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APPLICATIONS OF $f\pi g$-CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. In [8], fuzzy π-closed set is introduced. Using this concept as a basic tool, in [9] the notion of fuzzy π generalized closed set ($f\pi g$-closed set, for short) is introduced and studied. Afterwards, a new type of generalized version of fuzzy closure operator, viz., $f\pi g$-closure operator is introduced which is an idempotent operator. Next we introduce a new type of generalized version of fuzzy open and closed-like functions, viz., $f\pi g$-open and $f\pi g$-closed functions and characterize these two functions by using $f\pi g$-closure operator. Next we introduce $f\pi g$-continuous function and $f\pi g$-irresolute function. Next we introduce two new types of separation axioms, viz., $f\pi g$-regularity, $f\pi g$-normality and a new type of compactness, viz., $f\pi g$-compactness. It is shown that under $f\pi g$-irresolute function , $f\pi g$-regularity, $f\pi g$-normality and $f\pi g$-compactness remain invariant. Lastly, a new of fuzzy T_2-space, viz., $f\pi g$-T_2 space is introduced and it is shown that inverse image of fuzzy T_2-space [20] (resp., $f\pi g$-T_2 space) under $f\pi g$-continuous (resp., $f\pi g$-irresolute) function is $f\pi g$-T_2 space.

Keywords and phrases: Fuzzy π-open set, $f\pi g$-closed set, fuzzy regular open set, fuzzy semiopen set, $f\pi g$-continuous function, $f\pi g$-closed function, $f\pi g$-closure operator, $f\pi g$-irresolute function, fuzzy R-open function, fT_π-space.
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1. **Introduction**

In [2, 3], generalized version of fuzzy closed set is introduced. Afterwards, several types of generalized version of fuzzy closed sets are introduced and studied. In this context, we have to mention [3, 5, 6, 7, 8, 9, 10, 11]. In this paper we study fuzzy πg-closed set and several properties of this set are established and the mutual relationships of this newly defined set with the sets defined in [3, 5, 6, 7, 8, 9, 10, 11] are established. With the help of $f\pi$-closure operator a new type of neighbourhood structure in a fuzzy topological space is introduced and studied. Here we introduce $f\pi g$-continuous function, the collection of which is strictly larger than that of fuzzy continuous function [14], fg-continuous function [3], fgs^*-continuous function [5], fs^*g-continuous function [6], but weaker than $frwg$-continuous function [9]. Also it is shown that $f\pi g$-continuity is independent concept of fgs-continuous function [3], fsg-continuous function [3], $fg\alpha$-continuous function [3], $f\alpha g$-continuous function [3], $fg\beta$-continuous function [8], $f\beta g$-continuous function [8], fgp-continuous function [3], fpg-continuous function [3], $fg\gamma$-continuous function [10], $fg\gamma^*$-continuous function [11], $fswg$-continuous function [9], fmg-continuous function [9], fwg-continuous function [9].

2. Preliminaries

Throughout this paper (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [14]. In [26], L.A. Zadeh introduced fuzzy set as follows: A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The support [26] of a fuzzy set A, denoted by $suppA$ and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [26] of a fuzzy set A in X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [26] while AqB means A is quasi-coincident (q-coincident, for short) [24] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For a fuzzy point x_t and a fuzzy set A, $x_t \in A$ means $A(x) \geq t$, i.e., $x_t \leq A$. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [14] and fuzzy interior [14] respectively. A fuzzy set A is called a fuzzy neighbourhood (fuzzy nbd, for short) [24] of a fuzzy point x_α if there exists a fuzzy open set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd [24] of x_α. A fuzzy set A is called a fuzzy quasi neighbourhood (fuzzy q-nbd, for short) [24] of a fuzzy point x_α in an fts X if there is a fuzzy open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open q-nbd [24] of x_α. A fuzzy set A in X is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [23], fuzzy α-open [13], fuzzy β-open [17], fuzzy γ-open [4]) if $A = int(clA)$ (resp., $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$, $A \leq cl(intA) \bigvee int(clA)$). A fuzzy set A is called fuzzy π-open [8] if A is the union of finite number of fuzzy regular open sets. The complement of a fuzzy regular open (resp., fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open) set is called fuzzy regular closed [1] (resp., fuzzy semiclosed [1], fuzzy preclosed [23], fuzzy α-closed [13], fuzzy β-closed [17], fuzzy γ-closed [4]). The intersection of all fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed) sets containing a fuzzy set A is called fuzzy semiclosure [1] (resp., fuzzy preclosure [23], fuzzy α-closure [13], fuzzy β-closure [17], fuzzy γ-closure [4]) of A, to be denoted by $sclA$ (resp., $pclA$, αclA, βclA, γclA). The collection of all fuzzy open (resp., fuzzy regular open, fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open, fuzzy γ-open, fuzzy π-open) sets in an fts (X,τ) is denoted by τ (resp., $FRO(X,\tau)$, $FSO(X,\tau)$, $FPO(X,\tau)$, $F\alpha O(X,\tau)$, $F\beta O(X,\tau)$, $F\gamma O(X,\tau)$, $F\pi O(X,\tau)$). The collection of all fuzzy closed (resp., fuzzy regular closed, fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed, fuzzy γ-closed, fuzzy π-closed) sets in an fts X is denoted by τ^c (resp., $FRC(X,\tau)$, $FSC(X,\tau)$, $FPC(X,\tau)$, $F\alpha C(X,\tau)$, $F\beta C(X,\tau)$, $F\gamma C(X,\tau)$, $F\pi C(X,\tau)$).

3. $f\pi g$-Closed Set: Some Properties

In [9], $f\pi g$-closed set is introduced. In this section some important properties of this set is studied first. Then a new type of fuzzy neighbourhood system is introduced and studied using $f\pi g$-closed set as a basic tool. Lastly the mutual relationship of this set with the sets defined in [2, 3, 5, 6, 7, 9, 10, 11] are established.

First we recall the following definition from [9] for ready references.

Definition 3.1 [9]. Let (X,τ) be an fts and $A \in I^X$. Then A is called fuzzy π-generalized closed ($f\pi g$-closed, for short) set in X if $clA \leq U$ whenever $A \leq U \in F\pi O(X,\tau)$.

The complement of the above mentioned fuzzy set is called fuzzy

π-generalized open ($f\pi g$-open, for short) set.

Remark 3.2. It is clear from definition that union of two $f\pi g$-closed sets is also so. But the intersection of two $f\pi g$-closed sets need not be so, in general, as the following example shows.

Example 3.3. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then (X, τ) is an fts. Consider two fuzzy sets C and D defined by $C(a) = 0.4, C(b) = 0.5, D(a) = 0.3, D(b) = 0.6$. Here $1_X \in F\pi O(X, \tau)$ only containing C and D and so clearly C and D are $f\pi g$-closed sets in (X, τ). Let $E = C \bigwedge D$. Then $E(a) = 0.3, E(b) = 0.5$. Now $E \leq B \in F\pi O(X, \tau)$. But $clE = 1_X \setminus B \not\leq B$ which implies that E is not $f\pi g$-closed set in $(X.\tau)$.

So we can conclude that the set of all $f\pi g$-open sets cannot form a fuzzy topology.

Theorem 3.4. Let (X, τ) be an fts and $A, B \in I^X$. If $A \leq B \leq clA$ and A is $f\pi g$-closed set in X, then B is also $f\pi g$-closed set in X.

Proof. Let $U \in F\pi O(X, \tau)$ be such that $B \leq U$. Then by hypothesis, $A \leq B \leq U$. As A is $f\pi g$-closed set in X, $clA \leq U$ and so $A \leq B \leq clA \leq U$ implies that $clA \leq clB \leq cl(clA) = clA \leq U$ Then $clB \leq U$. Consequently, B is $f\pi g$-closed set in X.

Theorem 3.5. Let (X, τ) be an fts and $A, B \in I^X$. If $intA \leq B \leq A$ and A is $f\pi g$-open set in X, then B is also $f\pi g$-open set in X.

Proof. $intA \leq B \leq A$ So $1_X \setminus A \leq 1_X \setminus B \leq 1_X \setminus intA = cl(1_X \setminus A)$ where $1_X \setminus A$ is $f\pi g$-closed set in X. By Theorem 3.4, $1_X \setminus B$ is $f\pi g$-closed set in X. Hence B is $f\pi g$-open set in X.

Theorem 3.6. Let (X, τ) be an fts and $A \in I^X$. Then A is $f\pi g$-open set in X if and only if $K \leq intA$ whenever $K \leq A$ and $K \in F\pi C(X, \tau)$.

Proof. Let $A(\in I^X)$ be $f\pi g$-open set in X and $K \leq A$ where $K \in F\pi C(X, \tau)$. Then $1_X \setminus A \leq 1_X \setminus K$ where $1_X \setminus A$ is $f\pi g$-closed set in X and $1_X \setminus K \in F\pi O(X, \tau)$. So $cl(1_X \setminus A) \leq 1_X \setminus K$ implies that $1_X \setminus intA \leq 1_X \setminus K$ and so $K \leq intA$.

Conversely, let $K \leq intA$ whenever $K \leq A$, $K \in F\pi C(X, \tau)$. Then $1_X \setminus A \leq 1_X \setminus K \in F\pi O(X, \tau)$. Now $1_X \setminus intA \leq 1_X \setminus K$. Then $cl(1_X \setminus A) \leq 1_X \setminus K$ and so $1_X \setminus A$ is $f\pi g$-closed set in X. Hence A is $f\pi g$-open set in X.

Theorem 3.7. Let (X, τ) be an fts and $A(\in I^X)$. If A is fuzzy regular open set as well as $f\pi g$-closed set in X, then A is fuzzy closed set in X.

Proof. Now $A \leq A \in FRO(X, \tau) \subseteq F\pi O(X, \tau)$. By hypothesis,

$clA \leq A$ (as A is $f\pi g$-closed set in X) and so $A = clA$ Hence A is fuzzy closed set in X.

Theorem 3.8. Let (X, τ) be an fts and $A(\in I^X) \in F\pi O(X, \tau)$ as well as A is $f\pi g$-closed set in X, then A is fuzzy closed set in X.

Proof. Follows from Theorem 3.7.

Theorem 3.9. Let (X, τ) be an fts and $A(\in I^X)$ be $f\pi g$-closed set in X and $B \in F\pi C(X, \tau)$ with $A \not{q} B$. Then $clA \not{q} B$.

Proof. Now $A \not{q} B$ Then $A \leq 1_X \setminus B \in F\pi O(X, \tau)$. By assumption, $clA \leq 1_X \setminus B$ and so $clA \not{q} B$.

Remark 3.10. The converse of Theorem 3.9 may not be true, in general, as the following example shows.

Example 3.11. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ) is an fts. Here $F\pi o(X, \tau) = \tau$. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.2$. Then $C < B \in F\pi O(X, \tau)$. But $clC = 1_X \setminus A \not\leq B$ which implies that C is not $f\pi g$-closed set in (X, τ). Again $C \not{q}(1_X \setminus A) \in F\pi C(X, \tau)$ and $clC = (1_X \setminus A) \not{q}(1_X \setminus A)$.

Now we introduce a new type of generalized version of neighbourhood system in an fts.

Definition 3.12. Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called a fuzzy π-generalized neighbourhood ($f\pi g$-nbd, for short) of x_α, if there exists an $f\pi g$-open set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is $f\pi g$-open set in X, then A is called an $f\pi g$-open nbd of x_α.

Definition 3.13. Let (X, τ) be an fts and x_α, a fuzzy point in X. A fuzzy set A is called a fuzzy π-generalized quasi neighbourhood($f\pi g$-q-nbd, for short) of x_α if there is an $f\pi g$-open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is $f\pi g$-open set in X, then A is called an $f\pi g$-open q-nbd of x_α.

Note 3.14. It is clear from definitions that every $f\pi g$-open set is an $f\pi g$-open nbd of each of its points. But every $f\pi g$-nbd of x_α may not be an $f\pi g$-open set containing x_α as the following example shows.

Example 3.15. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $F\pi O(X, \tau) = \tau$. Now consider the fuzzy point $a_{0.4}$ and the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then $a_{0.4} \leq 1_X \setminus B$ and $B < A \in F\pi O(X, \tau)$, but $clB = 1_X \setminus A \not\leq A$ which implies that B is not $f\pi g$-closed set in (X, τ) and so $1_X \setminus B$ is not $f\pi g$-open set in (X, τ). Now consider the fuzzy set C defined by $C(a) = C(b) = 0.5$. Then $1_X \in F\pi O(X, \tau)$ only containing C and so C is $f\pi g$-closed set in X and so $1_X \setminus C = C$

is $f\pi g$-open set in X. Now $a_{0.4} \leq C < 1_X \setminus B$ which shows that $1_X \setminus B$ is an $f\pi g$-neighbourhood of $a_{0.4}$ though $1_X \setminus B$ is not $f\pi g$-open set containing $a_{0.4}$.

Note 3.16. Every fuzzy open nbd (resp., fuzzy open q-nbd) of a fuzzy point x_α is an $f\pi g$-open nbd (resp., $f\pi g$-open q-nbd) of x_α, but converses are not true, in general, as the following example shows.

Example 3.17. Consider Example 3.15 and the fuzzy set D defined by $D(a) = 0.4, D(b) = 0.7$. As $1_X \in F\pi O(X, \tau)$ containing D only, clearly D is $f\pi g$-closed set in (X, τ). Now consider the fuzzy point $a_{0.56}$. So $a_{0.56} \leq 1_X \setminus D$ implies that $1_X \setminus D$ is an $f\pi g$-open nbd of $a_{0.56}$. But $1_X \setminus D$ is not a fuzzy open nbd of $a_{0.56}$. Next consider the fuzzy point $a_{0.5}$. Then as $a_{0.5}q(1_X \setminus D)$, $1_X \setminus D$ is an $f\pi g$-open q-nbd of $a_{0.5}$. But $1_X \setminus D$ is not a fuzzy open q-nbd of $a_{0.5}$.

Theorem 3.18. Let (X, τ) be an fts and x_t, a fuzzy point in X. If $F(\in I^X)$ be an $f\pi g$-closed set in X with $x_t \in 1_X \setminus F$. Then there exists an $f\pi g$-nbd G of x_t in X such that $G \not{q} F$.

Proof. Let $x_t \in 1_X \setminus F$ where $1_X \setminus F$ be an $f\pi g$-open set in X. Then $1_X \setminus F$ is an $f\pi g$-open nbd of x_t. So by definition, there exists an $f\pi g$-open set G in X such that $x_t \in G \leq 1_X \setminus F$. Hence G is an $f\pi g$-nbd of x_t with $G \not{q} F$.

Definition 3.19. The set of all $f\pi g$-nbds of a fuzzy point x_t $(0 < t \leq 1)$ in an fts (X, τ) is called fuzzy π-generalized neighbourhood ($f\pi g$-nbd, for short) system at x_t, denoted by $f\pi g$-$N(x_t)$.

Theorem 3.20. For a fuzzy point x_t in an fts (X, τ), the following statements hold :
(i) $f\pi g$-$N(x_t) \neq \phi$,
(ii) $G \in f\pi g$-$N(x_t)$ implies $x_t \in G$,
(iii) $G \in f\pi g$-$N(x_t)$ and $F \geq G$ implies $F \in f\pi g$-$N(x_t)$,
(iv) $F, G \in f\pi g$-$N(x_t)$ implies $F \bigwedge G \in f\pi g$-$N(x_t)$,
(v) $G \in f\pi g$-$N(x_t)$. Then there exists $F \in f\pi g$-$N(x_t)$ such that $F \leq G$ and $F \in f\pi g$-$N(y_{t'})$ for every $y_{t'} \in F$.

Proof. (i) Since 1_X being an $f\pi g$-open set is an $f\pi g$-nbd of x_t $(0 < t \leq 1)$, $f\pi g$-$N(x_t) \neq \phi$.
(ii) and (iii) are obvious.
(iv) Since intersection of two $f\pi g$-open sets is $f\pi g$-open, (iv) is obvious.
(v) Follows from Note 3.16 and Definition 3.19.

Theorem 3.21. Let x_t be a fuzzy point in an fts (X, τ). Let $f\pi g$-$N(x_t)$ be a non-empty collection of fuzzy sets in X satisfying the following conditions :

(1) $G \in f\pi g\text{-}N(x_t)$ implies $x_t \in G$,
(2) $F, G \in f\pi g\text{-}N(x_t)$ implies $F \bigwedge G \in f\pi g\text{-}N(x_t)$.
Let τ consist of 0_X and all those non-empty fuzzy sets G of X having the property that $x_t \in G$. Then there exists an $F \in f\pi g\text{-}N(x_t)$ such that $x_t \in F \leq G$. Then τ is a fuzzy topology on X.

Proof. (i) By hypothesis, $0_X \in \tau$.
(ii) It is clear from the given property of τ that $1_X \in \tau$ as $1_X \in f\pi g\text{-}N(x_t)$ for any fuzzy point x_t $(0 < t \leq 1)$ in an fts X (by (1)).
(iii) Let $G_1, G_2 \in \tau$. If $G_1 \bigwedge G_2 = 0_X$, then by construction of τ, $G_1 \bigwedge G_2 \in \tau$. Suppose $G_1 \bigwedge G_2 \neq 0_X$. Let $x_t \in G_1 \bigwedge G_2$ where $0 < t \leq 1$. Then $G_1(x) \geq t, G_2(x) \geq t$. Since $G_1, G_2 \in \tau$, by definition of τ, there exist $F_1, F_2 \in f\pi g\text{-}N(x_t)$ such that $x_t \in F_1 \leq G_1$, $x_t \in F_2 \leq G_2$. Then $x_t \in F_1 \bigwedge F_2 \leq G_1 \bigwedge G_2$. By (2), $F_1 \bigwedge F_2 \in f\pi g\text{-}N(x_t)$ and so $G_1 \bigwedge G_2 \in \tau$ by construction of τ.
(iv) Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ where $G_\alpha \in \tau$, for each $\alpha \in \Lambda$. Let $x_t \in \bigvee\limits_{\alpha\in\Lambda} G_\alpha$. Then there exists $\beta \in \Lambda$ such that $x_t \in G_\beta$. By definition of τ, there exists $F_\beta \in f\pi g\text{-}N(x_t)$ such that $x_t \in F_\beta \leq G_\beta \leq \bigvee\limits_{\alpha\in\Lambda} G_\alpha$ which implies that $\bigvee\limits_{\alpha\in\Lambda} G_\alpha \in \tau$.

It follows that τ is a fuzzy topology on X.

Next we recall the following definitions of different types of fuzzy generalized version of closed sets from [2, 3, 5, 6, 7, 9, 10, 11] and then establish the mutual relationships of these sets with the set mentioned in this section.

Definition 3.22. Let (X, τ) be an fts and $A \in I^X$. Then A is called
(i) fg-closed set [2, 3] if $clA \leq U$ whenever $A \leq U \in \tau$,
the complement of fg-closed set is called fg-open set,
(ii) fgp-closed set [3] if $pclA \leq U$ whenever $A \leq U \in \tau$,
(iii) fpg-closed set [3] if $pclA \leq U$ whenever $A \leq U \in FPO(X, \tau)$,
(iv) $fg\alpha$-closed set [3] if $\alpha clA \leq U$ whenever $A \leq U \in \tau$,
(v) $f\alpha g$-closed set [3] if $\alpha clA \leq U$ whenever $A \leq U \in F\alpha O(X, \tau)$,
(vi) $fg\beta$-closed set [7] if $\beta clA \leq U$ whenever $A \leq U \in \tau$,
(vii) $f\beta g$-closed set [7] if $\beta clA \leq U$ whenever $A \leq U \in F\beta O(X, \tau)$,
(viii) fgs-closed set [3] if $sclA \leq U$ whenever $A \leq U \in \tau$,
(ix) fsg-closed set [3] if $sclA \leq U$ whenever $A \leq U \in FSO(X, \tau)$,
(x) fgs^*-closed set [5] if $clA \leq U$ whenever $A \leq U \in FSO(X, \tau)$,
(xi) fs^*g-closed set [6] if $clA \leq U$ whenever $A \leq U$ where U is fg-open

set in X,
(xii) $fswg$-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in FSO(X, \tau)$,
(xiii) $frwg$-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in FRO(X, \tau)$,
(xiv) fmg-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U$ where U is fg-open set in X,
(xv) fwg-closed set [9] if $cl(intA) \leq U$ whenever $A \leq U \in \tau$,
(xvi) $fg\gamma$-closed set [10] if $\gamma clA \leq U$ whenever $A \leq U \in \tau$,
(xvii) $fg\gamma^*$-closed set [11] if $\gamma clA \leq U$ whenever $A \leq U \in FSO(X, \tau)$.

Remark 3.23. It is clear from Definition 3.1 and Definition 3.19 that
(i) fuzzy closed set is $f\pi g$-closed set, fg-closed set is $f\pi g$-closed set, fgs^*-closed set is $f\pi g$-closed set, fs^*g-closed set is $f\pi g$-closed set and $f\pi g$-closed set is $frwg$-closed set. But the converses are not true, in general, as the following examples show.
(ii) $f\pi g$-closed set is independent concept of fgp-closed set, fpg-closed set, $fg\alpha$-closed set, $f\alpha g$-closed set, $fg\beta$-closed set, $f\beta g$-closed set, fgs-closed set, fsg-closed set, $fswg$-closed set, fmg-closed set, fwg-closed, $fg\gamma$-closed set, $fg\gamma^*$-closed set follow from the following examples.

Example 3.24. $f\pi g$-closed set may not be fuzzy closed set, fg-closed set, $fg\beta$-closed set, $f\beta g$-closed set, $fg\alpha$-closed set, $f\alpha g$-closed set, fgs-closed set, fsg-closed set, fgs^*-closed set, fpg-closed set, $fg\gamma^*$-closed set, $fswg$-closed set
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Here $F\pi O(X, \tau) = \{0_X, 1_X\}$, $FSO(X, \tau) = F\alpha O(X, \tau) = \{0_X, 1_X, U\}$ where $U \geq A$, $FPO(X, \tau) = \{0_X, 1_X, V\}$ where $V \not\leq 1_X \setminus A$. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. As $1_X \in F\pi O(X, \tau)$ only containing B, clearly B is $f\pi g$-closed set in (X, τ). Also B is not a fuzzy closed set in (X, τ). Now $B \leq A \in \tau$, but $clB = 1_X \not\leq A$ implies that B is not fg-closed set in (X, τ). Again $B \leq A \in F\alpha O(X, \tau)$ as well as $B \leq A \in \tau$. But as $\alpha clB = 1_X \not\leq A$, B is not $f\alpha g$-closed as well as $fg\alpha$-closed set in (X, τ). Again $B \leq A \in FSO(X, \tau)$ and $B \leq A \in \tau$. But $sclB = 1_X \not\leq A$ and so B is not fsg-closed as well as fgs-closed set in (X, τ). Also $clB = 1_X \not\leq A$ and so B is not fgs^*-closed set in (X, τ). Next consider the fuzzy set C defined by $C(a) = C(b) = 0.6$. Then as $1_X \in F\pi O(X, \tau)$ only containing C, clearly C is $f\pi g$-closed set in (X, τ). Now $C \leq C \in FPO(X, \tau)$. But $pclC = 1_X \not\leq C$.

So C is not fpg-closed set in (X,τ). Again $C \le C \in FSO(X,\tau)$ and $\gamma clC = 1_X \not\le C$ So C is not $fg\gamma^*$-closed set in (X,τ). Again $C \le C \in FSO(X,\tau)$. But $cl(intC) = 1_X \not\le C$. Consequently, C is not $fswg$-closed set in (X,τ). Now taking the fuzzy set A, we see that $1_X \in F\pi O(X,\tau)$ only containing A so that A is $f\pi g$-closed set in (X,τ). Now $A \le cl(int(clA))$ and so $A \in F\beta O(X,\tau)$ and so $A \le A \in F\beta O(X,\tau)$ as well $A \le A \in \tau$. But $\beta clA = 1_X \not\le A$. Hence A is not $f\beta g$-closed as well as $fg\beta$-closed set in (X,τ).

Example 3.25. $f\pi g$-closed set may not be fgp-closed set, $fg\gamma$-closed set, fwg-closed set

Let $X = \{a,b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.3, B(b) = 0.5$. Then (X,τ) is an fts. Here $F\pi O(X,\tau) = \{0_X, 1_X, B\}$. Consider the fuzzy set C defined by $C(a) = 0.4, C(b) = 0.5$. Then as $1_X \in F\pi O(X,\tau)$ only containing C, clearly C is $f\pi g$-closed set in (X,τ). Now $C < A \in \tau$. But $pclA \not\le A$. So C is not fgp-closed set in (X,τ). Again $cl(intC) = 1_X \setminus B \not\le A$ implies that C is not fwg-closed set in (X,τ). Next consider the fuzzy set D defined by $D(a) = 0.5, D(b) = 0.55$. Then as $1_X \in F\pi O(X,\tau)$ only containing D, clearly D is $f\pi g$-closed set in (X,τ). Now $D < A \in \tau$. But $\gamma clD = 1_X \setminus B \not\le A$. Then D is not $fg\gamma$-closed set in (X,τ). Now the collection of all fg-open sets in (X,τ) is $\{0_X, 1_X, W, T\}$ where $0.5 \le W(a) < 0.7, W(b) \ge 0.6$, $T \not\ge 1_X \setminus A$. Consider the fuzzy set F defined by $F(a) = F(b) = 0.5$. As $1_X \in F\pi O(X,\tau)$ only containing F, clearly F is $f\pi g$-closed set in (X,τ). Here $F < A$ where A is fg-open set in (X,τ). But $clF = 1_X \setminus B \not\le A$ implies that B is not fs^*g-closed set in (X,τ). Also $cl(intF) = 1_X \setminus B \not\le A$. Hence F is not fmg-closed set in (X,τ).

Example 3.26. $fg\alpha$-closed set, $f\alpha g$-closed set, $fg\gamma$-closed set, $fg\gamma^*$-closed set, $fswg$-closed set, $frwg$-closed set, fmg-closed set, fwg-closed set, $fg\beta$-closed set, $f\beta g$-closed set, fgs-closed set, fsg-closed set, fgp-closed set, fpg-closed set do not necessarily imply $f\pi g$-closed set

Consider Example 3.22 and the fuzzy set E defined by $E(a) = 0.2, E(b) = 0.4$. Since $E < B \in F\pi O(X,\tau)$, but $clE = 1_X \setminus A \not\le B$ implies that E is not $f\pi g$-closed set in (X,τ).

Now $cl(int(clE)) = 0_X$. Then $E \in F\alpha C(X,\tau)$ and so E is $fg\alpha$-closed as well as $f\alpha g$-closed set in (X,τ).

Again $int(cl(intE)) = 0_X$ implies that $E \in F\beta C(X,\tau)$ and so E is $fg\beta$-closed as well as $f\beta g$-closed set in (X,τ).

Also $(cl(intE)) \bigwedge (int(clE)) = 0_X < E$. Then $E \in F\gamma C(X,\tau)$ and so

E is $fg\gamma$-closed as well as $fg\gamma^*$-closed set in (X,τ).
Also $cl(intE) = 0_X < E$ Then $E \in FPC(X,\tau)$ and so E is fgp-closed as well as fpg-closed set in (X,τ). Again as $cl(intE) = 0_X$, E is fmg-closed, fwg-closed, $fswg$-closed, $frwg$-closed set in (X,τ).
Again $int(clE) = 0_X < E$ implies that $E \in FSC(X,\tau)$ and so E is fgs-closed as well as fsg-closed set in (X,τ).

Now we recall the definitions of some spaces from [3, 8, 9, 10, 11] in which the sets defined in [2, 3, 5, 6, 7, 9, 10, 11] are $f\pi g$-closed set and some partial converses are true.

Definition 3.27. An fts (X,τ) is said to be
(i) $f\beta T_b$-space [8] if every $f\beta g$-closed set in X is fuzzy closed set in X,
(ii) fT_β-space [8] if every $fg\beta$-closed set in X is fuzzy closed set in X,
(iii) fT_α-space [3] if every $fg\alpha$-closed set in X is fuzzy closed set in X,
(iv) $f\alpha T_b$-space [3] if every $f\alpha g$-closed set in X is fuzzy closed set in X,
(v) fT_b-space [3] if every fgs-closed set in X is fuzzy closed set in X,
(vi) fT_{sg}-space [3] if every fsg-closed set in X is fuzzy closed set in X,
(vii) fT_γ-space [10] if every $fg\gamma$-closed set in X is fuzzy closed set in X,
(viii) fT_{γ^*}-space [11] if every $fg\gamma^*$-closed set in X is fuzzy closed set in X,
(ix) frT_g-space [9] if every $frwg$-closed set in X is fuzzy closed set in X,
(x) fsT_g-space [9] if every $fswg$-closed set in X is fuzzy closed set in X,
(xi) fT_p-space [3] if every fgp-closed set in X is fuzzy closed set in X,
(xii) fpT_b-space [3] if every fpg-closed set in X is fuzzy closed set in X,
(xiii) fmT_g-space [9] if every fmg-closed set in X is fuzzy closed set in X,
(xiv) fT_w-space [8] if every fwg-closed set in X is fuzzy closed set in X,
(xv) fT_π-space [9] if every $f\pi g$-closed set in X is fuzzy closed set in X.

Remark 3.28 (i) In $f\beta T_b$-space (resp., fT_β-space, fT_α-space, $f\alpha T_b$-space, fT_b-space, fT_{sg}-space, fT_γ-space, fT_{γ^*}-space, frT_g-space, fsT_g-space, fT_p-space, fpT_b-space, fmT_g-space, fT_w-space) $f\beta g$-closed (resp., $fg\beta$-closed, $fg\alpha$-closed, $f\alpha g$-closed, fgs-closed,

fsg-closed, $fg\gamma$-closed, $fg\gamma^*$-closed, $frwg$-closed, $fswg$-closed, fgp-closed, fpg-closed, fmg-closed, fwg-closed) set is $f\pi g$-closed set.
(ii) In fT_π-space, $f\pi g$-closed set is fg-closed, fgs^*-closed, fs^*g-closed, $fg\beta$-closed, $f\beta g$-closed, $fg\alpha$-closed, $f\alpha g$-closed, fgp-closed, fpg-closed, fgs-closed, fsg-closed, $fg\gamma$-closed, $fg\gamma^*$-closed set.

4. $f\pi g$-Closure Operator and $f\pi g$-Open, $f\pi g$- Closed Functions

A new type of generalized version of closure operator in an fts, viz., $f\pi g$-closure operator is introduced here which is an idempotent operator. Then introduce $f\pi g$-open and $f\pi g$-closed functions which are characterized by $f\pi g$-closure operator.

Definition 4.1. Let (X,τ) be an fts and $A \in I^X$. Then $f\pi g$-closure and $f\pi g$-interior of A, denoted by $f\pi gcl(A)$ and $f\pi gint(A)$, are defined as follows:

$f\pi gcl(A) = \bigwedge\{F : A \leq F, F$ is $f\pi g$-closed set in $X\}$,

$f\pi gint(A) = \bigvee\{G : G \leq A, G$ is $f\pi g$-open set in $X\}$.

Remark 4.2. It is clear from definition that for any $A \in I^X$, $A \leq f\pi gcl(A) \leq clA$. If A is $f\pi g$-closed set in an fts X, then $A = f\pi gcl(A)$. Similarly, $intA \leq f\pi gint(A) \leq A$. If A is $f\pi g$-open set in an fts X, then $A = f\pi gint(A)$. It follows from Remark 3.2 that $f\pi gcl(A)$ (resp., $f\pi gint(A)$) may not be $f\pi g$-closed (resp., $f\pi g$-open) set in an fts X.

Theorem 4.3. Let (X,τ) be an fts and $A \in I^X$. Then for a fuzzy point x_t in X, $x_t \in f\pi gcl(A)$ if and only if every $f\pi g$-open q-nbd U of x_t, UqA.

Proof. Let $x_t \in f\pi gcl(A)$ for any fuzzy set A in an fts X and F be any $f\pi g$-open q-nbd of x_t. Then x_tqF implies that $x_t \notin 1_X \setminus F$ which is $f\pi g$-closed set in X. Then by Definition 4.1, $A \not\leq 1_X \setminus F$. Then there exists $y \in X$ such that $A(y) > 1 - F(y)$. Hence AqF.

Conversely, let for every $f\pi g$-open q-nbd F of x_t, FqA. If possible, let $x_t \notin f\pi gcl(A)$. Then by Definition 4.1, there exists an $f\pi g$-closed set U in X with $A \leq U$, $x_t \notin U$. Then $x_tq(1_X \setminus U)$ which being $f\pi g$-open set in X is $f\pi g$-open q-nbd of x_t. By assumption, $(1_X \setminus U)qA$. Then $(1_X \setminus A)qA$, a contradiction.

Theorem 4.4. Let (X,τ) be an fts and $A, B \in I^X$. Then the following statements are true:
(i) $f\pi gcl(0_X) = 0_X$,
(ii) $f\pi gcl(1_X) = 1_X$,
(iii) $A \leq B$ implies $f\pi gcl(A) \leq f\pi gcl(B)$,

(iv) $f\pi gcl(A \bigvee B) = f\pi gcl(A) \bigvee f\pi gcl(B)$,
(v) $f\pi gcl(A \wedge B) \leq f\pi gcl(A) \wedge f\pi gcl(B)$, equality does not hold, in general, follows from Example 3.3,
(vi) $f\pi gcl(f\pi gcl(A)) = f\pi gcl(A)$.

Proof. (i), (ii) and (iii) are obvious.
(iv) From (iii), $f\pi gcl(A) \bigvee f\pi gcl(B) \leq f\pi gcl(A \bigvee B)$.
To prove the converse, let $x_\alpha \in f\pi gcl(A \bigvee B)$. Then by Theorem 4.3, for any $f\pi g$-open set U in X with $x_\alpha qU$, $Uq(A \bigvee B)$, there exists $y \in X$ such that $U(y)+max\{A(y), B(y)\} > 1$. Then either $U(y)+A(y) > 1$ or $U(y)+B(y) > 1$. So either UqA or UqB. Then either $x_\alpha \in f\pi gcl(A)$ or $x_\alpha \in f\pi gcl(B)$. Hence $x_\alpha \in f\pi gcl(A) \bigvee f\pi gcl(B)$.
(v) Follows from (iii).
(vi) As $A \leq f\pi gcl(A)$, for any $A \in I^X$, $f\pi gcl(A) \leq f\pi gcl(f\pi gcl(A))$ (by (iii)).

Conversely, let $x_\alpha \in f\pi gcl(f\pi gcl(A)) = f\pi gcl(B)$ where $B = f\pi gcl(A)$. Let U be any $f\pi g$-open set in X with $x_\alpha qU$. Then UqB implies that there exists $y \in X$ such that $U(y) + B(y) > 1$. Let $B(y) = t$. Then $y_t qU$ and $y_t \in B = f\pi gcl(A)$ implies UqA. So $x_\alpha \in f\pi gcl(A)$. Then $f\pi gcl(f\pi gcl(A)) \leq f\pi gcl(A)$. Consequently, $f\pi gcl(f\pi gcl(A)) = f\pi gcl(A)$.

Theorem 4.5. Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold:
(i) $f\pi gcl(1_X \setminus A) = 1_X \setminus f\pi gint(A)$
(ii) $f\pi gint(1_X \setminus A) = 1_X \setminus f\pi gcl(A)$.

Proof (i). Let $x_t \in f\pi gcl(1_X \setminus A)$ for a fuzzy set A in an fts (X, τ). If possible, let $x_t \notin 1_X \setminus f\pi gint(A)$. Then $1 - (f\pi gint(A))(x) < t$ implies $[f\pi gint(A)](x) + t > 1$ and so $f\pi gint(A)qx_t$. Then there exists at least one $f\pi g$-open set $F \leq A$ with $x_t qF$ and so $x_t qA$. As $x_t \in f\pi gcl(1_X \setminus A), Fq(1_X \setminus A)$ which implies that $Aq(1_X \setminus A)$, a contradiction. Hence

$$f\pi gcl(1_X \setminus A) \leq 1_X \setminus f\pi gint(A)...(1)$$

Conversely, let $x_t \in 1_X \setminus f\pi gint(A)$. Then $1 - [(f\pi gint(A)](x) \geq t$. Then $x_t \not q(f\pi gint(A))$ and so $x_t \not q F$ for every $f\pi g$-open set F contained in A ... (2).
Let U be any $f\pi g$-closed set in X such that $1_X \setminus A \leq U$. Then $1_X \setminus U \leq A$. Now $1_X \setminus U$ is $f\pi g$-open set in X contained in A. By (2), $x_t \not q(1_X \setminus U)$. Then $x_t \in U$ implies $x_t \in f\pi gcl(1_X \setminus A)$ and so

$$1_X \setminus f\pi gint(A) \leq f\pi gcl(1_X \setminus A)...(3).$$

Combining (1) and (3), (i) follows.
(ii) Putting $1_X \setminus A$ for A in (i), we get $f\pi gcl(A) = 1_X \setminus f\pi gint(1_X \setminus A)$ implies $f\pi gint(1_X \setminus A) = 1_X \setminus f\pi gcl(A)$.

Let us now recall the following definition from [25] for ready references.

Definition 4.6 [25]. A function $f : X \to Y$ is called fuzzy open (resp., fuzzy closed) if $f(U)$ is fuzzy open (resp., fuzzy closed) set in Y for every fuzzy open (resp., fuzzy closed) set U in X.

Let us now introduce the following concept.

Definition 4.7. A function $h : X \to Y$ is called fuzzy π-generalized open ($f\pi g$-open, for short) function if $h(U)$ is $f\pi g$-open set in Y for every fuzzy open set U in X.

Remark 4.8. It is clear that fuzzy open function is $f\pi g$-open function. But the converse need not be true, as the following example shows.

Example 4.9. $f\pi g$-open function may not necessarily fuzzy open function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.4, A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_2) is $f\pi g$-open set in (X, τ_2), clearly i is $f\pi g$-open function. But $A \in \tau_1$, $i(A) = A \notin \tau_2$ implies that i is not a fuzzy open function.

Theorem 4.10. For a bijective function $h : X \to Y$, the following statements are equivalent:
(i) h is $f\pi g$-open,
(ii) $h(intA) \leq f\pi gint(h(A))$, for all $A \in I^X$,
(iii) for each fuzzy point x_α in X and each fuzzy open set U in X containing x_α, there exists an $f\pi g$-open set V in Y containing $h(x_\alpha)$ such that $V \leq h(U)$.

Proof (i) $\Rightarrow$ (ii). Let $A \in I^X$. Then $intA$ is a fuzzy open set in X. By (i), $h(intA)$ is $f\pi g$-open set in Y. Since $h(intA) \leq h(A)$ and $f\pi gint(h(A))$ is the union of all $f\pi g$-open sets contained in $h(A)$, we have $h(intA) \leq f\pi gint(h(A))$.
(ii) $\Rightarrow$ (i). Let U be any fuzzy open set in X. Then $h(U) = h(intU) \leq f\pi gint(h(U))$ (by (ii)) implies $h(U)$ is $f\pi g$-open set in Y and hence h is $f\pi g$-open function.
(ii) $\Rightarrow$ (iii). Let x_α be a fuzzy point in X, and U, a fuzzy open set in X such that $x_\alpha \in U$. Then $h(x_\alpha) \in h(U) = h(intU) \leq f\pi gint(h(U))$ (by (ii)). Then $h(U)$ is $f\pi g$-open set in Y. Let $V = h(U)$. Then $h(x_\alpha) \in V$ and $V \leq h(U)$.

(iii) $\Rightarrow$ (i). Let U be any fuzzy open set in X and y_α, any fuzzy point in $h(U)$, i.e., $y_\alpha \in h(U)$. Then there exists unique $x \in X$ such that $h(x) = y$ (as h is bijective). Then $[h(U)](y) \geq \alpha$ implies $U(h^{-1}(y)) \geq \alpha$ and so $U(x) \geq \alpha$. Then $x_\alpha \in U$. By (iii), there exists $f\pi g$-open set V in Y such that $h(x_\alpha) \in V$ and $V \leq h(U)$. Then $h(x_\alpha) \in V = f\pi gint(V) \leq f\pi gint(h(U))$. Since y_α is taken arbitrarily and $h(U)$ is the union of all fuzzy points in $h(U)$, $h(U) \leq f\pi gint(f(U))$ implies $h(U)$ is $f\pi g$-open set in YHence h is an $f\pi g$-open function.

Theorem 4.11. If $h : X \to Y$ is $f\pi g$-open, bijective function, then the following statements are true:
(i) for each fuzzy point x_α in X and each fuzzy open q-nbd U of x_α in X, there exists an $f\pi g$-open q-nbd V of $h(x_\alpha)$ in Y such that $V \leq h(U)$,
(ii) $h^{-1}(f\pi gcl(B)) \leq cl(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_α be a fuzzy point in X and U be any fuzzy open q-nbd of x_α in X. Then $x_\alpha qU = intU$ implies $h(x_\alpha)qh(intU) \leq f\pi gint(h(U))$ (by Theorem 4.10 (i)$\Rightarrow$(ii)) implies that there exists at least one $f\pi g$-open q-nbd V of $h(x_\alpha)$ in Y with $V \leq h(U)$.
(ii) Let x_α be any fuzzy point in X such that $x_\alpha \notin cl(h^{-1}(B))$ for any $B \in I^Y$. Then there exists a fuzzy open q-nbd U of x_α in X such that $U \not{q} h^{-1}(B)$. Now

$$h(x_\alpha)qh(U)...(1)$$

where $h(U)$ is $f\pi g$-open set in Y. Now $h^{-1}(B) \leq 1_X \setminus U$ which is a fuzzy closed set in X and so $B \leq h(1_X \setminus U)$ (as h is injective) $\leq 1_Y \setminus h(U)$. Then $B \not{q} h(U)$. Let $V = 1_Y \setminus h(U)$. Then $B \leq V$ which is $f\pi g$-closed set in Y. We claim that $h(x_\alpha) \notin V$. If possible, let $h(x_\alpha) \in V = 1_Y \setminus h(U)$. Then $1 - [h(U)](h(x)) \geq \alpha$. So $h(U) \not{q} h(x_\alpha)$, contradicting (1). So $h(x_\alpha) \notin V$. Then $h(x_\alpha) \notin f\pi gcl(B)$ and so $x_\alpha \notin h^{-1}(f\pi gcl(B))$ Hence $h^{-1}(f\pi gcl(B)) \leq cl(h^{-1}(B))$.

Theorem 4.12. An injective function $h : X \to Y$ is $f\pi g$-open if and only if for each $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$, there exists an $f\pi g$-closed set V in Y such that $B \leq V$ and $h^{-1}(V) \leq F$.

Proof. Let $B \in I^Y$ and F, a fuzzy closed set in X with $h^{-1}(B) \leq F$. Then $1_X \setminus h^{-1}(B) \geq 1_X \setminus F$ where $1_X \setminus F$ is a fuzzy open set in X. So $h(1_X \setminus F) \leq h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus B$ (as h is injective) where $h(1_X \setminus F)$ is an $f\pi g$-open set in Y. Let $V = 1_Y \setminus h(1_X \setminus F)$. Then V is $f\pi g$-closed set in Y such that $B \leq V$. Now $h^{-1}(V) = h^{-1}(1_Y \setminus h(1_X \setminus F)) = 1_X \setminus h^{-1}(h(1_X \setminus F)) \leq F$.

Conversely, let F be a fuzzy open set in X. Then $1_X \setminus F$ is a fuzzy closed set in X. We have to show that $h(F)$ is an $f\pi g$-open set in Y. Now $h^{-1}(1_Y \setminus h(F)) \leq 1_X \setminus F$ (as h is injective). By assumption, there exists an $f\pi g$-closed set V in Y such that

$$1_Y \setminus h(F) \leq V...(1)$$

and $h^{-1}(V) \leq 1_X \setminus F$. Therefore, $F \leq 1_X \setminus h^{-1}(V)$ implies that

$$h(F) \leq h(1_X \setminus h^{-1}(V)) \leq 1_Y \setminus V...(2)$$

(as h is injective). Combining (1) and (2), $h(F) = 1_Y \setminus V$ which is an $f\pi g$-open set in Y. Hence h is $f\pi g$-open function.

Definition 4.13. A function $h : X \to Y$ is called fuzzy π-generalized closed ($f\pi g$-closed, for short) function if $h(A)$ is $f\pi g$-closed set in Y for each fuzzy closed set A in X.

Remark 4.14. It is obvious that every fuzzy closed function is $f\pi g$-closed function, but the converse may not be true as it seen in Example 4.9. Here $1_X \setminus A \in \tau_1^c$, but $i(1_X \setminus A) = 1_X \setminus A \notin \tau_2^c$ and so i is not a fuzzy closed function. But since every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2), clearly i is $f\pi g$-closed function.

Theorem 4.15. A bijective function $h : X \to Y$ is $f\pi g$-closed function if and only if $f\pi gcl(h(A)) \leq h(clA)$, for all $A \in I^X$.

Proof. Let us suppose that $h : X \to Y$ be an $f\pi g$-closed function and $A \in I^X$. Then $h(cl(A))$ is $f\pi g$-closed set in Y. Since $h(A) \leq h(clA)$ and $f\pi gcl(h(A))$ is the intersection of all $f\pi g$-closed sets in Y containing $h(A)$, we have $f\pi gcl(h(A)) \leq h(clA)$.

Conversely, let for any $A \in I^X$, $f\pi gcl(h(A)) \leq h(clA)$. Let U be any fuzzy closed set in X. Then $h(U) = h(clU) \geq f\pi gcl(h(U))$ implies $h(U)$ is an $f\pi g$-closed set in Y. Hence h is an $f\pi g$-closed function.

Theorem 4.16. If $h : X \to Y$ is an $f\pi g$-closed bijective function, then the following statements hold:

(i) for each fuzzy point x_α in X and each fuzzy closed set U in X with $x_\alpha \not{q} U$, there exists an $f\pi g$-closed set V in Y with $h(x_\alpha) \not{q} V$ such that $V \geq h(U)$,

(ii) $h^{-1}(f\pi gint(B)) \geq int(h^{-1}(B))$, for all $B \in I^Y$.

Proof (i). Let x_α be a fuzzy point in X and U be any fuzzy closed set in X with $x_\alpha \not{q} U = clU$. So $h(x_\alpha) \not{q} h(clU) \geq f\pi gcl(h(U))$ (by Theorem 4.15). Then $h(x_\alpha) \not{q} V$ for some $f\pi g$-closed set V in Y with $V \geq h(U)$.

(ii). Let $B \in I^Y$ and x_α be any fuzzy point in X such that $x_\alpha \in int(h^{-1}(B))$. Then there exists a fuzzy open set U in X with $U \leq h^{-1}(B)$ such that $x_\alpha \in U$. Then $1_X \setminus U \geq 1_X \setminus h^{-1}(B)$ implies $h(1_X \setminus U) \geq h(1_X \setminus h^{-1}(B))$ where $h(1_X \setminus U)$ is an $f\pi g$-closed set in Y. Let $V = 1_Y \setminus h(1_X \setminus U)$. Then V is an $f\pi g$-open set in Y and

$V = 1_Y \setminus h(1_X \setminus U) \leq 1_Y \setminus h(1_X \setminus h^{-1}(B)) \leq 1_Y \setminus (1_Y \setminus B) = B$ (as h is injective). Now $U(x) \geq \alpha$. So $x_\alpha \not{q}(1_X \setminus U)$. Then $h(x_\alpha)\not{q}h(1_X \setminus U)$ implies that $h(x_\alpha) \leq 1_Y \setminus h(1_X \setminus U) = V$. Then $h(x_\alpha) \in V = f\pi gint(V) \leq f\pi gint(B)$. So $x_\alpha \in h^{-1}(f\pi gint(B))$. Since x_α is taken arbitrarily, $int(h^{-1}(B)) \leq h^{-1}(f\pi gint(B))$, for all $B \in I^Y$.

Remark 4.17. Composition of two $f\pi g$-closed (resp., $f\pi g$-open) functions need not be so, as it seen from the following example.

Example 4.18. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.8, A(b) = 0.5$, $B(a) = 0.3, B(b) = 0.5$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are $f\pi g$-closed functions. Let $i_3 = i_2 \circ i_1 : (X, \tau_1) \to (X, \tau_3)$. We claim that i_3 is not $f\pi g$-closed function. Here $F\pi O(X, \tau_3) = \tau_3$. Now $1_X \setminus A \in \tau_1^c$. $(i_2 \circ i_1)(1_X \setminus A) = 1_X \setminus A < B \in F\pi O(X, \tau_3)$. But $cl_{\tau_3}(1_X \setminus A) = 1_X \setminus B \not\leq B$ implies that $1_X \setminus A$ is not $f\pi g$-closed set in (X, τ_3). Hence $i_2 \circ i_1$ is not $f\pi g$-closed function.

Similarly we can show that $i_2 \circ i_1$ is not $f\pi g$-open function though i_1 and i_2 are so.

Theorem 4.19. If $h_1 : X \to Y$ is fuzzy closed (resp., fuzzy open) function and $h_2 : Y \to Z$ is $f\pi g$-closed (resp., $f\pi g$-open) function, then $h_2 \circ h_1 : X \to Z$ is $f\pi g$-closed (resp., $f\pi g$-open) function.

Proof. Obvious.

Now to establish the mutual relationships of $f\pi g$-closed function with the functions defined in [3, 5, 6, 7, 9, 10, 11]. We have to recall he following definitions first.

Definition 4.20. Let $(X, \tau_1) \to (Y, \tau_2)$ be a function. Then h is called an

(i) fg-closed function [3] if $h(A)$ is fg-closed set in Y for every $A \in \tau_1^c$,
(ii) $fg\beta$-closed function [7] if $h(A)$ is $fg\beta$-closed set in Y for every $A \in \tau_1^c$,
(iii) $f\beta g$-closed function [7] if $h(A)$ is $f\beta g$-closed set in Y for every $A \in \tau_1^c$,
(iv) $fg\alpha$-closed function [3] if $h(A)$ is $fg\alpha$-closed set in Y for every $A \in \tau_1^c$,
(v) $f\alpha g$-closed function [3] if $h(A)$ is $f\alpha g$-closed set in Y for every $A \in \tau_1^c$,
(vi) fgp-closed function [3] if $h(A)$ is fgp-closed set in Y for every $A \in \tau_1^c$,
(vii) fpg-closed function [3] if $h(A)$ is fpg-closed set in Y for every

$A \in \tau_1^c$,
(viii) fgs-closed function [3] if $h(A)$ is fgs-closed set in Y for every $A \in \tau_1^c$,
(ix) fsg-closed function [3] if $h(A)$ is fsg-closed set in Y for every $A \in \tau_1^c$,
(x) fgs^*-closed function [5] if $h(A)$ is fgs^*-closed set in Y for every $A \in \tau_1^c$,
(xi) fs^*g-closed function [6] if $h(A)$ is fs^*g-closed set in Y for every $A \in \tau_1^c$,
(xii) $fg\gamma$-closed function [10] if $h(A)$ is $fg\gamma$-closed set in Y for every $A \in \tau_1^c$,
(xiii) $fg\gamma^*$-closed function [11] if $h(A)$ is $fg\gamma^*$-closed set in Y for every $A \in \tau_1^c$,
(xiv) $fswg$-closed function [9] if $h(A)$ is $fswg$-closed set in Y for every $A \in \tau_1^c$,
(xv) $frwg$-closed function [9] if $h(A)$ is $frwg$-closed set in Y for every $A \in \tau_1^c$,
(xvi) fmg-closed function [9] if $h(A)$ is fmg-closed set in Y for every $A \in \tau_1^c$,
(xvii) fwg-closed function [9] if $h(A)$ is fwg-closed set in Y for every $A \in \tau_1^c$.

Remark 4.21. fg-closed function is $f\pi g$-closed function, fgs^*-closed function is $f\pi g$-closed function, fs^*g-closed function is $f\pi g$-closed function and $f\pi g$-closed function is $frwg$-closed function.
But the reverse implications are not true, in general, as it seen in the following examples.
(ii) $f\pi g$-closed function is independent concept of $fg\beta$-closed function, $f\beta g$-closed function, $fg\alpha$-closed function, $f\alpha g$-closed function, fgp-closed function, fpg-closed function, fgs-closed function, fsg-closed function, $fg\gamma$-closed function, $fg\gamma^*$-closed function, $fswg$-closed function, fmg-closed function, fwg-closed function.

Example 4.22. $f\pi g$-closed function not necessarily implies fg-closed function, fgs-closed function, fsg-closed function, fgs^*-closed function, fs^*g-closed function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.6, A(b) = 0.5, B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $F\pi O(X, \tau_2) = \{0_X, 1_X\}$ and so every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2) and hence i is $f\pi g$-closed function. Again $FSO(X, \tau_2) = \{0_X, 1_X, U\}$ where $U \geq B$. Now $1_X \backslash A \in \tau_1^c, i(1_X \backslash A) =$

$1_X \setminus A < B \in \tau_2$, but $cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$ implies that $1_X \setminus A$ is not fg-closed set in (X, τ_2) and hence i is not fg-closed function. Also $scl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not fgs-closed set in (X, τ_2). Hence i is not fgs-closed function. Again $1_X \setminus A \leq B \in FSO(X, \tau_2)$, but $scl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not fsg-closed set in (X, τ_2). Hence i is not fsg-closed function. Also $cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$ and so $1_X \setminus A$ is not fgs^*-closed set in (X, τ_2). Hence i is not fgs^*-closed function. Again B is fg-open set in (X, τ_2) and so $1_X \setminus A \leq B$, but $cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$ implies $1_X \setminus A$ is not fs^*g-closed set in (X, τ_2). Hence i is not fs^*g-closed function.

Example 4.23. $f\pi g$-closed function may not necessarily $fswg$-closed function, $fg\gamma^*$-closed function, fpg-closed function, fmg-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.4, B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $F\pi O(X, \tau_2) = \{0_X, 1_X\}$ and so every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2) and as a result i is clearly $f\pi g$-closed function. Now $FSO(X, \tau_2) = \{0_X, 1_X, U\}$ = the set of all fg-open set in (X, τ_2) where $U \geq B$, $FPO(X, \tau_2) = \{0_X, 1_X, V\}$ where $V \not\leq 1_X \setminus B$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A \in FSO(X, \tau_2)$. So $1_X \setminus A \leq 1_X \setminus A \in FSO(X, \tau_2)$, but $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 1_X \not\leq 1_X \setminus A$. Then $1_X \setminus A$ is not $fswg$-closed set in (X, τ_2). Hence i is not $fswg$-closed function. Again $\gamma cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq 1_X \setminus A$ and so $1_X \setminus A$ is not $fg\gamma^*$-closed set in (X, τ_2). Then i is not $fg\gamma^*$-closed function. Again $1_X \setminus A < B$ where B is an fg-open set in (X, τ_2), $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 1_X \not\leq B$. So $1_X \setminus A$ is not fmg-closed set in (X, τ_2). So i is not fmg-closed function. Furthermore, $1_X \setminus A \in FPO(X, \tau_2)$ and so $1_X \setminus A \leq 1_X \setminus A \in FPO(X, \tau_2)$, but $pcl_{\tau_2}(1_X \setminus A) = 1_X \not\leq 1_X \setminus A$ implies $1_X \setminus A$ is not fpg-closed set in (X, τ_2). Hence i is not fpg-closed function.

Example 4.24. $f\pi g$-closed function not necessarily $fg\alpha$-closed function, $f\alpha g$-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.5$, $B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $F\pi O(X, \tau_2) = \{0_X, 1_X\}$ and so every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2) and consequently, i is $f\pi g$-closed function. Now $F\alpha O(X, \tau_2) = \{0_X, 1_X, U\}$ where $U \geq B$ and $F\alpha C(X, \tau_2) = \{0_X, 1_X, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus B$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A$. Here $1_X \setminus A < B \in \tau_2$ as well as $1_X \setminus A < B \in F\alpha O(X, \tau_2)$.

But $\alpha cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not $fg\alpha$-closed as well as $f\alpha g$-closed set in (X, τ_2). Hence i is neither $fg\alpha$-closed nor $f\alpha g$-closed function.

Example 4.25. $f\pi g$-closed function may not necessarily $fg\gamma$-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B, C\}$ where $A(a) = 0.5, A(b) = 0.45$, $B(a) = 0.5, B(b) = 0.6$, $C(a) = 0.3, C(b) = 0.5$. Then $X, \tau_1)$ and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $F\pi O(X, \tau_2) = \{0_X, 1_X, C\}$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_2)$ only, $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_2). So i is $f\pi g$-closed function. Again $1_X \setminus A < B \in \tau_2$, but $\gamma cl_{\tau_2}(1_X \setminus A) = 1_X \setminus C \not\leq B$. Then $1_X \setminus A$ is not $fg\gamma$-closed set in (X, τ_2). Hence i is not $fg\gamma$-closed function.

Example 4.26. $f\pi g$-closed function may not necessarily $fg\beta$-closed function, $f\beta g$-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = A(b) = 0.4, B(a) = B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $F\pi O(X, \tau_2) = \{0_X, 1_X\}$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_2)$ only and so $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_2). Hence i is $f\pi g$-closed function. Again $cl_{\tau_2}(int_{\tau_2}(cl_{\tau_2} B)) = 1_X > B$ implies $B \in F\beta O(X, \tau_2)$. Now $1_X \setminus A < B \in \tau_2$ as well as $1_X \setminus A < B \in F\beta O(X, \tau_2)$. But $\beta cl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not $fg\beta$-closed as well as $f\beta g$-closed set in (X, τ_2). Hence i is not $fg\beta$-closed as well as $f\beta g$-closed function.

Example 4.27. $f\pi g$-closed function may not necessarily fgp-closed function, fwg-closed function

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B, C\}$ where $A(a) = A(b) = 0.5$, $B(a) = 0.5, B(b) = 0.6$, $C(a) = 0.3, C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $F\pi O(X, \tau_2) = \{0_X, 1_X, C\}$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_2)$ only and so $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_2) and hence i is $f\pi g$-closed function. Now $1_X \setminus A < B \in \tau_2$, but $pcl_{\tau_2}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not fgp-closed set in (X, τ_2). Then i is not fgp-closed function. Also $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 1_X \setminus B \not\leq B$. So $1_X \setminus A$ is not fwg-closed set in (X, τ_2) and hence i is not fwg-closed function.

Example 4.28. fgp-closed function, fpg-closed function, $fg\beta$-closed function, $f\beta g$-closed function, fgs-closed function, fsg-closed

function, $fg\gamma$-closed function, $fg\gamma^*$-closed function, $fswg$-closed function, fmg-closed function, fwg-closed function, $frwg$-closed function may not imply $f\pi g$-closed function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.7$, $B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $FSO(X, \tau_2) = \{0_X, 1_X, U\}$ where $B \leq U \leq 1_X \setminus B$ and the collection of all fg-open sets in (X, τ_2) is $\{0_X, 1_X, T\}$ where $T \ngeq 1_X \setminus B$. Here $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A < B \in F\pi O(X, \tau_2) = \tau_2$. But $cl_{\tau_2}(1_X \setminus A) = 1_X \setminus B \nleq B$ and so $1_X \setminus A$ is not $f\pi g$-closed set in (X, τ_2). Then i is not $f\pi g$-closed function.
Now $B \in FRO(X, \tau_2)$ and $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 0_X < B$. So $1_X \setminus A$ is $frwg$-closed set in (X, τ_2). Then i is $frwg$-closed function. Again $B \in FSO(X, \tau_2)$ and $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 0_X < B$. Then $1_X \setminus A$ is $fswg$-closed set in (X, τ_2) and hence i is $fswg$-closed function. Also $1_X \setminus A$ is fg-open set in (X, τ_2) and so $1_X \setminus A \leq 1_X \setminus A$. Now $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 0_X < 1_X \setminus A$ and so $1_X \setminus A$ is fmg-closed set in (X, τ_2). Then i is fmg-closed function. Again $1_X \setminus A < B \in \tau_2$ and $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 0_X < B$. Then $1_X \setminus A$ is fwg-closed set in (X, τ_2) and hence i is fwg-closed function. Since $cl_{\tau_2}(int_{\tau_2}(1_X \setminus A)) = 0_X < 1_X \setminus A$, $1_X \setminus A \in FPC(X, \tau_2)$ and so $1_X \setminus A$ is fgp-closed as well as fpg-closed set in (X, τ_2) and so i is fgp-closed as well as fpg-closed function. Also as $int_{\tau_2}(cl_{\tau_2}(int_{\tau_2}(1_X \setminus A))) = 0_X < 1_X \setminus A$, $1_X \setminus A \in F\beta C(X, \tau_2)$ and so $1_X \setminus A$ is $fg\beta$-closed as well as $f\beta g$-closed set in (X, τ_2). So i is $fg\beta$-closed as well as $f\beta g$-closed function. Also $(cl_{\tau_2}(int_{\tau_2}(1_X \setminus A))) \bigwedge (int_{\tau_2}(cl_{\tau_2}(1_X \setminus A))) = 0_X < 1_X \setminus A$, $1_X \setminus A \in F\gamma C(X, \tau_2)$ and so $1_X \setminus A$ is $fg\gamma$-closed as well as $fg\gamma^*$-closed set in (X, τ_2). Then i is $fg\gamma$-closed as well as $fg\gamma^*$-closed function. Furthermore, $1_X \setminus A < B \in \tau_2$ as well as $1_X \setminus A \leq B \in FSO(X, \tau_2)$ and $scl_{\tau_2}(1_X \setminus A) = B \leq B$. So $1_X \setminus A$ is fgs-closed as well as fsg-closed set in (X, τ_2). Hence i is fgs-closed as well as fsg-closed function.

Example 4.29. $fg\alpha$-closed function, $f\alpha g$-closed function may not necessarily $f\pi g$-closed function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B, C\}$ where $A(a) = 0.8, A(b) = 0.6$, $B(a) = 0.5, B(b) = 0.6$ and $C(a) = 0.3, C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_1^c$, $i(1_X \setminus A) = 1_X \setminus A$. Since $cl_{\tau_2}(int_{\tau_2}(cl_{\tau_2}(1_X \setminus A))) = 0_X < 1_X \setminus A$, $1_X \setminus A \in F\alpha C(X, \tau_2)$ and so $1_X \setminus A$ is $fg\alpha$-closed as well as $f\alpha g$-closed set in (X, τ_2), therefore i is $fg\alpha$-closed as well as $f\alpha g$-closed function. But $1_X \setminus A < C \in$

$F\pi O(X,\tau_2)$ and $cl_{\tau_2}(1_X \setminus A) = 1_X \setminus B \not\leq C$ and so $1_X \setminus A$ is not $f\pi g$-closed set in (X,τ_2). Hence i is not $f\pi g$-closed function.

Remark 4.30. (i) Let $h : X \to Y$ be a function where Y is an $f\beta T_b$-space (resp., fT_β-space, fT_α-space, $f\alpha T_b$-space, fT_b-space, fT_{sg}-space, fT_γ-space, fT_{γ^*}-space, frT_g-space, fsT_g-space, fT_p-space, fpT_b-space, fmT_g-space, fT_w-space). Then an $f\beta g$-closed (resp., $fg\beta$-closed, $fg\alpha$-closed, $f\alpha g$-closed, fgs-closed, fsg-closed, $fg\gamma$-closed, $fg\gamma^*$-closed, $frwg$-closed, $fswg$-closed, fgp-closed, fpg-closed, fmg-closed, fwg-closed) function is $f\pi g$-closed function.
(ii) Let $h : X \to Y$ be a function where Y is an fT_π-space. If h is an $f\pi g$-closed function, then h is an fg-closed function, fgs^*-closed function, fs^*g-closed function, $fg\beta$-closed function, $f\beta g$-closed function, $fg\alpha$-closed function, $f\alpha g$-closed function, fgp-closed function, fpg-closed function, fgs-closed function, fsg-closed function, $fg\gamma$-closed function, $fg\gamma^*$-closed function.

5. $f\pi g$-Regular, $f\pi g$-Normal and $f\pi g$-Compact Spaces

In this section a new type of generalized version of fuzzy regularity, fuzzy normality and fuzzy compactness are introduced and studied. It is also shown that these three concepts are weak concepts of fuzzy regularity [20], fuzzy normality [19] and fuzzy compactness [14].

Definition 5.1. An fts (X,τ) is said to be $f\pi g$-regular space if for any fuzzy point x_t in X and each $f\pi g$-closed set F in X with $x_t \notin F$, there exist $U, V \in FRO(X)$ such that $x_t \in U, F \leq V$ and $U \not{q} V$.

Theorem 5.2. In an fts (X,τ), the following statements are equivalent:
(i) X is $f\pi g$-regular,
(ii) for each fuzzy point x_t in X and any $f\pi g$-open q-nbd U of x_t, there exists $V \in FRO(X)$ such that $x_t \in V$ and $clV \leq U$,
(iii) for each fuzzy point x_t in X and each $f\pi g$-closed set A of X with $x_t \notin A$, there exists $U \in FRO(X)$ with $x_t \in U$ such that $clU \not{q} A$.

Proof (i) $\Rightarrow$ (ii). Let x_t be a fuzzy point in X and U, any $f\pi g$-open q-nbd of x_t. Then $x_t q U$ implies $U(x)+t > 1$ and so $x_t \notin 1_X \setminus U$ which is an $f\pi g$-closed set in X. By (i), there exist $V, W \in FRO(X)$ such that $x_t \in V, 1_X \setminus U \leq W$ and $V \not{q} W$. Then $V \leq 1_X \setminus W$ and so $clV \leq cl(1_X \setminus W) = 1_X \setminus W \leq U$.
(ii) $\Rightarrow$ (iii). Let x_t be a fuzzy point in X and A, an $f\pi g$-closed set in X with $x_t \notin A$. Then $A(x) < t$ and so $x_t q(1_X \setminus A)$ which being $f\pi g$-open set in X is $f\pi g$-open q-nbd of x_t. So by (ii), there exists $V \in FRO(X)$ such that $x_t \in V$ and $clV \leq 1_X \setminus A$. Then $clV \not{q} A$.

(iii) $\Rightarrow$ (i). Let x_t be a fuzzy point in X and F be any $f\pi g$-closed set in X with $x_t \notin F$. Then by (iii), there exists $U \in FRO(X)$ such that $x_t \in U$ and $clU \not{q}F$. Then $F \leq 1_X \setminus clU$ ($=V$, say). So $V \in FRO(X)$ and $V \not{q}U$ as $U \not{q}(1_X \setminus clU)$. Consequently, X is $f\pi g$-regular space.

Definition 5.3. An fts (X, τ) is called $f\pi g$-normal space if for each pair of $f\pi g$-closed sets A, B in X with $A \ /qB$, there exist $U, V \in FRO(X)$ such that $A \leq U, B \leq V$ and $U \not{q}V$.

Theorem 5.4. An fts (X, τ) is $f\pi g$-normal space if and only if for every $f\pi g$-closed set F and $f\pi g$-open set G in X with $F \leq G$, there exists $H \in FRO(X)$ such that $F \leq H \leq clH \leq G$.

Proof. Let X be $f\pi g$-normal space and let F be $f\pi g$-closed set and G be $f\pi g$-open set in X with $F \leq G$. Then $F \not{q}(1_X \setminus G)$ where $1_X \setminus G$ is $f\pi g$-closed set in X. By hypothesis, there exist $H, T \in FRO(X)$ such that $F \leq H, 1_X \setminus G \leq T$ and $H \not{q}T$. Then $H \leq 1_X \setminus T \leq G$. Therefore, $F \leq H \leq clH \leq cl(1_X \setminus T) = 1_X \setminus T \leq G$.

Conversely, let A, B be two $f\pi g$-closed sets in X with $A \not{q}B$. Then $A \leq 1_X \setminus B$. By hypothesis, there exists $H \in FRO(X)$ such that $A \leq H \leq clH \leq 1_X \setminus B$. Then $A \leq H, B \leq 1_X \setminus clH$ ($=V$, say). Then $V \in FRO(X)$ and so $B \leq V$. Also as $H \not{q}(1_X \setminus clH)$, $H \not{q}V$. Consequently, X is $f\pi g$-normal space.

Let us now recall the following definitions from [14, 18] for ready references.

Definition 5.5. Let (X, τ) be an fts and $A \in I^X$. A collection $\mathcal{U}$ of fuzzy sets in X is called a fuzzy cover of A if $\bigcup \mathcal{U} \geq A$ [18]. If each member of $\mathcal{U}$ is fuzzy open (resp., fuzzy regular open, $f\pi g$-open) in X, then $\mathcal{U}$ is called a fuzzy open [18] (resp., fuzzy regular open [1], $f\pi g$-open) cover of A. If, in particular, $A = 1_X$, we get the definition of fuzzy cover of X as $\bigcup \mathcal{U} = 1_X$ [14].

Definition 5.6. Let (X, τ) be an fts and $A \in I^X$. Then a fuzzy cover $\mathcal{U}$ of A (resp., of X) is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$ [18]. If, in particular $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$ [14].

Definition 5.7. Let (X, τ) be an fts and $A \in I^X$. Then A is called fuzzy compact [14] (resp., fuzzy almost compact [15], fuzzy nearly compact [21]) set if every fuzzy open (resp., fuzzy open, fuzzy regular open) cover $\mathcal{U}$ of A has a finite subcollection $\mathcal{U}_0$ such that $\bigcup \mathcal{U}_0 \geq A$ (resp., $\bigcup\limits_{U \in \mathcal{U}_0} clU \geq A$, $\bigcup \mathcal{U}_0 \geq A$). If, in particular, $A = 1_X$, we get the definition of fuzzy compact [14] (resp., fuzzy almost compact [15], fuzzy nearly compact [16]) space as $\bigcup \mathcal{U}_0 = 1_X$ (resp., $\bigcup\limits_{U \in \mathcal{U}_0} clU = 1_X$, $\bigcup \mathcal{U}_0 = 1_X$).

Let us now introduce the following concept.

Definition 5.8. Let (X, τ) be an fts and $A \in I^X$. Then A is called $f\pi g$-compact if every fuzzy cover $\mathcal{U}$ of A by $f\pi g$-open sets of X has a finite subcover. If, in particular, $A = 1_X$, we get the definition of $f\pi g$-compact space X.

Theorem 5.9. Every $f\pi g$-closed set in an $f\pi g$-compact space X is $f\pi g$-compact.

Proof. Let $A(\in I^X)$ be an $f\pi g$-closed set in an $f\pi g$-compact space X. Let $\mathcal{U}$ be a fuzzy cover of A by $f\pi g$-open sets of X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy cover of X by $f\pi g$-open sets of X. As X is $f\pi g$-compact space, $\mathcal{V}$ has a finite subcollection $\mathcal{V}_0$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcover of A. Hence A is $f\pi g$-compact set.

Next we recall the following two definitions from [20, 19] for ready references.

Definition 5.10 [20]. An fts (X, τ) is called fuzzy regular space if for each fuzzy point x_t in X and each fuzzy closed set F in X with $x_t \notin F$, there exist $U, V \in \tau$ such that $x_t \in U$, $F \leq V$ and $U \not{q} V$.

Definition 5.11 [19]. An fts (X, τ) is called fuzzy normal space if for each pair of fuzzy closed sets A, B of X with $A \not{q} B$, there exist $U, V \in \tau$ such that $A \leq U, B \leq V$ and $U \not{q} V$.

Remark 5.10. It is clear from above discussion that (i) $f\pi g$-regular (resp., $f\pi g$-normal) space is fuzzy regular (resp., fuzzy normal) space, (ii) $f\pi g$-compact space is fuzzy compact, fuzzy almost compact, fuzzy nearly compact space,
(iii) in fT_π-space, fuzzy compactness implies $f\pi g$-compactness.

6. $f\pi g$-Continuous and $f\pi g$-Irresolute Functions

In this section two new types of generalized version of fuzzy functions, viz,. $f\pi g$-continuous function and $f\pi g$-irresolute function are introduced and characterized by $f\pi g$-closed set. It is shown that $f\pi g$-continuous image of an $f\pi g$-regular (resp., $f\pi g$-normal, $f\pi g$-compact) space is fuzzy regular (resp., fuzzy normal, fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space. Also under $f\pi g$-irresolute function, $f\pi g$-regularity (resp., $f\pi g$-normality, $f\pi g$-compactness) remains invariant. Lastly, the mutual relationship of $f\pi g$-continuous function with the functions defined in [3, 5, 7, 9, 10, 11] are established.

Now we first introduce the following concept.

Definition 6.1. A function $h : X \to Y$ is said to be fuzzy π-generalized continuous ($f\pi g$-continuous, for short) function if $h^{-1}(V)$ is $f\pi g$-closed set in X for every fuzzy closed set V in Y.

Theorem 6.2. Let $h : (X, \tau) \to (Y, \sigma)$ be a function. Then the following statements are equivalent:
(i) h is $f\pi g$-continuous function,
(ii) for each fuzzy point x_α in X and each fuzzy open nbd V of $h(x_\alpha)$ in Y, there exists an $f\pi g$-open nbd U of x_α in X such that $h(U) \leq V$,
(iii) $h(f\pi gcl(A)) \leq cl(h(A))$, for all $A \in I^X$,
(iv) $f\pi gcl(h^{-1}(B)) \leq h^{-1}(clB)$, for all $B \in I^Y$.

Proof (i) $\Rightarrow$ (ii). Let x_α be a fuzzy point in X and V, any fuzzy open nbd of $h(x_\alpha)$ in Y. Then $x_\alpha \in h^{-1}(V)$ which is $f\pi g$-open in X (by (i)). Let $U = h^{-1}(V)$. Then $h(U) = h(h^{-1}(V)) \leq V$.
(ii) $\Rightarrow$ (i). Let A be any fuzzy open set in Y and x_α, a fuzzy point in X such that $x_\alpha \in h^{-1}(A)$. Then $h(x_\alpha) \in A$ where A is a fuzzy open nbd of $h(x_\alpha)$ in Y. By (ii), there exists an $f\pi g$-open nbd U of x_α in X such that $h(U) \leq A$. Then $x_\alpha \in U \leq h^{-1}(A)$. So $x_\alpha \in U = f\pi gint(U) \leq f\pi gint(h^{-1}(A))$. Since x_α is taken arbitrarily and $h^{-1}(A)$ is the union of all fuzzy points in $h^{-1}(A)$, $h^{-1}(A) \leq f\pi gint(h^{-1}(A))$. Then $h^{-1}(A)$ is an $f\pi g$-open set in X. Hence h is an $f\pi g$-continuous function.
(i) $\Rightarrow$ (iii). Let $A \in I^X$. Then $cl(h(A))$ is a fuzzy closed set in Y. By (i), $h^{-1}(cl(h(A)))$ is $f\pi g$-closed set in X. Now $A \leq h^{-1}(h(A)) \leq h^{-1}(cl(h(A)))$ and so $f\pi gcl(A) \leq f\pi gcl(h^{-1}(cl(h(A)))) = h^{-1}(cl(h(A)))$. Hence $h(f\pi gcl(A)) \leq cl(h(A))$.
(iii) $\Rightarrow$ (i). Let V be a fuzzy closed set in Y. Put $U = h^{-1}(V)$. Then $U \in I^X$. By (iii), $h(f\pi gcl(U)) \leq cl(h(U)) = cl(h(h^{-1}(V))) \leq clV = V$. Then $f\pi gcl(U) \leq h^{-1}(V) = U$ and so U is $f\pi g$-closed set in X. Hence h is $f\pi g$-continuous function.
(iii) $\Rightarrow$ (iv). Let $B \in I^Y$ and $A = h^{-1}(B)$. Then $A \in I^X$. By (iii), $h(f\pi gcl(A)) \leq cl(h(A))$ and so $h(f\pi gcl(h^{-1}(B))) \leq cl(h(h^{-1}(B))) \leq clB$. Hence $f\pi gcl(h^{-1}(B)) \leq h^{-1}(clB)$.
(iv) $\Rightarrow$ (iii). Let $A \in I^X$. Then $h(A) \in I^Y$. By (iv), $f\pi gcl(h^{-1}(h(A))) \leq h^{-1}(cl(h(A)))$. So $f\pi gcl(A) \leq f\pi gcl(h^{-1}(h(A))) \leq h^{-1}(cl(h(A)))$. Then $h(f\pi gcl(A)) \leq cl(h(A))$.

Remark 6.3. Composition of two $f\pi g$-continuous functions need not be so, as it seen from the following example.

Example 6.4. Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, A\}$ where $A(a) = 0.8, A(b) = 0.5, B(a) = 0.3, B(b) = 0.5$. Then (X,τ_1), (X,τ_2) and (X,τ_3) are fts's. Consider two identity functions $i_1 : (X,\tau_1) \to (X,\tau_2)$ and $i_2 : (X,\tau_2) \to (X,\tau_3)$. Then clearly i_1 and i_2 are $f\pi g$-continuous functions. Now $1_X \setminus A \in \tau_3^c$. So $(i_2 \circ i_1)^{-1}(1_X \setminus A) = 1_X \setminus A \leq B \in F\pi O(X,\tau_1)$. But $cl_{\tau_1}(1_X \setminus A) = 1_X \setminus B \not\leq B$ and so $1_X \setminus A$ is not $f\pi g$-closed set in (X,τ_1). Then $i_2 \circ i_1$ is not an $f\pi g$-continuous function.

Let us now recall the following definition from [14] for ready references.

Definition 6.5 [14]. A function $h : X \to Y$ is called fuzzy continuous function if $h^{-1}(V)$ is fuzzy closed set in X for every fuzzy closed set V in Y.

Remark 6.6. Since every fuzzy closed set is $f\pi g$-closed set, it is clear that fuzzy continuous function is $f\pi g$-continuous function. But the converse is not necessarily true, as the following example shows.

Example 6.7. Let $X = \{a,b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X,τ_1) and (X,τ_2) are fts's. Consider the identity function $i : (X,\tau_1) \to (X,\tau_2)$. Since every fuzzy set in (X,τ_1) is $f\pi g$-closed set in (X,τ), clearly i is $f\pi g$-continuous function. But $A \in \tau_2^c$, $i^{-1}(A) = A \notin \tau_1^c$. Hence i is not fuzzy continuous function.

Theorem 6.8. If $h_1 : X \to Y$ is $f\pi g$-continuous function and $h_2 : Y \to Z$ is fuzzy continuous function, then $h_2 \circ h_1 : X \to Z$ is $f\pi g$-continuous function.

Proof. Obvious.

Theorem 6.9. If a bijective function $h : X \to Y$ is $f\pi g$-continuous, fuzzy open function from an $f\pi g$-regular space X onto an fts Y, then Y is fuzzy regular space.

Proof. Let y_α be a fuzzy point in Y and F, a fuzzy closed set in Y with $y_\alpha \notin F$. As h is bijective, there exists unique $x \in X$ such that $h(x) = y$. So $h(x_\alpha) \notin F$ implies $x_\alpha \notin h^{-1}(F)$ where $h^{-1}(F)$ is $f\pi g$-closed set in X (as h is an $f\pi g$-continuous function). As X is $f\pi g$-regular space, there exist $U, V \in FRO(X)$ such that $x_\alpha \in U, h^{-1}(F) \leq V$ and $U \not{q} V$. Then $h(x_\alpha) \in h(U)$, $F = h(h^{-1}(F))$ (as h is bijective)$\leq h(V)$ and $h(U) \not{q} h(V)$ where $h(U)$ and $h(V)$ are fuzzy open sets in Y (as h is a fuzzy open function and fuzzy regular open set is fuzzy open set). (Indeed, $h(U) q h(V)$ implies the existence of $z \in Y$ such that $[h(U)](z) + [h(V)](z) > 1$, hence

$U(h^{-1}(z)) + V(h^{-1}(z)) > 1$ as h is bijective implies UqV, a contradiction). Hence Y is a fuzzy regular space.

In a similar manner we can state the following theorems easily the proofs of which are same as that of Theorem 6.9.

Theorem 6.10. If a bijective function $h : X \to Y$ is $f\pi g$-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal), fT_π-space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.11. If a bijective function $h : X \to Y$ is $f\pi g$-continuous, fuzzy open function from an $f\pi g$-normal space X onto an fts Y, then Y is fuzzy normal space.

Let us now recall the following definition from [12] for ready references.

Definition 6.12 [12]. A function $h : X \to Y$ is called fuzzy R-open function if $h(U) \in FRO(Y)$ for every $U \in FRO(X)$.

Now we state the following theorem easily the proof of which is same as that of Theorem 6.9.

Theorem 6.12. If a bijective function $h : X \to Y$ is $f\pi g$-continuous, fuzzy R-open function from an $f\pi g$-regular (resp., $f\pi g$-normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Definition 6.14. A function $h : X \to Y$ is called fuzzy π-generalized irresolute ($f\pi g$-irresolute, for short) function if $h^{-1}(U)$ is an $f\pi g$-open set in X for every $f\pi g$-open set U in Y.

Now we state the following two theorems easily the proofs of which are similar to that of Theorem 6.9.

Theorem 6.15. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute, fuzzy R-open function from an $f\pi g$-regular (resp., $f\pi g$-normal) space X onto an fts Y, then Y is $f\pi g$-regular (resp., $f\pi g$-normal) space.

Theorem 6.16. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute, fuzzy R-open function from an $f\pi g$-regular (resp., $f\pi g$-normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.17. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute, fuzzy open function from a fuzzy regular (resp., fuzzy normal), fT_π-space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.18. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute, fuzzy open function from an $f\pi g$-regular (resp., $f\pi g$-normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 6.19. A function $h : X \to Y$ is $f\pi g$-irresolute function iff for each fuzzy point x_α in X and each $f\pi g$-open nbd V in Y of $h(x_\alpha)$, there exists an $f\pi g$-open nbd U in X of x_α such that $h(U) \leq V$.

Proof. Let $h : X \to Y$ be an $f\pi g$-irresolute function. Let x_α be a fuzzy point in X and V be any $f\pi g$-open nbd of $h(x_\alpha)$ in Y. Then $h(x_\alpha) \in V$ implies $x_\alpha \in h^{-1}(V)$, but $h^{-1}(V)$ is an $f\pi g$-open set in X, therefore is an $f\pi g$-open nbd of x_α in X. Put $U = h^{-1}(V)$. Then U is an $f\pi g$-open nbd of x_α in X and $h(U) = h(h^{-1}(V)) \leq V$.

Conversely, let A be an $f\pi g$-open set in Y and x_α be any fuzzy point in X such that $x_\alpha \in h^{-1}(A)$. Then $h(x_\alpha) \in A$. By hypothesis, there exists an $f\pi g$-open nbd U of x_α in X such that $h(U) \leq A$ and so $x_\alpha \in U = f\pi gint(U) \leq f\pi gint(h^{-1}(A))$. Since x_α is taken arbitrarily and $h^{-1}(A)$ is the union of all fuzzy points in $h^{-1}(A)$, $h^{-1}(A) \leq f\pi gint(h^{-1}(A))$ implies $h^{-1}(A) = f\pi gint(h^{-1}(A))$. Then $h^{-1}(A)$ is $f\pi g$-open set in X. Hence h is an $f\pi g$-irresolute function.

Theorem 6.20. Let $h : X \to Y$ be an $f\pi g$-continuous function from X onto an fts Y and $A(\in I^X)$ be an $f\pi g$-compact set in X. Then $h(A)$ is a fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) set in Y.

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy cover of $h(A)$ by fuzzy open (resp., fuzzy open, fuzzy regular open) sets of Y. Then $h(A) \leq \bigcup_{\alpha\in\Lambda} U_\alpha$ implies $A \leq h^{-1}(\bigcup_{\alpha\in\Lambda} U_\alpha) = \bigcup_{\alpha\in\Lambda} h^{-1}(U_\alpha)$. Then $\mathcal{V} = \{h^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy cover of A by $f\pi g$-open sets of X as h is an $f\pi g$-continuous function. As A is $f\pi g$-compact set in X, there exists a finite subcollection Λ_0 of Λ such that $A \leq \bigcup_{\alpha\in\Lambda_0} h^{-1}(U_\alpha)$ implies $h(A) \leq h(\bigcup_{\alpha\in\Lambda_0} h^{-1}(U_\alpha) \leq \bigcup_{\alpha\in\Lambda_0} U_\alpha$. Hence $h(A)$ is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) set in Y.

Since fuzzy open set $f\pi g$-open, we can state the following theorems easily the proofs of which are same as that of Theorem 6.20.

Theorem 6.21. Let $h : X \to Y$ be an $f\pi g$-irresolute function from X onto an fts Y and $A(\in I^X)$ be an $f\pi g$-compact set in X. Then $h(A)$ is $f\pi g$-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) set in Y.

Theorem 6.22. Let $h : X \to Y$ be an $f\pi g$-continuous function from an $f\pi g$-compact space X onto an fts Y . Then Y is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.23. Let $h : X \to Y$ be an $f\pi g$-irresolute function from an $f\pi g$-compact space X onto an fts Y . Then Y is $f\pi g$-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.24. Let $h : X \to Y$ be an $f\pi g$-continuous function from a fuzzy compact, fT_π-space X onto an fts Y. Then Y is fuzzy compact (resp., fuzzy almost compact, fuzzy nearly compact) space.

Theorem 6.25. Let $h : X \to Y$ be an $f\pi g$-irresolute function from a fuzzy compact, fT_π-space X onto an fts Y. Then Y is $f\pi g$-compact (resp., fuzzy compact, fuzzy almost compact, fuzzy nearly compact) space.

Remark 6.26. It is clear from definitions that (i) $f\pi g$-irresolute function is $f\pi g$-continuous, but the converse may not be true, as the following example shows.
Also (ii) fuzzy continuity and $f\pi g$-irresoluteness are independent concepts as the following examples show.

Example 6.27. $f\pi g$-continuous function may not necessarily $f\pi g$-irresolute function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B, C\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.6, C(a) = 0.3, C(b) = 0.5$. then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_1)$ only and so $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_1). Hence i is $f\pi g$-continuous function. Now consider the fuzzy set D defined by $D(a) = 0.2, D(b) = 0.5$. Then $D < A \in F\pi O(X, \tau_2)$ and $cl_{\tau_2} D = A \leq A$. So D is $f\pi g$-closed set in (X, τ_2). Now $i^{-1}(D) = D < C \in F\pi O(X, \tau_1)$. But $cl_{\tau_1}(D) = 1_X \setminus C \not\leq C$. So D is not $f\pi g$-closed set in (X, τ_1). Hence i is not an $f\pi g$-irresolute function.

Example 6.28. Fuzzy continuity may not necessarily $f\pi g$-irresolute function
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2). Let us consider the fuzzy set B defined by $B(a) = 0.5, B(b) = 0.3$. Then B is $f\pi g$-closed set in (X, τ_2). Now $i^{-1}(B) = B < A \in F\pi O(X, \tau_1)$. But $cl_{\tau_1}(B) = 1_X \setminus A \not\leq A$. So B is not $f\pi g$-closed set in (X, τ_1). Hence i is not an $f\pi g$-irresolute function. But clearly i is fuzzy continuous function.

Example 6.29. $f\pi g$-irresoluteness may not necessarily imply fuzzy continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) =$

0.5. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1), clearly i is $f\pi g$-irresolute function. Also i is not fuzzy continuous function as $A \in \tau_2, i^{-1}(A) = A \notin \tau_1$.

Theorem 6.30. Let $h : X \to Y$ be an $f\pi g$-continuous function where Y is an fT_π-space. Then h is $f\pi g$-irresolute function.

Proof. Obvious.

Note 6.31. It is clear from definition that composition of two $f\pi g$-irresolute functions is $f\pi g$-irresolute function. Again if $h_1 : X \to Y$ is $f\pi g$-irresolute function and $h_2 : Y \to Z$ is $f\pi g$-continuous function, then $h_2 \circ h_1 : X \to Z$ is an $f\pi g$-continuous function.

To establish the mutual relationship of $f\pi g$-continuous function with the functions defined in [3, 5, 6, 7, 9, 10, 11], we first recall the definitions of the functions defined in [3, 5, 6, 7, 9, 10, 11].

Definition 6.32. Let $h : (X, \tau_1) \to (Y, \tau_2)$ be a function. Then h is called

(i) fg-continuous function [3] if $h^{-1}(V)$ is fg-closed set in X for every $V \in \tau_2^c$,

(ii) $fg\beta$-continuous function [7] if $h^{-1}(V)$ is $fg\beta$-closed set in X for every $V \in \tau_2^c$,

(iii) $f\beta g$-continuous function [7] if $h^{-1}(V)$ is $f\beta g$-closed set in X for every $V \in \tau_2^c$,

(iv) fgp-continuous function [3] if $h^{-1}(V)$ is fgp-closed set in X for every $V \in \tau_2^c$,

(v) fpg-continuous function [3] if $h^{-1}(V)$ is fpg-closed set in X for every $V \in \tau_2^c$,

(vi) $fg\alpha$-continuous function [3] if $h^{-1}(V)$ is $fg\alpha$-closed set in X for every $V \in \tau_2^c$,

(vii) $f\alpha g$-continuous function [3] if $h^{-1}(V)$ is $f\alpha g$-closed set in X for every $V \in \tau_2^c$,

(viii) fgs-continuous function [3] if if $h^{-1}(V)$ is fgs-closed set in X for every $V \in \tau_2^c$,

(ix) fsg-continuous function [3] if $h^{-1}(V)$ is fsg-closed set in X for every $V \in \tau_2^c$,

(x) fgs^*-continuous function [5] if $h^{-1}(V)$ is fgs^*-closed set in X for every $V \in \tau_2^c$,

(xi) fs^*g-continuous function [6] if $h^{-1}(V)$ is fs^*g-closed set in X for every $V \in \tau_2^c$,

(xii) $fg\gamma$-continuous function [10] if $h^{-1}(V)$ is $fg\gamma$-closed set in X for every $V \in \tau_2^c$,

(xiii) $fg\gamma^*$-continuous function [11] if $h^{-1}(V)$ is $fg\gamma^*$-closed set in X for every $V \in \tau_2^c$,
(xiv) $frwg$-continuous function [9] if $h^{-1}(V)$ is $frwg$-closed set in X for every $V \in \tau_2^c$,
(xv) $fswg$-continuous function [9] if $h^{-1}(V)$ is $fswg$-closed set in X for every $V \in \tau_2^c$,
(xvi) fmg-continuous function [9] if $h^{-1}(V)$ is fmg-closed set in X for every $V \in \tau_2^c$,
(xvii) fwg-continuous function [9] if $h^{-1}(V)$ is fwg-closed set in X for every $V \in \tau_2^c$.

Remark 6.33. (i) fg-continuity implies $f\pi g$-continuity, fgs^*-continuity implies $f\pi g$-continuity, fs^*g-continuity implies $f\pi g$-continuity and $f\pi g$-continuity implies $frwg$-continuity, $f\pi g$-irresoluteness implies $frwg$-continuity. But the reverse implications are not necessarily true, in general, as the following examples show.
(ii) $f\pi g$-continuity is an independent concept of $fg\beta$-continuity, $f\beta g$-continuity, fgp-continuity, fpg-continuity, $fg\alpha$-continuity, $f\alpha g$-continuity, fgs-continuity, fsg-continuity, $fg\gamma$-continuity, $fg\gamma^*$-continuity, $fswg$-continuity, fmg-continuity, fwg-continuity.

Example 6.34. $f\pi g$-continuity may not necessarily fg-continuity, fgs-continuity, fsg-continuity, fgs^*-continuity, fs^*g-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.6, A(b) = 0.5, B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1 \to (X, \tau_2)$. Since $F\pi O(X, \tau_1) = \{0_X, 1_X\}$, every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1) implies i is $f\pi g$-continuous function. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < B \in \tau_1$ (also $B \in FSO(X, \tau_1) = \{0_X, 1_X, U\}$ where $U \geq B$) and $cl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$ (resp., $scl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$) implies $1_X \setminus A$ is not fg-closed (resp., fgs-closed, fsg-closed, fgs^*-closed) set in (X, τ_1). Hence i is not fg-continuous (resp., fgs-continuous, fsg-continuous, fgs^*-continuous) function. Again B is fg-open set in (X, τ_1). Then $1_X \setminus A < B$, but $cl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not fs^*g-closed set in (X, τ_1). Hence i is not fs^*g-continuous function.

Example 6.35. $f\pi g$-continuity may not necessarily $fg\beta$-continuity, $f\beta g$-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.4, B(a) = B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1), i is clearly $f\pi g$-continuous

function. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A \leq B \in \tau_1$ (also $B \in F\beta O(X, \tau_1) = \{0_X, 1_X, U\}$ where $U \geq B$). But $\beta cl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$. Hence $1_X \setminus A$ is not $fg\beta$-closed (resp., $f\beta g$-closed) set in (X, τ_1) and so i is not $fg\beta$-continuous (resp., $f\beta g$-continuous) function.

Example 6.36. $f\pi g$-continuity may not necessarily $fswg$-continuity, $fg\gamma^*$-continuity, fpg-continuity, fmg-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.4, B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and $(X.\tau_2$ are fts's. Consider the identity function $i : (X.\tau_1) \to (X, \tau_2)$. As every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1), clearly i is $f\pi g$-continuous function. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < B \in FSO(X, \tau_1)$. But $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 1_X \not\leq B$. So $1_X \setminus A$ is not $fswg$-closed set in (X, τ_1) and so i is not $fswg$-continuous function. Again $\gamma cl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$ implies $1_X \setminus A$ is not $fg\gamma^*$-closed set in (X, τ_1) and so i is not $fg\gamma^*$-continuous function. Again $1_X \setminus A \in FPO(X, \tau_1)$ and so $1_X \setminus A \leq 1_X \setminus A \in FPO(X, \tau_1)$. But $1_X \setminus A \notin FPC(X, \tau_1)$ and so $pcl_{\tau_1}(1_X \setminus A) \not\leq 1_X \setminus A$. So $1_X \setminus A$ is not fpg-closed set in (X, τ_1). Hence i is not fpg-continuous function. Again $1_X \setminus A$ is fg-open set in (X, τ_1) and so $1_X \setminus A \leq 1_X \setminus A$. But $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 1_X \not\leq 1_X \setminus A$. Then $1_X \setminus A$ is not fmg-closed set in (X, τ_1). Consequently, i is not fmg-continuous function.

Example 6.37. $f\pi g$-continuity may not necessarily $fg\alpha$-continuity, $f\alpha g$-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. As every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1), clearly i is $f\pi g$-continuous function. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A$. Here $1_X \setminus A < B \in \tau_1$ (also $B \in F\alpha O(X, \tau_1) = \{0_X, 1_X, U\}$ where $U \geq B$). So $\alpha cl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$. Then $1_X \setminus A$ is not $fg\alpha$-closed (resp., $f\alpha g$-closed) set in (X, τ_1). Hence i is not $fg\alpha$-continuous (resp., $f\alpha g$-continuous) function.

Example 6.38. $f\pi g$-continuity may not necessarily fwg-continuity
Consider Example 6.27. Here i is $f\pi g$-continuous function. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < B \in \tau_1$. But $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 1_X \setminus C \not\leq B$ and so $1_X \setminus A$ is not fwg-closed set in (X, τ_1). Hence i is not fwg-continuous function.

Example 6.39. $f\pi g$-continuity may not necessarily $fg\gamma$-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B, C\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) =$

$0.5, A(b) = 0.45, B(a) = 0.5, B(b) = 0.6, C(a) = 0.3, C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_1)$ only and so $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_1) implies i is $f\pi g$-continuous function. But $1_X \setminus A < B \in \tau_1$ and $\gamma cl_{\tau_1}(1_X \setminus A) = 1_X \setminus C \not\leq B$. So $1_X \setminus A$ is not $fg\gamma$-closed set in (X, τ_1). Hence i is not $fg\gamma$-continuous function.

Example 6.40. $f\pi g$-continuity may not necessarily fgp-continuity Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B, C\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5, B(a) = 0.5, B(b) = 0.6, C(a) = 0.3, C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < 1_X \in F\pi O(X, \tau_1)$ only, clearly $1_X \setminus A$ is $f\pi g$-closed set in (X, τ_1). So i is $f\pi g$-continuous function. Now $1_X \setminus A < B \in \tau_1$. But $pcl_{\tau_1}(1_X \setminus A) = 1_X \not\leq B$. So $1_X \setminus A$ is not fgp-closed set in (X, τ_1). Hence i is not fgp-continuous function.

Example 6.41. $frwg$-continuity, $fg\gamma$-continuity, $fg\gamma^*$-continuity, fgp-continuity, fpg-continuity, $fg\beta$-continuity, $f\beta g$-continuity, $fswg$-continuity, fmg-continuity, fwg-continuity may not necessarily $f\pi g$-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.7, B(a) = 0.5, B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_2)$. Now $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A < B \in F\pi O(X, \tau_1)$, but $cl_{\tau_1}(1_X \setminus A) = 1_X \setminus B \not\leq B$. So $1_X \setminus A$ is not $f\pi g$-closed set in (X, τ_1). Hence i is not $f\pi g$-continuous function. Now $1_X \setminus A < B \in FRO(X, \tau_1)$ and $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 0_X < B$. Then $1_X \setminus A$ is $frwg$-closed set in (X, τ_1). Hence i is $frwg$-continuous function. Now $\gamma cl_{\tau_1}(1_X \setminus A) = 1_X \setminus A$ implies $1_X \setminus A \in F\gamma C(X, \tau_1)$. Then $1_X \setminus A$ is $fg\gamma$-closed as well as $fg\gamma^*$-closed set in (X, τ_1). So i is $fg\gamma$-continuous as well as $fg\gamma^*$-continuous function. Again $1_X \setminus A \in FPC(X, \tau_1)$ implies $1_X \setminus A$ is fgp-closed as well as fpg-closed set in (X, τ_1). Hence i is fgp-continuous as well as fpg-continuous function. Also $1_X \setminus A \in F\beta C(X, \tau_1)$. So $1_X \setminus A$ is $fg\beta$-closed as well as $f\beta g$-closed set in (X, τ_1). Hence i is $fg\beta$-continuous as well as $f\beta g$-continuous function. Now $1_X \setminus A < B \in FSO(X, \tau_1)$ and $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 0_X < B$. So $1_X \setminus A$ is $fswg$-closed set in (X, τ_1). Then i is $fswg$-continuous function. Again $1_X \setminus A$ is fg-open set in (X, τ_1) and so $1_X \setminus A \leq 1_X \setminus A$ and $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 0_X < 1_X \setminus A$ implies $1_X \setminus A$ is fmg-closed set in (X, τ_1) implies i is fmg-continuous function. Again $1_X \setminus A < B \in \tau_1$ and $cl_{\tau_1}(int_{\tau_1}(1_X \setminus A)) = 0_X < B$

and so $1_X \setminus A$ is fwg-closed set in (X, τ_1). Hence i is fwg-continuous function.

Example 6.42. $fg\alpha$-continuity, $f\alpha g$-continuity, fgs-continuity, fsg-continuity may not necessarily $f\pi g$-continuity

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, B, C\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.8, A(b) = 0.6, B(a) = 0.5, B(b) = 0.6, C(a) = 0.3, C(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Here $1_X \setminus A \in \tau_2^c$, $i^{-1}(1_X \setminus A) = 1_X \setminus A$. As $cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}(1_X \setminus A))) = 0_X < 1_X \setminus A$, $1_X \setminus A \in F\alpha C(X, \tau_1)$. So $1_X \setminus A$ is $fg\alpha$-closed as well as $f\alpha g$-closed set in (X, τ_1). Hence i is $fg\alpha$-continuous as well as $f\alpha g$-continuous function. Also as $int_{\tau_1}(cl_{\tau_1}(1_X \setminus A)) = 0_X < 1_X \setminus A$, so $1_X \setminus A \in FSC(X, \tau_1)$. Then $1_X \setminus A$ is fgs-closed as well as fsg-closed set in (X, τ_1). Hence i is fgs-continuous as well as sfg-continuous function. But $1_X \setminus A < C \in F\pi O(X, \tau_1)$ and $cl_{\tau_1}(1_X \setminus A) = 1_X \setminus B \not\leq C$. Then $1_X \setminus A$ is not $f\pi g$-closed set in (X, τ_1). Hence i is not $f\pi g$-continuous function.

Example 6.43. $frwg$-continuity may not necessarily imply $f\pi g$-irresoluteness

Consider Example 6.40. Here i is $frwg$-continuous function. Here every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2). Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.3$. Then C is $f\pi g$-closed set in (X, τ_2). Now $i^{-1}(C) = C < B \in F\pi O(X, \tau_1)$. But $cl_{\tau_1}(C) = 1_X \setminus B \not\leq C$ and so C is not $f\pi g$-closed set in (X, τ_1). Hence i is not $f\pi g$-irresolute function.

7. $f\pi g$-T_2 Space

A new type of fuzzy T_2-property is introduced here. Then we introduce a strong form of $f\pi g$-continuity which implies $f\pi g$-continuity and the converse is true on fT_π-space.

We first recall the definition and theorem from [20, 21] for ready references.

Definition 7.1 [20]. An fts (X, τ) is called fuzzy T_2-space if for any two distinct fuzzy points x_α and y_β; when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1, U_1 \not{q} V_1$ and $x_\alpha q U_2, y_\beta \in V_2, U_2 \not{q} V_2$; when $x = y$ and $\alpha < \beta$ (say), there exist fuzzy open sets U and V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Theorem 7.2 [21]. An fts (X, τ) is fuzzy T_2-space if and only if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist fuzzy open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), x_α has a fuzzy open nbd U and y_β has a fuzzy open q-nbd V such that $U \not{q} V$.

Now we introduce the following concept.

Definition 7.3. An fts (X, τ) is called fuzzy π-generalized T_2 space ($f\pi g$-T_2 Space, for short), if for any two distinct fuzzy points x_α and y_β in X; when $x \neq y$, there exist $f\pi g$-open sets U, V in X such that $x_\alpha q U$, $y_\beta q V$ and $U \not{q} V$; when $x = y$ and $\alpha < \beta$ (say), x_α has an $f\pi g$-open nbd U and y_β has an $f\pi g$-open q-nbd V such that $U \not{q} V$.

Theorem 7.4. If an injective function $h : X \to Y$ is $f\pi g$-continuous function from an fts X onto a fuzzy T_2-space Y, then X is $f\pi g$-T_2 space.

Proof. Let x_α and y_β be two distinct fuzzy points in X. Then $h(x_\alpha)$ ($= z_\alpha$, say) and $h(y_\beta)$ ($= w_\beta$, say) are two distinct fuzzy points in Y.

Case I. Suppose $x \neq y$. Then $z \neq w$. Since Y is fuzzy T_2-space, there exist fuzzy open sets U, V in Y such that $z_\alpha q U, w_\beta q V$ and $U \not{q} V$. As h is $f\pi g$-continuous function, $h^{-1}(U)$ and $h^{-1}(V)$ are $f\pi g$-open sets in X with $x_\alpha q h^{-1}(U), y_\beta q h^{-1}(V)$ and $h^{-1}(U) \not{q} h^{-1}(V)$ [Indeed, $z_\alpha q U$ imply $U(z) + \alpha > 1$, so $U(h(x)) + \alpha > 1$. Then $[h^{-1}(U)](x) + \alpha > 1$. Now $x_\alpha q h^{-1}(U)$. Again, $h^{-1}(U) q h^{-1}(V)$. Then there exists $t \in X$ such that $[h^{-1}(U)](t) + [h^{-1}(V)](t) > 1$ implies $U(h(t)) + V(h(t)) > 1$. So UqV, a contradiction].

Case II. Suppose $x = y$ and $\alpha < \beta$ (say). Then $z = w$ and $\alpha < \beta$. Since Y is fuzzy T_2-space, there exist a fuzzy open nbd U of x_α and a fuzzy open q-nbd V of w_β such that $U \not{q} V$. Then $U(z) \geq \alpha$. So $[h^{-1}(U)](x) \geq \alpha$. Then $x_\alpha \in h^{-1}(U), y_\beta q h^{-1}(V)$ and $h^{-1}(U) \not{q} h^{-1}(V)$ where $h^{-1}(U)$ and $h^{-1}(V)$ are $f\pi g$-open sets in X as h is $f\pi g$-continuous function. Consequently, X is $f\pi g$-T_2-space.

Similarly we can state the following theorems easily the proofs of which are similar to that of Theorem 7.4.

Theorem 7.5. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute function from an fts X onto an $f\pi g$-T_2 space Y, then X is $f\pi g$-T_2 space.

Theorem 7.6. If a bijective function $h : X \to Y$ is $f\pi g$-continuous function from an fT_π-space X onto a fuzzy T_2-space Y, then X is fuzzy T_2 space.

Theorem 7.7. If a bijective function $h : X \to Y$ is $f\pi g$-irresolute function from an fT_π-space X onto an $f\pi g$-T_2 space Y, then X is fuzzy T_2 space.

Theorem 7.8. If a bijective function $h : X \to Y$ is $f\pi g$-open function from a fuzzy T_2-space X onto an fts Y, then Y is $f\pi g$-T_2-space.

Theorem 7.9. If a bijective function $h : X \to Y$ is $f\pi g$-open function from a fuzzy T_2-space X onto an fT_π-space Y, then Y is fuzzy T_2-space.

Now we introduce the following concept.

Definition 7.10. A function $h : X \to Y$ is called strongly fuzzy π-generalized continuous (strongly $f\pi g$-continuous, for short) function if $h^{-1}(V)$ is fuzzy closed set in X for every $f\pi g$-closed set V in Y.

Remark 7.11. It is clear from above discussion that strongly $f\pi g$-continuous function implies fuzzy continuous, $f\pi g$-continuous and $f\pi g$-irresolute functions. But the converses are not true, in general, as the following examples show.

Example 7.12. Fuzzy continuity, $f\pi g$-continuity may not necessarily imply strongly $f\pi g$-continuity
Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since 0_X and 1_X are the only fuzzy closed sets in (X, τ_2), clearly i is fuzzy continuous as well as $f\pi g$-continuous function. As every fuzzy set in (X, τ_2) is $f\pi g$-closed set in (X, τ_2), considering the fuzzy set B, defined by $B(a) = 0.5, B(b) = 0.3$, B is $f\pi g$-closed set in (X, τ_2). Now $i^{-1}(B) = B \notin \tau_1^c$. Hence i is not strongly $f\pi g$-continuous function.

Example 7.13. $f\pi g$-irresoluteness may not necessarily imply strongly $f\pi g$-continuity
Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is $f\pi g$-closed set in (X, τ_1), clearly i is $f\pi g$-irresolute function. Now $A \in \tau_2$ is $f\pi g$-closed set in (X, τ_2). $i^{-1}(A) = A \notin \tau_1^c$. Hence i is not strongly $f\pi g$-continuous function.

Remark 7.14. Clearly composition of two $f\pi g$-irresolute functions is also so.

Theorem 7.15. If $h_1 : X \to Y$ is strongly $f\pi g$-continuous function and $h_2 : Y \to Z$ is $f\pi g$-continuous function, then $h_2 \circ h_1 : X \to Z$ is fuzzy continuous function.

Proof. Obvious.

Since fuzzy open set is fuzzy $f\pi g$-open set, we have the following theorems.

Theorem 7.16. If a bijective function $h : X \to Y$ is strongly $f\pi g$-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal) space X onto an fts Y, then Y is $f\pi g$-regular (resp., $f\pi g$-normal) space.

Theorem 7.17. If a bijective function $h : X \to Y$ is strongly $f\pi g$-continuous, fuzzy open function from a fuzzy regular (resp., fuzzy normal) space X onto an fts Y, then Y is fuzzy regular (resp., fuzzy normal) space.

Theorem 7.18. If a bijective function $h : X \to Y$ is strongly $f\pi g$-continuous function from an fts X onto an $f\pi g$-T_2 space Y, then X is fuzzy T_2 space.

Theorem 7.19. If a bijective function $h : X \to Y$ is strongly $f\pi g$-continuous function from a fuzzy compact space X onto an fts Y, then Y is $f\pi g$-compact (resp., fuzzy almost compact, fuzzy nearly compact) space.

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