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HYPER CHROMATIC AND AUGMENTED CHROMATIC ZAGREB INDICES OF WHEEL RELATED GRAPHS AND CYCLE RELATED GRAPHS

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Abstract. In this paper, we introduce the chromatic variance of a graph. This notion is a counterpart for the variance of a graph introduced by F. K. Bell. Here, in the setting of graph coloring, the role played by the degrees of graph vertices is replaced by the products between the indices of colors and the cardinality of the corresponding color class. We compute the chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices of wheel related graphs and cycle related graphs.

1. Introduction

Topological graph indices are the numerical parameters associated with the structure of molecular compound and play an important role in determining physico-chemical properties of chemical graphs. Let $G = (V, E)$ be a simple, undirected graph. Let $V(G)$ be the vertex set and $E(G)$ be the edge set of G. Let $|V(G)| = n$ and $|E(G)| = m$, in which case G is said to be an (n, m) graph. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. A set S of vertices in a graph G is called an *independent* set if no two vertices in S are adjacent. An independent set S is called a *maximal independent set* [5] if any vertex set properly containing S is not independent.

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The independence number $\beta_0(G)$ is the maximum cardinality of maximal independent set in G. An independent set S of G such that $|S| = \beta_0(G)$ is called a *maximum independent set* [5] of G.
F. K. Bell [1], defined the variance of graph G as

$$Var(G) = \frac{1}{n}\sum_{u\in V(G)} d_G^2(u) - \frac{1}{n^2}\left(\sum_{u\in V(G)} d_G(u)\right)^2.$$

In 2013, Shridel et al [10], defined the hyper Zagreb index as

$$HM(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^2.$$

Fortula et al [3], introduced the augmented Zagreb index as

$$AZI(G) = \sum_{uv\in E(G)} \left(\frac{d_G(u)\cdot d_G(v)}{d_G(u) + d_G(v) - 2}\right)^3.$$

Graph coloring is a function of the vertices of a graph by taking to a set of colors $C = \{c_1, c_2, c_3, \ldots, c_l\}$. A proper vertex coloring of a graph G is a coloring in which adjacent vertices of G have different colors. The minimum number of colors required to apply a proper vertex coloring to G is called the chromatic number of G and is denoted by $\chi(G)$. The set of all vertices of G which have the color c_i is named as the color class of that color c_i in G. The strength of the color class, denoted by $\theta(c_i)$ is the cardinality of each color class of color c_i. A vertex coloring consisting of the colors having minimum subscripts may be called a minimum parameter coloring [6]. If we colour the vertices of G in such a way that c_1 is assigned to maximum possible number of vertices, then c_2 is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a coloring is called φ^- coloring of G. In a similar manner, if c_l is assigned to maximum possible number of vertices, then $c_{(l-1)}$ is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a coloring is called φ^+ coloring of G [6]. For computational convenience, function $\zeta : V(G) \to \{1, 2, 3, \ldots, l\}$ such that $\zeta(v_i) = s$ if and only if $\varphi(v_i) = c_s, c_s \in C$. The total number of edges with end points having colors c_t and c_s is denoted by η_{ts}, where $t \leq s$, $1 \leq t$, $s \leq \chi(G)$. Chromatic Zagreb indices has been proposed by J. Kok et al [6]. Recently, the study of chromatic Zagreb indices are reported in [6, 7, 8, 9].

Motivated by previous research on chromatic Zagreb indices, we now define the chromatic variance of G, which is defined as

Definition 1. *Let G be a graph of order n and let $C = \{c_1, c_2, c_3, \dots, c_l\}$ be a proper coloring of G such that $\varphi(v_i) = c_s; 1 \leq i \leq n, 1 \leq s \leq l$. Then for $1 \leq t \leq l!$.*
The chromatic variance of G, denoted by $CVar^{\varphi_t}(G)$ is defined as

$$CVar^{\varphi_t}(G) = \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2.$$

We defined in [2], the hyper chromatic Zagreb and augmented chromatic Zagreb indices. The definitions of hyper and augmented chromatic Zagreb indices are as follows:

Definition 2. *Let G be a graph and let $C = \{c_1, c_2, c_3, \dots, c_l\}$ be a proper coloring of G such that $\varphi(v_i) = c_s; 1 \leq i \leq n, 1 \leq s \leq l$. Then for $1 \leq t \leq l!$.*

(i) The hyper chromatic Zagreb index of G, denoted by $HM_1^{\varphi_t}(G)$ is defined as $HM_1^{\varphi_t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i) + \zeta(v_j))^2, v_iv_j \in E(G)$.
(ii) The augmented chromatic Zagreb index of G, denoted by $AZI^{\varphi_t}(G)$ is defined as $AZI^{\varphi_t}(G) = \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3, v_iv_j \in E(G)$.

The minimum and maximum values of the above chromatic topological indices are denoted by $CVar^{\varphi^-}(G)$, $HM_1^{\varphi^-}(G)$, $AZI^{\varphi^-}(G)$, $CVar^{\varphi^+}(G)$, $HM_1^{\varphi^+}(G)$, $AZI^{\varphi^+}(G)$, respectively.

2. New Results

2.1. **The chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices of wheel related graphs.** In this section, we obtain the chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices of wheel related graphs viz., wheel graph, double wheel graph, helm graph, closed helm graph.

Definition 3. [4] *A wheel graph W_n of order n, sometimes called an n-wheel, is a graph that contains a cycle C_{n-1} of order $n-1$ and a vertex adjacent to each vertex of C_{n-1}.*

Theorem 4. *For a wheel graph $W_n = C_{n-1} + K_1$, we have*

$$\text{(i) } CVar^{\varphi^-}(W_n) = \begin{cases} \frac{5n+40}{2n} - \frac{(3n+8)^2}{4n^2} & \text{if } n \text{ is even,} \\ \frac{5n+13}{2n} - \frac{9(n+1)^2}{4n^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^-}(W_n) = \begin{cases} \frac{79n-59}{2} & \text{if } n \text{ is even,} \\ \frac{59(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^-}(W_n) = \begin{cases} 8n + \frac{140(n-2)}{27} - \frac{1199}{1000} & \text{if } n \text{ is even,} \\ \frac{219(n-1)}{16} & \text{if } n \text{ is odd.} \end{cases}$$

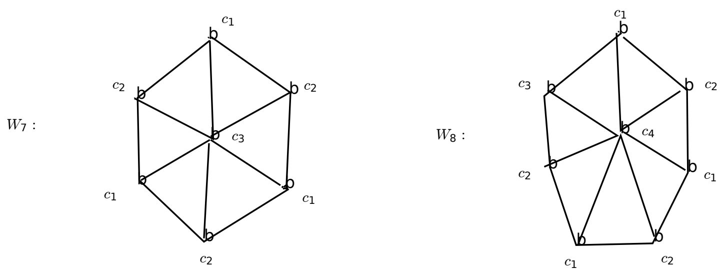

FIGURE 1. φ^- coloring of wheel graph W_7 and W_8

Proof. The wheel graph W_n has chromatic number 4 when n is even and chromatic number 3 when n is odd. Let $x_1, x_2, \dots x_n$ be the vertices of C_n on the rim of the wheel and y be the central vertex. In order to calculate $CVar^{\varphi^-}(G)$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of W_n, we consider the following cases.

Case 1. Assume that n is even. Then, the vertices on the rim are colored with c_1 and c_2, remaining one rim vertex is colored with c_3 and central vertex y is colored with c_4. So, $\theta(c_1) = \theta(c_2) = \frac{n-2}{2}$, $\theta(c_3) = \theta(c_4) = 1$. Also, $\eta_{12} = n-3$, $\eta_{13} = \eta_{23} = \eta_{34} = 1$ and $\eta_{14} = \eta_{24} = \frac{n-2}{2}$. Therefore, the corresponding chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices are

$$\begin{aligned} CVar^{\varphi^-}(W_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\ &= \frac{1}{n}\left(\frac{n-2}{2} + 2n + 21\right) - \frac{1}{n^2}\left(\frac{n-2}{2} + n + 5\right)^2 \\ &= \frac{5n+40}{2n} - \frac{(3n+8)^2}{4n^2}. \\ HM_1^{\varphi^-}(W_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i) + \zeta(v_j))^2 \end{aligned}$$

$$
\begin{aligned}
&= 9(n-3)+90+\frac{61(n-2)}{2}\\
&= \frac{79n-59}{2}.
\end{aligned}
$$

$$
\begin{aligned}
AZI^{\varphi^-}(W_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3\\
&= 8(n-3)+\frac{91}{8}+\frac{1728}{125}+\frac{280(n-2)}{54}\\
&= 8n+\frac{140(n-2)}{27}-\frac{1199}{1000}.
\end{aligned}
$$

Case 2. Assume that n is odd. Then, the vertices on the rim are colored with c_1 and c_2 and central vertex y is colored with c_3. So, $\theta(c_1)=\theta(c_2)=\frac{n-1}{2}$, $\theta(c_3)=1$. Also, $\eta_{13}=\eta_{23}=\frac{n-1}{2}$ and $\eta_{12}=n-1$. Therefore, the corresponding chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices are

$$
\begin{aligned}
CVar^{\varphi^-}(W_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2-\frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2\\
&= \frac{1}{n}\left(\frac{n-1}{2}+2(n-1)+9\right)-\frac{1}{n^2}\left(\frac{n-1}{2}+(n-1)+3\right)^2\\
&= \frac{5n+13}{2n}-\frac{9(n+1)^2}{4n^2}.\\
HM_1^{\varphi^-}(W_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2\\
&= \frac{41(n-1)}{2}+9n-9\\
&= \frac{59(n-1)}{2}.\\
AZI^{\varphi^-}(W_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3\\
&= \frac{91(n-1)}{16}+8n-8\\
&= \frac{219(n-1)}{16}.
\end{aligned}
$$

$\square$

Using minimum parameter coloring, we can also get φ^+ coloring of wheel graph W_n. The results obtained are charted below as next Theorem.

Theorem 5. *For a wheel graph $W_n = C_{n-1} + K_1$, we have*

$$\text{(i) } CVar^{\varphi^+}(W_n) = \begin{cases} \frac{25n-40}{2n} - \frac{(7n-4)^2}{4n^2} & \text{if } n \text{ is even,} \\ \frac{13n-11}{2n} - \frac{(5n-3)^2}{4n^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^+}(W_n) = \begin{cases} \frac{90n-159}{2} & \text{if } n \text{ is even,} \\ \frac{75(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^+}(W_n) = \begin{cases} \frac{1728(n-3)}{125} + \frac{1241n-7886}{432} & \text{if } n \text{ is even,} \\ \frac{219(n-1)}{16} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. When n is even, the wheel graph W_n has chromatic number 4. Then, the vertices on the rim are colored with c_4 and c_3, remaining one rim vertex is colored with c_2 and central vertex y is colored with c_1. So, $\theta(c_4) = \theta(c_3) = \frac{n-2}{2}$, $\theta(c_2) = \theta(c_1) = 1$. Also, $\eta_{34} = n-3$, $\eta_{12} = \eta_{24} = \eta_{23} = 1$ and $\eta_{14} = \eta_{13} = \frac{n-2}{2}$. Let n be odd, we have chromatic number 3. The vertices on the rim are colored with c_3 and c_2 and central vertex y is colored with c_1. So, $\theta(c_3) = \theta(c_2) = \frac{n-1}{2}$, $\theta(c_1) = 1$. Also, $\eta_{12} = \eta_{13} = \frac{n-1}{2}$ and $\eta_{23} = n-1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 4. □

Definition 6. [9] *Joining all the vertices of two disjoint cycles to an external vertex will give double wheel graph. A double wheel graph DW_n is a graph defined by $2C_n + K_1$.*

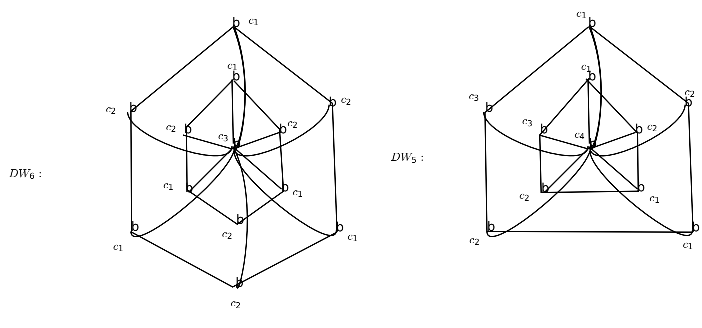

FIGURE 2. φ^- coloring of double wheel graph DW_6 and DW_5.

Theorem 7. *For a double wheel graph $DW_n = 2C_n + K_1$, we have*

$$\text{(i) } CVar^{\varphi^-}(DW_n) = \begin{cases} \frac{14n^2+15n-1}{n^2} & \text{if } n \text{ is even,} \\ \frac{14n^2+71n+49}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $HM_1^{\varphi^-}(DW_n) = \begin{cases} 59n & \text{if } n \text{ is even,} \\ 79n + 83 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $AZI^{\varphi^-}(DW_n) = \begin{cases} \frac{219n}{8} & \text{if } n \text{ is even,} \\ 16(n-2) + \frac{25199}{500} + \frac{280(n-1)}{27} & \text{if } n \text{ is odd.} \end{cases}$

Proof. The double wheel graph $DW_n = 2C_n + K_1$ has chromatic number 4 when n is odd and chromatic number 3 when n is even. Let $x'_1, x'_2 \dots x'_n$ be the vertices of inner cycle, $x_1, x_2 \dots x_n$ be the vertices of outer cycle and y be the central vertex. When n is even we will get two maximum independent sets $D_1 = \{x_1, x_3, \dots x_{n-1}, x'_1, x'_3, \dots x'_{n-1}\}$ and $D_2 = \{x_2, x_4, \dots x_n, x'_2, x'_4, \dots x'_n\}$ are colored with c_1 and c_2 respectively and central vertex y gets the color c_3.
Let n be odd, here we will get two maximum independent sets $D_1 = \{x_1, x_3, \dots x_{n-1}, x'_1, x'_3, \dots x'_{n-1}\}$ and $D_2 = \{x_2, x_4, \dots x_{n-2}, x'_2, x'_4, \dots x'_{n-2}\}$ are colored with c_1 and c_2, the vertices x'_n, x_n gets the color c_3 and central vertex y gets the color c_4. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of DW_n, we consider the following cases.
Case 1. Assume that n is even. Then, $\theta(c_1) = \theta(c_2) = n$ and $\theta(c_3) = 1$. Also, $\eta_{12} = 2n$, $\eta_{23} = \eta_{13} = n$.

$$\begin{aligned} CVar^{\varphi^-}(DW_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\ &= \frac{n+4n+9}{n} - \frac{n+2n+1}{n^2} \\ &= \frac{14n^2+15n-1}{n^2}. \\ HM_1^{\varphi^-}(DW_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\ &= 18n + 41n \\ &= 59n. \\ AZI^{\varphi^-}(DW_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\ &= 16n + \frac{91n}{8} \\ &= \frac{219n}{8}. \end{aligned}$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = \theta(c_2) = n-1$, $\theta(c_3) = 2$ and $\theta(c_4) = 1$. Also, $\eta_{12} = 2(n-2)$, $\eta_{23} = \eta_{13} = \eta_{34} = 2$ and $\eta_{14} = \eta_{24} = n-1$.

$$\begin{aligned}
CVar^{\varphi^-}(DW_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{5n+29}{n} - \frac{3n+7}{n^2} \\
&= \frac{14n^2+71n+49}{n^2}. \\
HM_1^{\varphi^-}(DW_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 18n-36+180+61n-61 \\
&= 79n+83. \\
AZI^{\varphi^-}(DW_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 16(n-2) + \left(\frac{91}{8}+\frac{1728}{128}\right)2 + \frac{280(n-1)}{27} \\
&= 16(n-2) + \frac{25199}{500} + \frac{280(n-1)}{27}.
\end{aligned}$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of double wheel graph DW_n.

Theorem 8. *For a double wheel graph $DW_n = 2C_n + K_1$, we have*

(i) $CVar^{\varphi^+}(DW_n) = \begin{cases} \frac{13n^2-34n-1}{n^2} & \text{if } n \text{ is even,} \\ \frac{25n^2-37n-4}{n^2} & \text{if } n \text{ is odd.} \end{cases}$

(ii) $HM_1^{\varphi^+}(DW_n) = \begin{cases} 75n & \text{if } n \text{ is even,} \\ 139n-97 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $AZI^{\varphi^+}(DW_n) = \begin{cases} \frac{219n}{8} & \text{if } n \text{ is even,} \\ \frac{3456(n-2)}{125} + \frac{1241n-9127}{216} & \text{if } n \text{ is odd.} \end{cases}$

Proof. When n is even, the double wheel graph DWn has chromatic number 3. Then, two maximum independent sets $D_1 = \{x_1, x_3, \dots x_{n-1}, x'_1, x'_3, \dots x'_{n-1}\}$ and $D_2 = \{x_2, x_4, \dots x_n, x'_2, x'_4, \dots x'_n\}$ are colored with c_3 and c_2

respectively. Therefore, central vertex y gets the color c_1. Also, we get $\theta(c_3) = \theta(c_2) = n$, $\theta(c_1) = 1$ and $\eta_{23} = 2n$, $\eta_{12} = \eta_{13} = n$. When n is odd, the double wheel graph DWn has chromatic number 4. Here we will get two maximum independent sets $D_1 = \{x_1, x_3, \ldots x_{n-1}, x'_1, x'_3, \ldots x'_{n-1}\}$ and $D_2 = \{x_2, x_4, \ldots x_{n-2}, x'_2, x'_4, \ldots x'_{n-2}\}$ are colored with c_4 and c_3, the vertices x'_n, x_n gets the color c_2 and central vertex y gets the color c_1. Then, $\theta(c_4) = \theta(c_3) = n - 1$, $\theta(c_2) = 2$ and $\theta(c_1) = 1$. Also, $\eta_{34} = 2(n-2)$, $\eta_{12} = \eta_{23} = \eta_{24} = 2$ and $\eta_{13} = \eta_{14} = n - 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 7. □

Definition 9. [9] *A helm graph H_n is a graph obtained by attaching a pendent edge to every vertex of the rim C_n of a wheel graph W_n.*

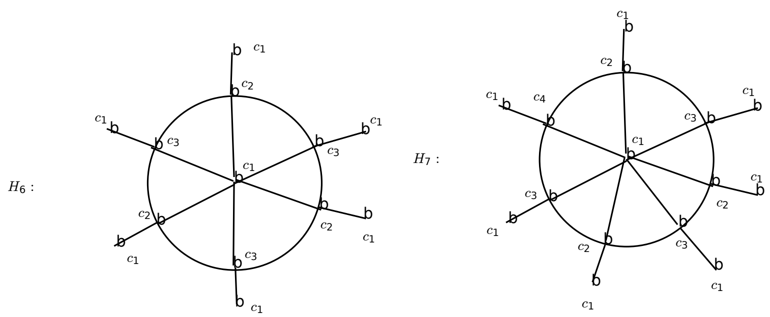

FIGURE 3. φ^- coloring of helm graph H_6 and H_7.

Theorem 10. *For a helm graph H_n, we have*

(i) $$CVar^{\varphi^-}(H_n) = \begin{cases} \frac{15n+2}{2n} - \frac{(7n+2)^2}{4n^2} & \text{if } n \text{ is even,} \\ \frac{15n+34}{2n} - \frac{(7n+10)^2}{4n^2} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $$HM_1^{\varphi^-}(H_n) = \begin{cases} 50n & \text{if } n \text{ is even,} \\ 50n + 60 & \text{if } n \text{ is odd.} \end{cases}$$

(iii) $$AZI^{\varphi^-}(H_n) = \begin{cases} \frac{155n}{8} & \text{if } n \text{ is even,} \\ \frac{91(n-1)}{8} + 8(n-2) + \frac{344}{27} + \frac{1728}{125} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The helm H_n graph has chromatic number 3 when n is even, chromatic number 4 when n is odd. Let the vertices $x_1, x_2, \ldots x_n$ be the vertices on the rim of the wheel, $x'_1, x'_2, \ldots x'_n$ be the pendent vertices and y be the central vertex. When n is even, the maximum independent set $D_1 = \{x'_1, x'_2, \ldots x'_n, y\}$ gets the color c_1, the independent sets $D_2 = \{x_1, x_3, \ldots x'_{n-1}\}$, $D_3 = \{x_2, x_4, \ldots x'_n\}$ gets the colors c_2 and c_3 respectively. When n is odd, we color c_1 to the maximum independent set D_1, the independent sets $D_2 = \{x_1, x_3, \dots x'_{n-2}\}$, $D_3 = \{x_2, x_4, \dots x'_{n-1}\}$ are colored with c_2 and c_3 respectively and vertex x_n gets the color c_4. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of H_n, we consider the following cases.

Case 1. Assume that n is even. Then, $\theta(c_1) = n+1$, $\theta(c_2) = \theta(c_3) = \frac{n}{2}$. Also, $\eta_{12} = n$, $\eta_{23} = \eta_{13} = n$.

$$
\begin{aligned}
CVar^{\varphi^-}(H_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{6n+9n+2}{2n} - \left(\frac{(4n+2+3n)^2}{4n^2}\right) \\
&= \frac{15n+2}{2n} - \frac{(7n+2)^2}{4n^2}. \\
HM_1^{\varphi^-}(H_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 9n + 25n + 16n \\
&= 50n. \\
AZI^{\varphi^-}(H_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= \left(16 + \frac{27}{8}\right)n \\
&= \frac{155n}{8}.
\end{aligned}
$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = n+1$, $\theta(c_2) = \theta(c_3) = \frac{n}{2}$ and , $\theta(c_4) = 1$. Also, $\eta_{23} = n-2$, $\eta_{12} = \eta_{13} = n-1$ and $\eta_{14} = 2$, $\eta_{24} = \eta_{34} = 1$.

$$
\begin{aligned}
CVar^{\varphi^-}(H_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{6n+34+9n}{2n} - \frac{(4n+3n+10)^2}{4n^2} \\
&= \frac{15n+34}{2n} - \frac{(7n+10)^2}{4n^2}.
\end{aligned}
$$

$$\begin{aligned}
HM_1^{\varphi^-}(H_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2\\
&= 25n-50+25n-25+50+85\\
&= 50n+60.\\
AZI^{\varphi^-}(H_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3\\
&= \left(8+\frac{27}{8}\right)(n-1)+8(n-2)+\frac{128}{27}+\frac{1728}{125}+8\\
&= \frac{91(n-1)}{8}+8(n-2)+\frac{344}{27}+\frac{1728}{125}.
\end{aligned}$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of helm graph H_n.

Theorem 11. *For a helm graph H_n, we have*

(i) $CVar^{\varphi^+}(H_n) = \begin{cases} \frac{23n+18}{2n} - \frac{(9n+6)^2}{4n^2} & \text{if } n \text{ is even},\\ \frac{45n+34}{2n} - \frac{(13n+10)^2}{4n^2} & \text{if } n \text{ is odd}.\end{cases}$

(ii) $HM_1^{\varphi^+}(H_n) = \begin{cases} 50n & \text{if } n \text{ is even},\\ 110n-60 & \text{if } n \text{ is odd}.\end{cases}$

(iii) $AZI^{\varphi^+}(H_n) = \begin{cases} \frac{155n}{8} & \text{if } n \text{ is even},\\ \frac{2728(n-1)}{125} + 8(n-2) + \frac{3481}{216} & \text{if } n \text{ is odd}.\end{cases}$

Proof. The helm H_n graph has chromatic number 3 when n is even, chromatic number 4 when n is odd. When n is even, the maximum independent set $D_1 = \{x'_1, x'_2, \dots x'_n, y\}$ gets the color c_3, the independent sets $D_2 = \{x_1, x_3, \dots x'_{n-1}\}$, $D_3 = \{x_2, x_4, \dots x'_n\}$ gets the colors c_2 and c_1 respectively. Then, $\theta(c_3) = n+1$, $\theta(c_2) = \theta(c_1) = \frac{n}{2}$. Also, $\eta_{12} = n$, $\eta_{23} = \eta_{13} = n$. when n is odd, we color c_4 to the maximum independent set D_1, the independent sets $D_2 = \{x_1, x_3, \dots x'_{n-2}\}$, $D_3 = \{x_2, x_4, \dots x'_{n-1}\}$ are colored with c_3 and c_2 respectively and vertex x_n gets the color c_1. Then, $\theta(c_4) = n+1$, $\theta(c_3) = \theta(c_2) = \frac{n}{2}$ and , $\theta(c_1) = 1$. Also, $\eta_{23} = n-2$, $\eta_{34} = \eta_{24} = n-1$ and $\eta_{14} = 2$, $\eta_{12} = \eta_{13} = 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 10. □

Definition 12. [9] *A closed helm graph CH_n is a graph obtained from the helm graph H_n, by joining a pendant vertex v_i to the pendant vertex*

v_{i+1}, where $1 \leq i \leq n$ and $v_{n+i} = v_i$. That is, the pendant vertices in H_n induce a cycle in CH_n.

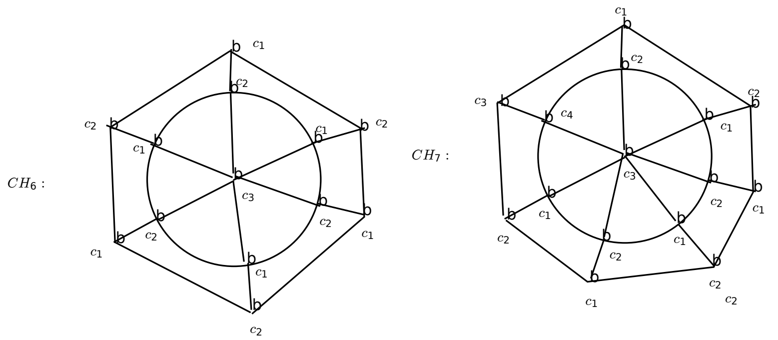

FIGURE 4. φ^- coloring of closed helm graph CH_6 and CH_7.

Theorem 13. *For a closed helm graph CH_n, we have*

$$\text{(i) } CVar^{\varphi^-}(CH_n) = \begin{cases} \frac{5n+9}{n} - \frac{(3n+3)^2}{n^2} & \text{if } n \text{ is even,} \\ \frac{5n+29}{n} - \frac{(3n+7)^2}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^-}(CH_n) = \begin{cases} \frac{95n}{2} & \text{if } n \text{ is even,} \\ \frac{95n+253}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^-}(CH_n) = \begin{cases} \frac{475n}{16} & \text{if } n \text{ is even,} \\ 28n + \frac{27(n+1)}{16} + \frac{1728}{125} - \frac{628}{27} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The closed helm CH_n graph has chromatic number 3 when n is even, chromatic number 4 when n is odd. Let the vertices $x_1, x_2, \ldots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \ldots x'_n$ on the inner cycle and y be the central vertex.
When n is even there exits two maximum independent sets $D_1 = \{x_1, x_3, \ldots x_{n-1}, x'_2, x'_4 \ldots x'_n\}$, $D_2 = \{x_2, x_4, \ldots x_n, x'_1, x'_3 \ldots x'_{n-1}\}$ gets the color c_1 and c_2 respectively. The vertex y gets the color c_3. When n is odd there exits two maximum independent sets $D_1 = \{x_1, x_3, \ldots x_{n-2}, x'_2, x'_4 \ldots x'_{n-1}\}$, $D_2 = \{x_2, x_4, \ldots x_{n-1}, x'_3 \ldots x'_n\}$ gets the color c_1 and c_2 respectively. The vertices y, x_n gets the color c_3, the vertex x'_1 gets the color c_4. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of CH_n, we consider the following cases.
Case 1. Assume that n is even. Then, $\theta(c_1) = \theta(c_2) = n$ and

$\theta(c_1) = 1$. Also, $\eta_{12} = 3n$, $\eta_{23} = \eta_{13} = \frac{n}{2}$.

$$\begin{aligned}
CVar^{\varphi^-}(CH_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{n+4n+9}{n} - \frac{(n+2n+3)^2}{n^2} \\
&= \frac{5n+9}{n} - \frac{(3n+3)^2}{n^2}. \\
HM_1^{\varphi^-}(CH_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 27n + \frac{41n}{2} \\
&= \frac{95n}{2}. \\
AZI^{\varphi^-}(CH_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 24n + 4n + \frac{27n}{16} \\
&= \frac{475n}{16}.
\end{aligned}$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = \theta(c_2) = n-1$, $\theta(c_3) = 2$ and $\theta(c_4) = 1$. Also, $\eta_{12} = 3(n-2)$, $\eta_{13} = \frac{n+1}{2}$, $\eta_{23} = \frac{n+3}{2}$, $\eta_{14} = 2$, $\eta_{34} = \eta_{24} = 1$.

$$\begin{aligned}
CVar^{\varphi^-}(CH_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{n-1+4n-4+18+16}{n} - \frac{(n-1+2n-2+6+4)^2}{n^2} \\
&= \frac{5n+29}{n} - \frac{(3n+7)^2}{n^2}. \\
HM_1^{\varphi^-}(CH_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 27n - 54 + 8n + 8 + \frac{25(n+3)}{2} + 135 \\
&= \frac{95n+253}{2}.
\end{aligned}$$

$$
\begin{aligned}
AZI^{\varphi^-}(CH_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 24n+\frac{27(n+1)}{16}+4n+12+\frac{128}{27}+\frac{1728}{125}-40 \\
&= 28n+\frac{27(n+1)}{16}+\frac{1728}{125}-\frac{628}{27}.
\end{aligned}
$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of closed helm graph CH_n.

Theorem 14. *For a closed helm graph CH_n, we have*

(i) $CVar^{\varphi^+}(CH_n) = \begin{cases} \frac{13n+1}{n} - \frac{(5n+1)^2}{n^2} & \text{if } n \text{ is even,} \\ \frac{25n-16}{n} - \frac{(7n-2)^2}{n^2} & \text{if } n \text{ is odd.} \end{cases}$

(ii) $HM_1^{\varphi^+}(CH_n) = \begin{cases} \frac{175n}{2} & \text{if } n \text{ is even,} \\ \frac{355n-327}{2} & \text{if } n \text{ is odd.} \end{cases}$

(iii) $AZI^{\varphi^+}(CH_n) = \begin{cases} \frac{475n}{16} & \text{if } n \text{ is even,} \\ 8n+\frac{5184(n-2)}{125}+\frac{6937}{216} & \text{if } n \text{ is odd.} \end{cases}$

Proof. The closed helm CH_n graph has chromatic number 3 when n is even, chromatic number 4 when n is odd. Let the vertices $x_1, x_2, \ldots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \ldots x'_n$ on the inner cycle and y be the central vertex.
When n is even there exits two maximum independent sets $D_1 = \{x_1, x_3, \ldots x_{n-1}, x'_2, x'_4 \ldots x'_n\}$, $D_2 = \{x_2, x_4, \ldots x_n, x'_1, x'_3 \ldots x'_{n-1}\}$ gets the color c_3 and c_2 respectively. The vertex y gets the color c_1. Therefore, $\theta(c_3) = \theta(c_2) = n$ and $\theta(c_1) = 1$. Also, $\eta_{13} = 3n$, $\eta_{12} = \eta_{13} = \frac{n}{2}$
When n is odd there exits two maximum independent sets $D_1 = \{x_1, x_3, \ldots x_{n-2}, x'_2, x'_4 \ldots x'_{n-1}\}$, $D_2 = \{x_2, x_4, \ldots x_{n-1}, x'_3 \ldots x'_n\}$ gets the color c_4 and c_3 respectively. The vertices y, x_n gets the color c_2, the vertex x'_1 gets the color c_1. Therefore, $\theta(c_4) = \theta(c_3) = n-1$, $\theta(c_2) = 2$ and $\theta(c_1) = 1$. Also, $\eta_{43} = 3(n-2)$, $\eta_{24} = \frac{n+1}{2}$, $\eta_{23} = \frac{n+3}{2}$, $\eta_{14} = 2$, $\eta_{12} = \eta_{13} = 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 13. □

2.2. **The chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices of cycle related graphs.** In this section, we obtain the chromatic variance, the hyper chromatic and augmented chromatic Zagreb indices of cycle related graphs viz., flower graph, sunflower graph, closed sunflower graph, blossom graph.

Definition 15. [8] *A flower graph F_n is a graph which is obtained by joining the pendant vertices of a helm graph H_n to its central vertex. A flower graph F_n has $2n + 1$ vertices and $4n$ edges.*

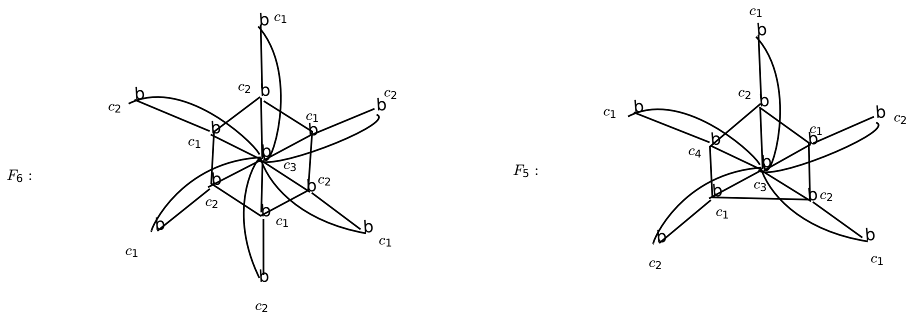

FIGURE 5. φ^- coloring of flower graph F_6 and F_5

.

Theorem 16. *For a flower graph F_n, we have*

$$\text{(i) } CVar^{\varphi^-}(F_n) = \begin{cases} \frac{5n+9}{n} - \frac{(3n+3)^2}{n^2} & \text{if } n \text{ is even,} \\ \frac{6n+2}{n} - \frac{(4n+5)^2}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^-}(F_n) = \begin{cases} 59n & \text{if } n \text{ is even,} \\ 59n + 83 & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^-}(F_n) = \begin{cases} \frac{219}{8} & \text{if } n \text{ is even,} \\ \frac{219n}{8} + \frac{1728}{125} - \frac{520}{27} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. A flower graph has chromatic number 3 when n is even, the chromatic number 4 when n is odd. Let the independent set of vertices $x_1, x_2, \dots x_n$ on the outer cycle of the F_n. The vertices $x'_1, x'_2, \dots x'_n$ on the inner cycle of the F_n and y be the central vertex of F_n. When n is even, the vertex sets $V_1 = \{x_1, x_3, \dots x_{n-1}, x'_2, x'_4, \dots x'_n\}$, $V_2 = \{x_2, x_4, \dots x_n, x'_1, x'_3, \dots x'_{n-1}\}$ gets the color c_1 and c_2, the vertex y gets the color c_3. When n is odd, the vertex sets $V_1 = \{x_1, x_3, \dots x_{n-2}, x'_2, x'_4, \dots x'_{n-1}\}$, $V_2 = \{x_2, x_4, \dots x_{n-1}, x'_1, x'_3, \dots x'_{n-2}\}$ gets the color c_1 and c_2, the central vertex y gets the color c_3, x'_n gets the color c_4. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of F_n, we consider the following cases.

Case 1. Assume that n is even. Then, $\theta(c_1) = \theta(c_2) = n$, $\theta(c_3) = 1$. Also, $\eta_{12} = 2n$, $\eta_{23} = \eta_{13} = n$.

$$\begin{aligned}
CVar^{\varphi^-}(F_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{n+4n+9}{n} - \frac{n+2n+3}{n^2} \\
&= \frac{5n+9}{n} - \frac{(3n+3)^2}{n^2}. \\
HM_1^{\varphi^-}(F_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 18n + 25n + 9n \\
&= 59n. \\
AZI^{\varphi^-}(F_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 16n + 8n + \frac{27n}{8} \\
&= \frac{219}{8}.
\end{aligned}$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = 2n$, $\theta(c_2) = n-1$, $\theta(c_3) = \theta(c_4) = 1$. Also, $\eta_{12} = 2n-3$, $\eta_{13} = n$, $\eta_{23} = n-1$, $\eta_{14} = 2$, $\eta_{34} = \eta_{24} = 1$.

$$\begin{aligned}
CVar^{\varphi^-}(F_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{2n+4n-4+9+16}{n} - \frac{(2n+2n+-2+3+4)^2}{n^2} \\
&= \frac{6n+2}{n} - \frac{(4n+5)^2}{n^2}. \\
HM_1^{\varphi^-}(F_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 18n - 27 + 16n + 25n - 25 + 50 + 49 + 36 \\
&= 59n + 83. \\
AZI^{\varphi^-}(F_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3
\end{aligned}$$

$$= \quad 24n + \frac{27n}{8} - 32 + \frac{344}{27} + \frac{1728}{125}$$
$$= \quad \frac{219n}{8} + \frac{1728}{125} - \frac{520}{27}.$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of flower graph F_n.

Theorem 17. *For a flower graph F_n, we have*

$$\text{(i) } CVar^{\varphi^+}(F_n) = \begin{cases} \frac{13n+1}{n} - \frac{(5n+1)^2}{n^2} & \text{if } n \text{ is even,} \\ \frac{162n-4}{n} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^+}(F_n) = \begin{cases} 75n & \text{if } n \text{ is even,} \\ 159n - 97 & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^+}(F_n) = \begin{cases} \frac{219n}{8} & \text{if } n \text{ is even,} \\ 16n + \frac{1728(2n-3)}{125} + \frac{1753}{216} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. A flower graph has chromatic number 3 when n is even, 4 when n is odd. Let the independent set of vertices $x_1, x_2, \dots x_n$ on the outer cycle of the F_n. the vertices $x'_1, x'_2, \dots x'_n$ on the inner cycle of the F_n and y be the central vertex of F_n. When n is even the vertex sets $V_1 = \{x_1, x_3, \dots x_{n-1}, x'_2, x'_4, \dots x'_n\}$, $V_2 = \{x_2, x_4, \dots x_n, x'_1, x'_3, \dots x'_{n-1}\}$ gets the color c_3 and c_2, the vertex y gets the color c_1. Therefore, $\theta(c_3) = \theta(c_2) = n$, $\theta(c_1) = 1$. Also, $\eta_{12} = \eta_{13} = n$, $\eta_{23} = 2n$. When n is odd, then the vertex sets $V_1 = \{x_1, x_3, \dots x_{n-2}, x'_2, x'_4, \dots x'_{n-1}\}$, $V_2 = \{x_2, x_4, \dots x_{n-1}, x'_1, x'_3, \dots x'_{n-2}\}$ gets the color c_4 and c_3, the central vertex y gets the color c_2, x'_n gets the color c_1. Therefore, $\theta(c_4) = 2n$, $\theta(c_3) = n - 1$, $\theta(c_2) = \theta(c_1) = 1$. Also, $\eta_{34} = 2n - 3$, $\eta_{24} = n$, $\eta_{23} = n - 1$, $\eta_{14} = 2$, $\eta_{12} = \eta_{13} = 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 16. □

Definition 18. [8] *A sunflower graph SF_n is a graph obtained by replacing each edge of the rim of a wheel graph W_n by a triangle such that two triangles share a common vertex if and only if the corresponding edges in W_n are adjacent in W_n. Sunflower graph SF_n has $2n+1$ vertices and $4n$ edges.*

Theorem 19. *For a sunflower graph SF_n, we have*

$$\text{(i) } CVar^{\varphi^-}(SF_n) = \begin{cases} \frac{15n+2}{2n} - \frac{(7n+2)^2}{4n^2} & \text{if } n \text{ is even,} \\ \frac{15n+47}{2n} - \frac{(7n+15)^2}{4n^2} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $HM_1^{\varphi^-}(SF_n) = \begin{cases} \frac{125n}{2} & \text{if } n \text{ is even,} \\ \frac{125n+145}{2} & \text{if } n \text{ is odd.} \end{cases}$

(iii) $AZI^{\varphi^-}(SF_n) = \begin{cases} \frac{401n}{16} & \text{if } n \text{ is even,} \\ \frac{281n+289}{16} + \frac{408}{27} + \frac{1728}{125} & \text{if } n \text{ is odd.} \end{cases}$

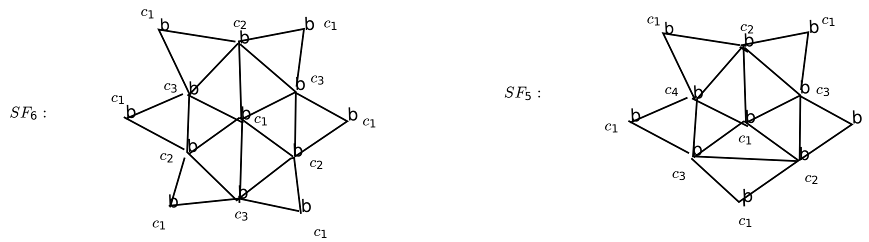

FIGURE 6. φ^- coloring of sunflower graph SF_6 and SF_5

Proof. A sunflower graph has chromatic number 3 when n is even, the chromatic number 4 when n is odd. Let the vertices $x_1, x_2, \ldots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \ldots x'_n$ on the inner cycle, y be the central vertex. When n is even, the maximum independent set $D_1 = \{x_1, x_2, \ldots x_n, y\}$ receives the color c_1, the vertex sets $V_1 = \{x'_1, x'_3, \ldots x'_{n-1}\}$, $V_2 = \{x'_2, x'_4, \ldots x'_n\}$ receives the color c_2 and c_3 respectively. When n is odd, the maximum independent set $D_1 = \{x_1, x_2, \ldots x_n, y\}$ receives the color c_1, the maximum independent sets $V_1 = \{x'_1, x'_3, \ldots x'_{n-2}\}$, $V_2 = \{x'_2, x'_4, \ldots x'_{n-1}\}$ receives the color c_2 and c_3 respectively and x'_n gets the color c_4. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of SF_n, we consider the following cases.

Case 1. Assume that n is even. Then, $\theta(c_1) = n+1$, $\theta(c_2) = \theta(c_3) = \frac{n}{2}$. Also, $\eta_{23} = n$, $\eta_{12} = \eta_{13} = \frac{3n}{2}$.

$$\begin{aligned} CVar^{\varphi^-}(SF_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\ &= \frac{1}{n}\left(3n + \frac{9n}{2} + 1\right) - \frac{1}{n^2}\left(n + 1 + n + \frac{3n}{2}\right)^2 \\ &= \frac{15n+2}{2n} - \frac{(7n+2)^2}{4n^2}. \\ HM_1^{\varphi^-}(SF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i) + \zeta(v_j))^2 \end{aligned}$$

$$
\begin{aligned}
&= 25n + \frac{75n}{2} \\
&= \frac{125n}{2}. \\
AZI^{\varphi^-}(SF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 8n + \frac{273n}{16} \\
&= \frac{401n}{16}.
\end{aligned}
$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = n+1$, $\theta(c_2) = \theta(c_3) = \frac{n+1}{2}$, $\theta(c_4) = 1$. Also, $\eta_{12} = \eta_{13} = \frac{3(n-1)}{2}$, $\eta_{23} = n-2$, $\eta_{14} = 3$, $\eta_{24} = \eta_{34} = 1$.

$$
\begin{aligned}
CVar^{\varphi^-}(SF_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{1}{n}\left(n+1+2n+2+\frac{9(n+1)}{2}+16\right) \\
&\quad -\frac{1}{n^2}\left(n+1+n+1+\frac{3(n+1)}{2}+4\right)^2 \\
&= \frac{15n+47}{2n} - \frac{(7n+15)^2}{4n^2}. \\
HM_1^{\varphi^-}(SF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= \frac{75(n-1)}{2} + 25(n-2) + 160 \\
&= \frac{125n+145}{2}. \\
AZI^{\varphi^-}(SF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= \frac{273(n-1)}{16} + 8(n-2) + \frac{408}{27} + \frac{1728}{125} \\
&= \frac{281n+289}{16} + \frac{408}{27} + \frac{1728}{125}.
\end{aligned}
$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of sunflower graph SF_n.

Theorem 20. *For a sunflower graph SF_n, we have*

$$\text{(i) } CVar^{\varphi^+}(SF_n) = \begin{cases} \frac{23n+18}{2n} - \frac{(9n+6)^2}{4n^2} & \text{if } n \text{ is even,} \\ \frac{45n+47}{2n} - \frac{(13n+15)^2}{4n^2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^+}(SF_n) = \begin{cases} \frac{141n}{2} & \text{if } n \text{ is even,} \\ \frac{305n-155}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{(iii) } AZI^{\varphi^+}(SF_n) = \begin{cases} \frac{401n}{16} & \text{if } n \text{ is even,} \\ \frac{3092n-4092}{125} + \frac{537}{216} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. A sunflower graph has chromatic number 3 when n is even, 4 when n is odd. Let the vertices $x_1, x_2, \dots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \dots x'_n$ on the inner cycle, y be the central vertex. When n is even, the maximum independent set $D_1 = \{x_1, x_2, \dots x_n, y\}$ receives the color c_3, the vertex sets $V_1 = \{x'_1, x'_3, \dots x'_{n-1}\}$, $V_2 = \{x'_2, x'_4, \dots x'_n\}$ receives the color c_2 and c_1 respectively. Then, $\theta(c_3) = n+1$, $\theta(c_2) = \theta(c_1) = \frac{n}{2}$. Also, $\eta_{12} = n$, $\eta_{13} = \eta_{23} = \frac{3n}{2}$. When n is odd, the maximum independent set $D_1 = \{x_1, x_2, \dots x_n, y\}$ receives the color c_4, the maximum independent sets $V_1 = \{x'_1, x'_3, \dots x'_{n-2}\}$, $V_2 = \{x'_2, x'_4, \dots x'_{n-1}\}$ receives the color c_3 and c_2 respectively and x'_n gets the color c_1. Then, $\theta(c_4) = n+1$, $\theta(c_3) = \theta(c_2) = \frac{n+1}{2}$, $\theta(c_1) = 1$. Also, $\eta_{24} = \eta_{34} = \frac{3(n-1)}{2}$, $\eta_{23} = n-2$, $\eta_{14} = 3$ and $\eta_{12} = \eta_{13} = 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 19. □

Definition 21. [8] *A closed sunflower graph CSF_n is a graph obtained by joining the independent vertices of a sunflower graph SF_n, which are not adjacent to its central vertex so that these vertices induces a cycle on n vertice*

Theorem 22. *For a closed sunflower graph CSF_n, we have*

$$\text{(i) } CVar^{\varphi^-}(CSF_n) = \begin{cases} \frac{16n^2+60n+64}{3n} & \text{if } n \equiv 0 \ (mod\ 3), \\ \frac{28n+143}{3n} - \frac{(4n-9)^2}{n^2} & \text{if } n \equiv 1 \ (mod\ 3), \\ \frac{28n+95}{3n} - \frac{(4n+5)^2}{n^2} & \text{if } n \equiv -1 \ (mod\ 3). \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^-}(CSF_n) = \begin{cases} \frac{310n}{3} & \text{if } n \equiv 0 \ (mod\ 3), \\ \frac{349n+578}{3} & \text{if } n \equiv 1 \ (mod\ 3), \\ \frac{310n+439}{3} & \text{if } n \equiv -1 \ (mod\ 3). \end{cases}$$

$$(iii)\ AZI^{\varphi^-}(CSF_n) = \begin{cases} \frac{155n}{6} + \frac{81656n}{10125} & \textit{if } n \equiv 0\ (mod\ 3), \\ \frac{16(4n-7)}{3} + \frac{9(4n-10)}{8} & \textit{if } n \equiv 1\ (mod\ 3), \\ +\frac{44199(n-1)}{5184} + \frac{163968}{3375} + \frac{13488}{343} & \\ \frac{18980n-20221}{648} + \frac{576(n+1)}{125} & \textit{if } n \equiv -1\ (mod\ 3). \\ +\frac{6750}{216} + \frac{125}{64} + \frac{8000}{343} & \end{cases}$$

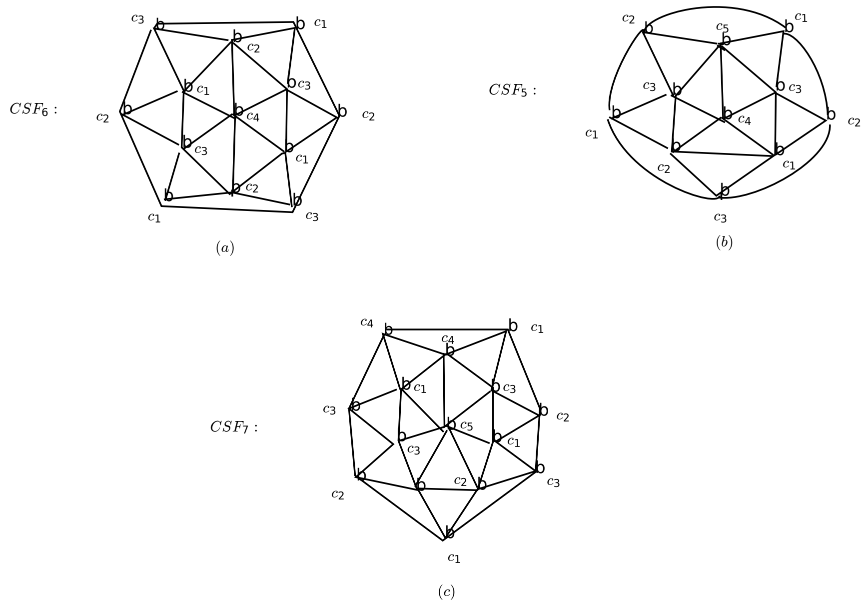

FIGURE 7. φ^- coloring of closed sunflower graphs (a) CSF_6, (b) CSF_5 and (c) CSF_7.

Proof. A closed sunflower graph CSF_n has $\chi(4)$ when $n \equiv 0\ (mod\ 3)$ and has $\chi(5)$ when $n \equiv 1\ (mod\ 3)$ and $n \equiv -1\ (mod\ 3)$. Let the vertices $x_1, x_2, \dots x_n$ on the rim of the outer cycle, the vertices $x'_1, x'_2, \dots x'_n$ on the rim of the inner cycle and y be the central vertex. Assume that $n \equiv 0\ (mod\ 3)$. Then, we will get three color classes with same cardinality $\frac{3n}{2}$ and they will gets the colors c_1, c_2, c_3 and central vertex y is colored with c_4. Refer Fig.7 (a).
Assume that $n \equiv 1\ (mod\ 3)$. Then, we will get three color classes with the maximum independent sets having the same cardinality $\frac{2(n-1)}{3}$ they will gets the colors c_1, c_2, c_3. Also, the vertices x'_1, x_n are receives the color c_4 and the central vertex y is coloured with c_5. Refer Fig.7 (c).

Assume that $n \equiv -1$ $(mod\ 3)$. Then, we will get three color classes with the maximum independent sets having the same cardinality $\frac{2(n-1)}{3}$ they will gets the colors c_1, c_2, c_3. Remaining two vertices y, x'_n gets the colors c_4 and c_5 respectively. Refer Fig.7 (b). In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of CSF_n, we consider the following cases.

Case 1. Assume that $n \equiv 0$ $(mod\ 3)$. Then, $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2n}{3}$, $\theta(c_4) = 1$. Also, $\eta_{12} = \eta_{23} = \eta_{13} = \frac{4n}{3}$, $\eta_{14} = \eta_{24} = \eta_{34} = \frac{n}{3}$.

$$
\begin{aligned}
CVar^{\varphi^-}(CSF_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{28n+48}{n} - \frac{(12n+12)^2}{9n^2} \\
&= \frac{16n^2+60n+64}{3n}. \\
HM_1^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= \frac{200n}{3} + \frac{110n}{3} \\
&= \frac{310n}{3}. \\
AZI^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= \frac{620n}{3} + \frac{280n}{81} + \frac{1728n}{375} \\
&= \frac{155n}{6} + \frac{81656n}{10125}.
\end{aligned}
$$

Case 2. Assume that $n \equiv 1$ $(mod\ 3)$. Then, $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{2(n-1)}{3}$, $\theta(c_4) = 2$, $\theta(c_5) = 1$ Also, $\eta_{12} = \eta_{23} = \frac{4n-7}{3}$, $\eta_{13} = \frac{4n-10}{3}$, $\eta_{15} = \eta_{25} = \eta_{35} = \frac{n-1}{3}$, $\eta_{14} = \eta_{34} = 3$, $\eta_{24} = 2$ and $\eta_{45} = 1$.

$$
\begin{aligned}
CVar^{\varphi^-}(CSF_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{28n-28+171}{3n} - \frac{(12n-12+39)^2}{9n^2} \\
&= \frac{28n+143}{3n} - \frac{(4n-9)^2}{n^2}.
\end{aligned}
$$

$$\begin{aligned}
HM_1^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2\\
&= \frac{34(4n-7)}{3}+\frac{16(4n-10)}{3}+\frac{149(n-1)}{3}+375\\
&= \frac{349n+578}{3}.\\
AZI^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3\\
&= \frac{16(4n-7)}{3}+\frac{9(4n-10)}{8}+\left(\frac{637}{64}+\frac{3375}{216}\right)\frac{n-1}{3}\\
&\quad +\frac{163959}{3375}+\frac{13488}{343}\\
&= \frac{16(4n-7)}{3}+\frac{9(4n-10)}{8}+\frac{44199(n-1)}{5184}\\
&\quad +\frac{163968}{3375}+\frac{13488}{343}.
\end{aligned}$$

Case 3. Assume that $n \equiv -1 \ (mod\ 3)$. Then, $\theta(c_1)=\theta(c_2)=\theta(c_3)=\frac{2(n-1)}{3}$, $\theta(c_4)=\theta(c_5)=1$. Also, $\eta_{12}=\frac{4n-2}{3}$, $\eta_{13}=\eta_{23}=\frac{4n-5}{3}$, $\eta_{14}=\eta_{24}=\frac{n-2}{3}$, $\eta_{34}=\frac{n+1}{3}$, $\eta_{35}=2$, $\eta_{15}=\eta_{45}=1$.

$$\begin{aligned}
CVar^{\varphi^-}(CSF_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2-\frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2\\
&= \frac{28n-28+123}{3n}-\frac{(12n+15)^2}{9n^2}\\
&= \frac{28n+95}{3n}-\frac{(4n+5)^2}{n^2}.\\
HM_1^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2\\
&= 12n+239+\frac{274n-278}{3}\\
&= \frac{310n+439}{3}.\\
AZI^{\varphi^-}(CSF_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3\\
&= \frac{8(4n-2)}{8}+\frac{91(4n-5)}{24}+\frac{280(n-2)}{81}+\frac{576(n+1)}{125}\\
&\quad +\frac{6750}{216}+\frac{125}{64}+\frac{8000}{343}\\
&= \frac{18980n-20221}{648}+\frac{576(n+1)}{125}+\frac{6750}{216}+\frac{125}{64}+\frac{8000}{343}.
\end{aligned}$$

$\square$

Using minimum parameter coloring, we can also get φ^+ coloring of closed sunflower graph CSF_n.

Theorem 23. *For a closed sunflower graph CSF_n, we have*

$$\text{(i) } CVar^{\varphi^+}(CSF_n) = \begin{cases} \frac{58n+3}{3n} - \frac{(6n+1)^2}{n^2} & \text{if } n \equiv 0 \ (mod\ 3), \\ \frac{100n-73}{3n} - \frac{(8n-3)^2}{n^2} & \text{if } n \equiv 1 \ (mod\ 3), \\ \frac{100n-85}{3n} - \frac{(8n-5)^2}{n^2} & \text{if } n \equiv -1 \ (mod\ 3). \end{cases}$$

$$\text{(ii) } HM_1^{\varphi^+}(CSF_n) = \begin{cases} \frac{490n}{3} & \text{if } n \equiv 0 \ (mod\ 3), \\ \frac{853n-1627}{3} + 303 & \text{if } n \equiv 1 \ (mod\ 3), \\ \frac{886n-566}{3} & \text{if } n \equiv -1 \ (mod\ 3). \end{cases}$$

$$\text{(iii) } AZI^{\varphi^+}(CSF_n) = \begin{cases} \frac{14912n}{375} + \frac{2969n}{648} & \text{if } n \equiv 0 \ (mod\ 3), \\ \left(\frac{8000}{343} + \frac{1728}{125}\right)\frac{4n-7}{3} & \text{if } n \equiv 1 \ (mod\ 3), \\ +\frac{3375(4n-10)}{648} + \frac{13303(n-1)}{5184} + 72 & \\ \frac{8000(4n-2)}{1029} + \left(\frac{1728}{125} + \frac{3375}{216}\right)\frac{4n-5}{3} & \text{if } n \equiv -1 \ (mod\ 3). \\ +\frac{24(n-1)}{3} + \frac{1849}{108} + \frac{125}{64} & \end{cases}$$

Proof. A closed sunflower graph CSF_n has $\chi(4)$ when $n \equiv 0$ $(mod\ 3)$ and has $\chi(5)$ when $n \equiv 1$ $(mod\ 3)$ and $n \equiv -1$ $(mod\ 3)$. Let the vertices $x_1, x_2, \ldots x_n$ on the rim of the outer cycle, the vertices $x'_1, x'_2, \ldots x'_n$ on the rim of the inner cycle and y be the central vertex. Assume that $n \equiv 0$ $(mod\ 3)$. Then, we will get three color classes with same cardinality $\frac{3n}{2}$ and they will gets the colors c_4, c_3, c_2 and central vertex y is colored with c_1. Then, $\theta(c_4) = \theta(c_3) = \theta(c_2) = \frac{2n}{3}$, $\theta(c_1) = 1$. Also, $\eta_{23} = \eta_{24} = \eta_{34} = \frac{4n}{3}$, $\eta_{12} = \eta_{13} = \eta_{14} = \frac{n}{3}$.
Assume that $n \equiv 1$ $(mod\ 3)$. Then, we will get three color classes with the maximum independent sets having the same cardinality $\frac{2(n-1)}{3}$ they will gets the colors c_5, c_4, c_3. Also the vertices x'_1, x_n are receives the color c_2 and the central vertex y is coloured with c_1. Then, $\theta(c_5) = \theta(c_4) = \theta(c_3) = \frac{2(n-1)}{3}$, $\theta(c_2) = 2$, $\theta(c_1) = 1$ Also, $\eta_{45} = \eta_{34} = \frac{4n-7}{3}$, $\eta_{35} = \frac{4n-10}{3}$, $\eta_{13} = \eta_{14} = \eta_{15} = \frac{n-1}{3}$, $\eta_{23} = \eta_{25} = 3$, $\eta_{24} = 2$ and $\eta_{12} = 1$.
Assume that $n \equiv -1$ $(mod\ 3)$. Then, we will get three color classes with the maximum independent sets having the same cardinality $\frac{2(n-1)}{3}$ they will gets the colors c_5, c_4, c_3. Remaining two vertices y, x'_n gets the colors c_2 and c_1 respectively. $\theta(c_5) = \theta(c_4) = \theta(c_3) = \frac{2(n-1)}{3}$,

$\theta(c_2) = \theta(c_1) = 1$. Also, $\eta_{45} = \frac{4n-2}{3}$, $\eta_{34} = \eta_{35} = \frac{4n-5}{3}$, $\eta_{25} = \eta_{24} = \frac{n-2}{3}$, $\eta_{23} = \frac{n+1}{3}$, $\eta_{13} = 2$, $\eta_{12} = \eta_{14} = \eta_{15} = 1$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 22. □

Definition 24. [8] *A blossom graph BL_n is the graph obtained by joining all vertices of the outer cycle of a closed sunflower graph CSF_n to its central vertex. It is clear that Bl_n has $2n+1$ vertices, $5n$ edges. It is clear that Bl_n has $2n+1$ vertices, $5n$ edges and chromatic number 5.*

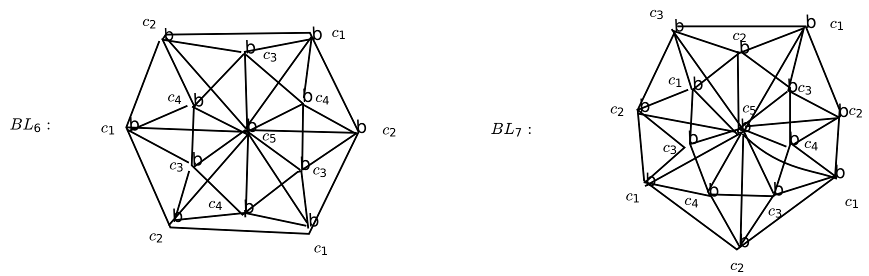

FIGURE 8. φ^- coloring of blossom graph BL_6 and BL_7.

Theorem 25. *For a blossom graph BL_n, we have*

(i) $$CVar^{\varphi^-}(BL_n) = \begin{cases} \frac{5(5n^3+10n^2+8n+5)}{n} & \text{if } n \text{ is even,} \\ \frac{15n+8}{n} - \frac{(5n-27)^2}{n^2} & \text{if } n \text{ is odd.} \end{cases}$$

(ii) $$HM_1^{\varphi^-}(BL_n) = \begin{cases} 224n & \text{if } n \text{ is even,} \\ 2(112n-95) & \text{if } n \text{ is odd.} \end{cases}$$

(iii) $$AZI^{\varphi^-}(BL_n) = \begin{cases} \frac{2728n}{125} + \left(\frac{4616}{216} + \frac{1661}{64} + \frac{8000}{343}\right)\frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{219n+367}{16} + \frac{1728(n-3)}{125} + \left(\frac{280}{27} + \frac{8000}{343}\right)\frac{n-3}{2} + \left(\frac{637}{64} + \frac{3375}{216}\right)\frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. A blossom graph BL_n has chromatic number 5. Let the vertices $x_1, x_2, \ldots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \ldots x'_n$ on the inner cycle, y be the central vertex. When n is even. Then, the vertex sets $V_1 = \{x_2, x_4, \ldots x_n\}$, $V_2 = \{x_1, x_3, \ldots x_{n-1}\}$, $V_3 = \{x'_2, x'_4, \ldots x'_n\}$ and $V_4 = \{x'_1, x'_3, \ldots x'_{n-1}\}$ receives the colors c_1, c_2, c_3 and c_4 respectively. Also central vertex gets the color c_5. When n is odd, the vertex sets $V_1 = \{x_1, x_3, \ldots x_{n-2}, x'_{n-1}\}$, $V_2 = \{x_2, x_4, \ldots x_{n-1}, x'_n\}$,

$V_3 = \{x'_1, x'_3, \ldots x'_{n-2}, x_1\}$ and $V_4 = \{x'_2, x'_4, \ldots x'_{n-3}\}$ receives the colors c_1, c_2, c_3 and c_4 respectively. Also, the central vertex gets the color c_5. In order to calculate $CVar^{\varphi^-}$, $HM_1^{\varphi^-}$ and AZI^{φ^-} of BL_n, we consider the following cases.

Case 1. Assume that n is even. Then, $\theta(c_1) = \theta(c_2) = \theta(c_3) = \theta(c_4) = \frac{n}{2}$, $\theta(c_5) = 1$ Also, $\eta_{12} = \eta_{34} = n$, $\eta_{13} = \eta_{14} = \eta_{15} = \eta_{23} = \eta_{24} = \eta_{25} = \eta_{35} = \eta_{45} = \frac{n}{2}$.

$$\begin{aligned}
CVar^{\varphi^-}(BL_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{1}{n}\left(\frac{n}{2} + \frac{4n}{2} + \frac{9n}{2} + \frac{16n}{2} + 25\right) \\
&\quad -\frac{1}{n^2}\left(\frac{n}{2} + n + \frac{3n}{2} + 2n + 5\right)^2 \\
&= \frac{5(5n^3 + 10n^2 + 8n + 5)}{n}. \\
HM_1^{\varphi^-}(BL_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i) + \zeta(v_j))^2 \\
&= 58n + \frac{332n}{2} \\
&= 224n. \\
AZI^{\varphi^-}(BL_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i) + \zeta(v_j) - 2}\right)^3 \\
&= \left(8 + \frac{1728}{125}\right)n + \left(\frac{1241}{216} + \frac{1661}{64} + \frac{3375}{216} + \frac{8000}{343}\right)\frac{n}{2} \\
&= \frac{2728n}{125} + \left(\frac{4616}{216} + \frac{1661}{64} + \frac{8000}{343}\right)\frac{n}{2}.
\end{aligned}$$

Case 2. Assume that n is odd. Then, $\theta(c_1) = \theta(c_2) = \theta(c_3) = \frac{n+1}{2}.$, $\theta(c_4) = \frac{n-3}{2}$, $\theta(c_5) = 1$. Also, $\eta_{12} = n + 1$, $\eta_{23} = \frac{n+5}{2}$, $\eta_{13} = \frac{n+3}{2}$, $\eta_{34} = n - 3$, $\eta_{15} = \eta_{25} = \eta_{35} = \frac{n+1}{2}$, $\eta_{14} = \eta_{24} = \eta_{45} = \frac{n-3}{2}$.

$$\begin{aligned}
CVar^{\varphi^-}(BL_n) &= \frac{1}{n}\sum_{i=1}^{l}\theta(c_i)\cdot i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{l}\theta(c_i)\cdot i\right)^2 \\
&= \frac{30n - 34 + 50}{2n} - \frac{(10n - 6 + 60)^2}{4n^2} \\
&= \frac{15n + 8}{n} - \frac{(5n - 27)^2}{n^2}.
\end{aligned}$$

$$\begin{aligned}
HM_1^{\varphi^-}(BL_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}(\zeta(v_i)+\zeta(v_j))^2 \\
&= 137n + \frac{174n+274}{2} - 327 \\
&= 2(112n-95). \\
AZI^{\varphi^-}(BL_n) &= \sum_{i=1}^{n-1}\sum_{j=2}^{n}\left(\frac{\zeta(v_i)\cdot\zeta(v_j)}{\zeta(v_i)+\zeta(v_j)-2}\right)^3 \\
&= 8(n+1) + \frac{27(n-3)}{16} + 4(n+5) + \frac{1728(n-3)}{125} \\
&\quad + \left(\frac{280}{27}+\frac{8000}{343}\right)\frac{n-3}{2} + \left(\frac{637}{64}+\frac{3375}{216}\right)\frac{n+1}{2} \\
&= \frac{219n+367}{16} + \frac{1728(n-3)}{125} + \left(\frac{280}{27}+\frac{8000}{343}\right)\frac{n-3}{2} \\
&\quad + \left(\frac{637}{64}+\frac{3375}{216}\right)\frac{n+1}{2}.
\end{aligned}$$

□

Using minimum parameter coloring, we can also get φ^+ coloring of blossom graph BL_n.

Theorem 26. *For a blossom graph BL_n, we have*

(i) $CVar^{\varphi^+}(BL_n) = \begin{cases} \frac{54n+1}{n} - \frac{(7n+1)^2}{n^2} & \text{if } n \text{ is even,} \\ \frac{27n+20}{n} - \frac{(7n+4)^2}{n^2} & \text{if } n \text{ is odd.} \end{cases}$

(ii) $HM_1^{\varphi^+}(BL_n) = \begin{cases} 248n & \text{if } n \text{ is even,} \\ 248n+122 & \text{if } n \text{ is odd.} \end{cases}$

(iii) $AZI^{\varphi^+}(BL_n) = \begin{cases} \frac{10744n}{343} + \left(\frac{6344}{216}+\frac{1149}{64}+\frac{1728}{125}\right)\frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{8000(n+1)}{343} + \frac{864(n+3)-7500}{125} & \text{if } n \text{ is odd.} \\ +\frac{3375(n+3)}{432} + \left(\frac{125}{64}+\frac{1241}{216}\right)\frac{n+1}{2} + 20n & \end{cases}$

Proof. A blossom graph BL_n has chromatic number 5. Let the vertices $x_1, x_2, \dots x_n$ on the outer cycle, the vertices $x'_1, x'_2, \dots x'_n$ on the inner cycle, y be the central vertex. When n is even, the vertex sets $V_1 = \{x_2, x_4, \dots x_n\}$, $V_2 = \{x_1, x_3, \dots x_{n-1}\}$, $V_3 = \{x'_2, x'_4, \dots x'_n\}$ and $V_4 = \{x'_1, x'_3, \dots x'_{n-1}\}$ receives the colors c_5, c_4, c_3 and c_2 respectively. Also central vertex gets the color c_1. Therefore, $\theta(c_1) = 1$,

$\theta(c_2) = \theta(c_3) = \theta(c_4) = \theta(c_5) = \frac{n}{2}$ and $\eta_{45} = \eta_{23} = n$, $\eta_{35} = \eta_{25} = \eta_{15} = \eta_{34} = \eta_{24} = \eta_{14} = \eta_{13} = \eta_{12} = \frac{n}{2}$.
When n is odd, the vertex sets $V_1 = \{x_1, x_3, \dots x_{n-2}, x'_{n-1}\}$, $V_2 = \{x_2, x_4, \dots x_{n-1}, x'_n\}$, $V_3 = \{x'_1, x'_3, \dots x'_{n-2}, x_1\}$ and $V_4 = \{x'_2, x'_4, \dots x'_{n-3}\}$ receives the colors c_5, c_4, c_3 and c_1 respectively. Also, the central vertex gets the color c_1. Therefore, $\theta(c_1) = 1$, $\theta(c_5) = \theta(c_4) = \theta(c_3) = \frac{n+1}{2}, \theta(c_2) = \frac{n-3}{2}$ and $\eta_{45} = n + 1$, $\eta_{34} = \frac{n+5}{2}$, $\eta_{35} = \frac{n+3}{2}$, $\eta_{23} = n - 3$, $\eta_{15} = \eta_{14} = \eta_{13} = \frac{n+1}{2}$ and $\eta_{25} = \eta_{24} = \eta_{12} = \frac{n-3}{2}$. Remaining part of the proof follows exactly as mentioned in the proof of Theorem 25. $\square$

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