**"Vasile Alecsandri" University of Bacău**
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# DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA BHATIA AND KITTANEH-MANASRAH RESULTS

SILVESTRU SEVER DRAGOMIR

**Abstract.** In this paper we prove among others that, if $A$ and $B$ are positive definite matrices, then

$$
\begin{aligned}
0 &\leq \int_0^1 \left[\det\left((1-t)A+tB\right)\right]^{-1} dt - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1} \\
&\leq \frac{1}{3}\left[\frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right] \\
&\leq \frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \int_0^1 \left[\det\left((1-t)A+tB\right)\right]^{-1} dt \\
&\leq \frac{4}{3}\left[\frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right].
\end{aligned}
$$

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## 1. Introduction

We call Heinz means, the mean defined by

$$H_\nu(a,b) := \frac{1}{2}\left(a^{1-\nu}b^\nu + a^\nu b^{1-\nu}\right)$$

We call *Heron means*, the means defined by

$$F_\alpha(a,b) := (1-\alpha)\sqrt{ab} + \alpha\frac{a+b}{2},$$

where $a$, $b > 0$ and $\alpha \in [0,1]$.

In [2], Bhatia obtained the following interesting inequality between the Heinz and Heron means

$$H_\nu(a,b) \le F_{(2\nu-1)^2}(a,b) \tag{1.1}$$

where $a$, $b > 0$ and $\alpha \in [0,1]$.

This inequality can be written as

$$(0 \le)\, H_\nu(a,b) - \sqrt{ab} \le (2\nu-1)^2\left(\frac{a+b}{2} - \sqrt{ab}\right), \tag{1.2}$$

where $a$, $b > 0$ and $\alpha \in [0,1]$.

Kittaneh and Manasrah [3], [4] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le R\left(\sqrt{a}-\sqrt{b}\right)^2 \tag{1.3}$$

where $a,b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu,\nu\}$ and $R = \max\{1-\nu,\nu\}$.

If we replace in (1.3) $\nu$ with $1-\nu$, add the obtained inequalities and divide by 2, then we get

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 \le \frac{a+b}{2} - H_\nu(a,b) \le R\left(\sqrt{a}-\sqrt{b}\right)^2, \tag{1.4}$$

where $a,b > 0$, $\nu \in [0,1]$.

A real square matrix $A = (a_{ij})$, $i$, $j = 1,...,n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i$, $j = 1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^n a_{ij}x_ix_j$ is positive for all $x = (x_1,...,x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix $A$ to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det(a_{ij}),\ i,\ j = 1,...,k;\ k = 1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

(1.5)
$$\begin{aligned} J_n(A) &:= \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx \\ &= \frac{\pi^{n/2}}{[\det(A)]^{1/2}}, \end{aligned}$$

where $A$ is a positive definite matrix of order $n$ and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.5) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det((1-\lambda)A + \lambda B) \geq [\det(A)]^{1-\lambda} [\det(B)]^{\lambda} \tag{1.6}$$

for any positive definite matrices $A$, $B$ and $\lambda \in [0, 1]$.

By mathematical induction we can get a generalization of (1.6) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^{m} \lambda_j A_j\right) \geq \prod_{j=1}^{m} [\det(A_j)]^{\lambda_j}, \ m \geq 2, \tag{1.7}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

If we write (1.7) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m} \lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m} \left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m} [\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m} [\det(A_j)]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m} \lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.8}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (1.5) one can also prove the result, see [11, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k}) \det\left(A_{(k+1)n}\right), \ k = 1, ..., n; \tag{1.9}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i, \ j = r, ..., s.$$

In particular,

$$\det(A) \le a_{11}a_{22}...a_{nn}. \tag{1.10}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n} \ge [\det(A)]^{1/n} + [\det(B)]^{1/n} \tag{1.11}$$

for $A$, $B$ positive definite matrices of order $n$. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if $A$ and $B$ are positive definite matrices, then

$$\begin{aligned}
0 &\le \int_0^1 [\det((1-t)A+tB)]^{-1}dt - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1} \\
&\le \frac{1}{3}\left[\frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right] \\
&\le \frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \int_0^1 [\det((1-t)A+tB)]^{-1}dt \\
&\le \frac{4}{3}\left[\frac{1}{2}\left([\det(A)]^{-1}+[\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right].
\end{aligned}$$

## 2. Main Results

Our first main result is as follows:

**Theorem 1.** *For any positive definite matrices $A$, $B$ and $t \in [0,1]$,*

(2.1)

$$\begin{aligned}
0 &\le \frac{1}{2}\left[[\det((1-t)A+tB)]^{-1/2} + [\det(tA+(1-t)B)]^{-1/2}\right] \\
&\quad - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2} \\
&\le (2t-1)^2\left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right]
\end{aligned}$$

*and*

(2.2)

$$\begin{aligned}
&0 \le r\left[\frac{1}{2}\left([\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right] \\
&\quad \le \frac{1}{2}\left([\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right) \\
&\quad - \frac{1}{2}\left[[\det((1-t)A + tB)]^{-1/2} + [\det(tA + (1-t)B)]^{-1/2}\right] \\
&\quad \le R\left[\frac{1}{2}\left([\det(A)]^{-1/2} + [\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right],
\end{aligned}$$

*where* $r = \min\{1-t, t\}$ *and* $R = \max\{1-t, t\}$.

*Proof.* From (1.2) we get for $a = \exp(-\langle Ax, x\rangle)$, $b = \exp(-\langle Bx, x\rangle)$ with $x \in \mathbb{R}^n$ that

(2.3)

$$\begin{aligned}
&0 \le \frac{1}{2}\left(\exp(-\langle((1-t)A + tB)x, x\rangle) + \exp(-\langle(tA + (1-t)B)x, x\rangle)\right) \\
&\quad - \exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle\right) \\
&\quad \le (2t-1)^2 \\
&\quad \times \left(\frac{1}{2}\left(\exp(-\langle Ax, x\rangle) + \exp(-\langle Bx, x\rangle)\right) - \exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle\right)\right).
\end{aligned}$$

By taking the integral on $\mathbb{R}^n$ in (2.3), then we get

$$\begin{aligned}
&0 \le \frac{1}{2}\left(\int_{\mathbb{R}^n}\exp(-\langle((1-t)A + tB)x, x\rangle)\,dx\right. \\
&\quad \left. + \int_{\mathbb{R}^n}\exp(-\langle(tA + (1-t)B)x, x\rangle)\,dx\right) - \int_{\mathbb{R}^n}\exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle\right)dx \\
&\quad \le (2t-1)^2\left(\frac{1}{2}\left(\int_{\mathbb{R}^n}\exp(-\langle Ax, x\rangle)\,dx + \int_{\mathbb{R}^n}\exp(-\langle Bx, x\rangle)\,dx\right)\right. \\
&\quad \left. - \exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle\right)\right)
\end{aligned}$$

and by (1.5) we obtain

$$
\begin{aligned}
0 &\leq \frac{1}{2}\left[J_n\left((1-t)A+tB\right)+J_n\left(tA+(1-t)B\right)\right]-J_n\left(\frac{A+B}{2}\right)\\
&\leq (2t-1)^2\left[\frac{1}{2}\left(J_n(A)+J_n(B)\right)-J_n\left(\frac{A+B}{2}\right)\right]
\end{aligned}
$$

which by (1.5), we derive (2.1).

From (1.4) we have

$$
\begin{aligned}
&r\left(\frac{1}{2}\left(\exp\left(-\langle Ax,x\rangle\right)+\exp\left(-\langle Bx,x\rangle\right)\right)-\exp\left(-\left\langle\frac{A+B}{2}x,x\right\rangle\right)\right)\\
&\leq \frac{1}{2}\left(\exp\left(-\langle Ax,x\rangle\right)+\exp\left(-\langle Bx,x\rangle\right)\right)\\
&-\frac{1}{2}\left(\exp\left(-\langle((1-t)A+tB)x,x\rangle\right)+\exp\left(-\langle(tA+(1-t)B)x,x\rangle\right)\right)\\
&\leq R\left(\frac{1}{2}\left(\exp\left(-\langle Ax,x\rangle\right)+\exp\left(-\langle Bx,x\rangle\right)\right)-\exp\left(-\left\langle\frac{A+B}{2}x,x\right\rangle\right)\right),
\end{aligned}
$$

for $x\in\mathbb{R}^n$.

By taking the integral on $\mathbb{R}^n$ we deduce the desired result (2.2). ∎

**Corollary 1.** *For any positive definite matrices $A$, $B$ we have*

$$
\begin{aligned}
0 &\leq \int_0^1\left[\det\left((1-t)A+tB\right)\right]^{-1/2}dt-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\\
&\leq \frac{1}{3}\left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right)-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right]
\end{aligned}
\tag{2.4}
$$

*and*

$$
\begin{aligned}
&\frac{1}{3}\left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right)-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right]\\
&\leq \frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right)-\int_0^1\left[\det\left((1-t)A+tB\right)\right]^{-1/2}dt\\
&\leq \frac{4}{3}\left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right)-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right].
\end{aligned}
\tag{2.5}
$$

*Proof.* Taking the integral over $t$ in (2.1), we get

(2.6)
$$0 \le \frac{1}{2}\left[\int_0^1 [\det((1-t)A+tB)]^{-1/2}\,dt + \int_0^1 [\det(tA+(1-t)B)]^{-1/2}\,dt\right] - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}$$
$$\le \left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right] \times \int_0^1 (2t-1)^2\,dt.$$

Observe that

$$\int_0^1 [\det((1-t)A+tB)]^{-1/2}\,dt = \int_0^1 [\det(tA+(1-t)B)]^{-1/2}\,dt$$

and

$$\int_0^1 (2t-1)^2\,dt = \frac{1}{3},$$

then by (2.6) we obtain (2.4).

From (2.2) we get

(2.7)
$$0 \le \left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right] \times \int_0^1 \min\{1-t,t\}\,dt$$
$$\le \frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right) - \frac{1}{2}\left[\int_0^1 [\det((1-t)A+tB)]^{-1/2}\,dt + \int_0^1 [\det(tA+(1-t)B)]^{-1/2}\,dt\right]$$
$$\le \left[\frac{1}{2}\left([\det(A)]^{-1/2}+[\det(B)]^{-1/2}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1/2}\right] \times \int_0^1 \max\{1-t,t\}\,dt.$$

Since

$$\int_0^1 \min\{1-t,t\}\,dt = \frac{1}{3} \text{ and } \int_0^1 \max\{1-t,t\}\,dt = \frac{3}{4},$$

hence by (2.7) we get (2.5). ∎

If we take the square in the representation (1.5), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)dxdy,\end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)dxdy = \frac{\pi^n}{\det(A)} \tag{2.8}$$

for $A$ a positive definite matrix of order $n$ and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

**Theorem 2.** *For any positive definite matrices $A$, $B$ and $t \in [0,1]$,*

(2.9)

$$\begin{aligned}0 &\le \frac{1}{2}\left[[\det((1-t)A + tB)]^{-1} + [\det(tA + (1-t)B)]^{-1}\right] \\ &\quad - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1} \\ &\le (2t-1)^2\left[\frac{1}{2}\left([\det(A)]^{-1} + [\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right]\end{aligned}$$

*and*

$$(2.10)\quad 0 \le r\left[\frac{1}{2}\left([\det(A)]^{-1} + [\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right]$$
$$\le \frac{1}{2}\left([\det(A)]^{-1} + [\det(B)]^{-1}\right)$$
$$- \frac{1}{2}\left[[\det((1-t)A + tB)]^{-1} + [\det(tA + (1-t)B)]^{-1}\right]$$
$$\le R\left[\frac{1}{2}\left([\det(A)]^{-1} + [\det(B)]^{-1}\right) - \left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right],$$

*where* $r = \min\{1-t, t\}$ *and* $R = \max\{1-t, t\}$.

*Proof.* From (1.2) we get for $a = \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)$, $b = \exp(-\langle Bx, x\rangle - \langle Bx, x\rangle)$ with $x, y \in \mathbb{R}^n$ that

$$(2.11)$$
$$0 \le \frac{1}{2}[\exp(-\langle((1-t)A + tB)x, x\rangle - \langle((1-t)A + tB)y, y\rangle)$$
$$+ \exp(-\langle(tA + (1-t)B)x, x\rangle - \langle(tA + (1-t)B)y, y\rangle)]$$
$$- \exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle - \left\langle\frac{A+B}{2}y, y\right\rangle\right)$$
$$\le (2t-1)^2$$
$$\times\left[\frac{1}{2}(\exp(-\langle Ax, x\rangle - \langle Ay, y\rangle) + \exp(-\langle Bx, x\rangle - \langle By, y\rangle))\right.$$
$$\left.- \exp\left(-\left\langle\frac{A+B}{2}x, x\right\rangle - \left\langle\frac{A+B}{2}y, xy\right\rangle\right)\right].$$

If we take the double integral on $\mathbb{R}^n \times \mathbb{R}^n$, the we get

$$
\begin{aligned}
&\frac{1}{2}\left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle\left(\left(1-t\right)A+tB\right)x,x\right\rangle-\left\langle\left(\left(1-t\right)A+tB\right)y,y\right\rangle\right)dxdy\right.\\
&\left.+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle\left(tA+\left(1-t\right)B\right)x,x\right\rangle-\left\langle\left(tA+\left(1-t\right)B\right)y,y\right\rangle\right)dxdy\right]\\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle\frac{A+B}{2}x,x\right\rangle-\left\langle\frac{A+B}{2}y,y\right\rangle\right)dxdy\\
&\leq\left(2t-1\right)^2\\
&\times\left[\frac{1}{2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle Ax,x\right\rangle-\left\langle Ay,y\right\rangle\right)dxdy\right.\right.\\
&\left.+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle Bx,x\right\rangle-\left\langle By,y\right\rangle\right)dxdy\right)\\
&\left.-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\left\langle\frac{A+B}{2}x,x\right\rangle-\left\langle\frac{A+B}{2}y,xy\right\rangle\right)dxdy\right].
\end{aligned}
\tag{2.12}
$$

By utilising the representation (2.8) we get

$$
\begin{aligned}
0&\leq\frac{1}{2}\left[K_n\left(\left(1-t\right)A+tB\right)+K_n\left(tA+\left(1-t\right)B\right)\right]-K_n\left(\frac{A+B}{2}\right)\\
&\leq\left(2t-1\right)^2\left(\frac{1}{2}\left[K_n\left(A\right)+K_n\left(B\right)\right]-K_n\left(\frac{A+B}{2}\right)\right),
\end{aligned}
$$

which is equivalent to (2.9).

The inequality (2.10) follows in a similar way from (1.4). ∎

**Corollary 2.** *For any positive definite matrices $A$, $B$ we have*

$$
\begin{aligned}
0&\leq\int_0^1\left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1}dt-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\\
&\leq\frac{1}{3}\left[\frac{1}{2}\left(\left[\det\left(A\right)\right]^{-1}+\left[\det\left(B\right)\right]^{-1}\right)-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right]\\
&\leq\frac{1}{2}\left(\left[\det\left(A\right)\right]^{-1}+\left[\det\left(B\right)\right]^{-1}\right)-\int_0^1\left[\det\left(\left(1-t\right)A+tB\right)\right]^{-1}dt\\
&\leq\frac{4}{3}\left[\frac{1}{2}\left(\left[\det\left(A\right)\right]^{-1}+\left[\det\left(B\right)\right]^{-1}\right)-\left[\det\left(\frac{A+B}{2}\right)\right]^{-1}\right].
\end{aligned}
\tag{2.13}
$$

## 3. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all $i$, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix $H$, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{3.1}$$

where $z = x + iy$ and $dx$ and $dy$ denote integration over real $n$-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

On making use of a similar argument to the one in Theorem 2 for the representation $K_n(\cdot)$ we can state the following result as well:

**Theorem 3.** *For any positive definite Hrmitian matrices $H$, $K$ and $t \in [0, 1]$,*

$$\begin{aligned} 0 &\le \frac{1}{2}\left[\left[\det\left((1-t)H + tK\right)\right]^{-1} + \left[\det\left(tH + (1-t)K\right)\right]^{-1}\right] \\ &\quad - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1} \\ &\le (2t-1)^2 \left[\frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1}\right] \end{aligned} \tag{3.2}$$

*and*

$$\begin{aligned} 0 &\le r\left[\frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1}\right] \\ &\le \frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) \\ &\quad - \frac{1}{2}\left[\left[\det\left((1-t)H + tK\right)\right]^{-1} + \left[\det\left(tH + (1-t)K\right)\right]^{-1}\right] \\ &\le R\left[\frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1}\right], \end{aligned} \tag{3.3}$$

*where $r = \min\{1-t, t\}$ and $R = \max\{1-t, t\}$.*

*Also*

$$
\begin{aligned}
&0 \le \int_0^1 \left[\det\left((1-t)\,H + tK\right)\right]^{-1} dt - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1} \\
&\le \frac{1}{3}\left[\frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1}\right] \\
&\le \frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \int_0^1 \left[\det\left((1-t)\,H + tK\right)\right]^{-1} dt \\
&\le \frac{4}{3}\left[\frac{1}{2}\left([\det(H)]^{-1} + [\det(K)]^{-1}\right) - \left[\det\left(\frac{H+K}{2}\right)\right]^{-1}\right].
\end{aligned} \tag{3.4}
$$

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Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.
sever.dragomir@vu.edu.au

DST-NRF Centre of Excellence in the Mathematical
and Statistical Sciences, School of Computer Science
& Applied Mathematics, University of the Witwatersrand,
Private Bag 3, Johannesburg 2050, South Africa
*URL*: http://rgmia.org/dragomir