered at origin. Furthermore points P_i $(i = 1, 2, ..., 8)$ are the vertices of a truncated triakis octahedron's octagonal face. ψ maps points P_i $(i = 1, 2, ..., 8)$ to the vertices of a truncated triakis octahedron by Corollary 17. Since ψ preserves the lenghts of the edges and truncated triakis octahedron has 6 octagonal faces and for each face there are 8 possibilities to the points which they can map to, the total number of possibilities is 48. By dealing with each possibility it would seem that all of the elements of pertinent subgroup are found. $\square$

Theorem 19. *Let* $\psi : \mathbb{R}^3_{\Delta_i} \longrightarrow \mathbb{R}^3_{\Delta_i}$, $i = 1, 2$, *be an isometry. Then there exists a unique* $T_A \in T(3)$ *and* $\phi \in O_h$ *where* $\phi = T_A \circ \psi$.

Proof. Let ψ maps O to A such that $A = (a_1, a_2, a_3)$. Also ϕ is defined suct that $\phi = T_{-A} \circ \psi$. furthermore it is known that $\psi(O) = O$ and ϕ is an isometry. Thus, $\psi \in O_h$ and $\phi = T_A \circ \phi$ by Proposition 18. The uniqueness of T_A is obvious. □

Acknowledgements

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