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UPPER AND LOWER (τ, M)-J-CONTINUOUS MULTIFUNCTIONS

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Abstract. We introduce the notions of upper/lower (τ, m)-J-continuous multifunctions and obtain many characterizations of such multifunctions. The notion is obtained from a multifunction $F : (X, \tau) \to (Y, \sigma, J)$ and several generalizations of J-open sets on the ideal topological space (Y, σ, J). If F is single valued, $m = \sigma$ and $J = \{\emptyset\}$, then the above multifunction is a (τ, m)-continuous function.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the research of generalizations of continuity for functions and multifunctions. By using these sets, many authors introduced and studied various types of continuous functions and multifunctions. The notions of minimal structures, m-spaces, m-continuity and M-continuity are introduced and investigated in [27], [28], [30], and [31]. By using these notions, the present authors obtained the unified forms of continuity for multifunctions in [23], [24], and [29].

The present authors introduced the notion of (τ, m)-continuous functions in [30]. In [21], the authors introduced and studied the notions of upper/lower ultra continuous multifunctions. Recently, the present authors [25], [26] have generalized the results in [21].

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The notion of ideal topological spaces were introduced in [16], [32]. As generalizations of open sets, the notions of I-open [15] (resp. semi-I-open [12], pre-I-open [9], α-I-open [12], b-I-open [20]) sets are introduced. The notion of I-continuous multifunctions is introduced in [2]. Other relarated forms of I-continuous multifunctions are introduced and studied in [3], [4], and [7] and other papers.

The purpose of this paper is to extend the results of (τ, m)-continuous function $f : (X, \tau) \to (Y, m)$ to a multifunction $F : (X, \tau) \to (Y, \sigma, J)$, where (Y, σ, J) is an ideal topological space.

Throughout the present paper, (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. singlevalued) function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *semi-open* [17] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(2) *preopen* [19] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(3) α*-open* [22] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(4) β*-open* [1] or *semi-preopen* [5] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) b*-open* [6] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all semi-open (resp. preopen, α-open, β-open, b-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$), $\mathrm{BO}(X)$). The complement of a semi-open (resp. preopen, α-open, β-open, b-open) set is said to be *semi-closed* (resp. *preclosed*, α*-closed*, β*-closed* or *semi-preclosed*, b-closed).

Definition 2.2. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly m*-structure*) on X [27], [28] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty subset X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be m_X-*open* (briefly m-open) and the complement of an m_X-open set is said to be m_X-*closed* (briefly m-closed).

Remark 2.1. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$, $\mathrm{BO}(X)$ are all m-structures on X.

Definition 2.3. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-*closure* of A and the m_X-*interior* of A are defined in [18] as follows:

(1) $\mathrm{mCl}(A) = \bigcap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \bigcup\{U : U \subset A, U \in m_X\}$.

Lemma 2.1. [18] *Let X be a nonempty set and m_X an m-structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,

(2) If $(X \setminus A) \in m_X$, then $\mathrm{mCl}(A) = A$ *and if $A \in m_X$, then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) If $A \subset B$, then $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 2.4. An m-structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [18] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 2.2. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{SPO}(X)$ and $\mathrm{BO}(X)$ have property $\mathcal{B}$.

Lemma 2.2. [31] *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ if and only if $\mathrm{mInt}(A) = A$,

(2) A is m_X-closed if and only if $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 2.5. A function $f(X, \tau) \to (Y, m_Y)$ is said to be (τ, m)-*continuous* [30] at $x \in X$ if for each $V \in m_Y$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(U) \subset V$. The function f is said to be (τ, m)-*continuous* if it has the property at each point $x \in X$.

Theorem 2.1. [30] *For a function $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) f is (τ, m)-continuous;

(2) $f^{-1}(V)$ *is open in* X *for each* m_Y*-open set* V *of* Y*;*
(3) $f^{-1}(K)$ *is closed in* X *for each* m_Y*-closed set* K *of* Y*;*
(4) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl}(B))$ *for every subset* B *of* Y*;*
(5) $f(\mathrm{Cl}(A)) \subset \mathrm{mCl}(f(A))$ *for every subset* A *of* X*;*
(6) $f^{-1}(\mathrm{mInt}(B)) \subset \mathrm{Int}(f^{-1}(B))$ *for every subset* B *of* Y*.*

3. Upper and lower (τ, m)-continuous multifunctions

In this section, we recall several properties of upper and lower (τ, m)-continuous multifunctions obtained by the present authors [25].

Definition 3.1. [25] A multifunction $F : (X, \tau) \to (Y, m_Y)$ is said to be

(1) *upper* (τ, m)*-continuous* at $x \in X$ if for each $V \in m_Y$ containing $F(x)$, there exists $U \in \tau$ containing x such that $F(U) \subset V$,

(2) *lower* (τ, m)*-continuous* at $x \in X$ if for each $V \in m_Y$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \tau$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,

(3) *upper/lower* (τ, m)*-continuous* if F has this property at each $x \in X$.

Theorem 3.1. [25] *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$*, where* m_Y *has property* $\mathcal{B}$*, the following properties are equivalent:*

(1) F *is upper* (τ, m)*-continuous at* $x \in X$*;*

(2) $x \in \mathrm{Int}(F^+(V))$ *for each* m_Y*-open set* V *of* Y *containing* $F(x)$*;*

(3) $x \in F^-(\mathrm{mCl}(B))$ *for every subset* B *of* Y *such that* $x \in \mathrm{Cl}(F^-(B))$*;*

(4) $x \in \mathrm{Int}(F^+(B))$ *for every subset* B *of* Y *such that* $x \in F^+(\mathrm{mInt}(B))$*.*

Theorem 3.2. [25] *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$*, where* m_Y *has property* $\mathcal{B}$*, the following properties are equivalent:*

(1) F *is lower* (τ, m)*-continuous at* $x \in X$*;*

(2) $x \in \mathrm{Int}(F^-(V))$ *for each* m*-open set* V *of* Y *meeting* $F(x)$*;*

(3) $x \in F^+(\mathrm{mCl}(B))$ *for every subset* B *of* Y *such that* $x \in \mathrm{Cl}(F^+(B))$*;*

(4) $x \in \mathrm{Int}(F^-(B))$ *for every subset* B *of* Y *such that* $x \in F^-(\mathrm{mInt}(B))$*;*

(5) $x \in F^+(\mathrm{mCl}(F(A)))$ *for every subset* A *of* X *such that* $x \in \mathrm{Cl}(A)$*.*

Corollary 3.1. [25] *For a function* $f : (X, \tau) \to (Y, m_Y)$*, where* m_Y *has property* $\mathcal{B}$*, the following properties are equivalent:*

(1) f is (τ, m)-continuous at $x \in X$;
(2) $x \in \mathrm{Int}(f^{-1}(V))$ for each m-open set V of Y containing $f(x)$;
(3) $x \in f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(f^{-1}(B))$;
(4) $x \in \mathrm{Int}(f^{-1}(B))$ for every subset B of Y such that $x \in f^{-1}(\mathrm{mInt}(B))$,
(5) $x \in f^{-1}(\mathrm{mCl}(f(A)))$ for every subset A of X such that $x \in \mathrm{Cl}(A))$,

Theorem 3.3. [25] *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where (Y, m_Y) satisfies property $\mathcal{B}$, the following properties are equivalent:*
(1) F is upper (τ, m)-continuous;
(2) $F^+(V)$ is open for every $V \in m_Y$;
(3) $F^-(K)$ is closed for every m_Y-closed set K of Y;
(4) $\mathrm{Cl}(F^-(B)) \subset F^-\mathrm{mCl}(B))$ for every subset B of Y;
(5) $F^+(\mathrm{mInt}(B)) \subset \mathrm{Int}(F^+(B))$ for every subset B of Y.

Theorem 3.4. [25] *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where (Y, m_Y) satisfies property $\mathcal{B}$, the following properties are equivalent:*
(1) F is lower (τ, m)-continuous;
(2) $F^-(V)$ is open for every $V \in m_Y$;
(3) $F^+(K)$ is closed for every m_Y-closed set K of Y;
(4) $\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{mCl}(B))$ for every subset B of Y;
(5) $F(\mathrm{Cl}(A)) \subset \mathrm{mCl}(F(A))$ for every subset A of X;
(6) $F^-(\mathrm{mInt}(B)) \subset \mathrm{Int}(F^-(B))$ for every subset B of Y.

Corollary 3.2. [30] *For a function $f : (X, \tau) \to (Y, m_Y)$, where (Y, m_Y) satisfies property $\mathcal{B}$, the following properties are equivalent:*
(1) f is (τ, m)-continuous;
(2) $f^{-1}(V)$ is open for every $V \in m_Y$;
(3) $f^{-1}(K)$ is closed for every m_Y-closed set K of Y;
(4) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl}(B))$ for every subset B of Y;
(5) $f(\mathrm{Cl}(A)) \subset \mathrm{mCl}(f(A))$ for every subset A of X;
(6) $f^{-1}(\mathrm{mInt}(B)) \subset \mathrm{Int}(f^{-1}(B))$ for every subset B of Y.

For a multifunction $F : (X, \tau) \to (Y, m_Y)$, we define $D^+_{\tau m}(F)$ and $D^-_{\tau m}(F)$ as follows:

$D^+_{\tau m}(F) = \{x \in X : F \text{ is not upper } (\tau, m)\text{-continuous at } x \in X\}$,
$D^-_{\tau m}(F) = \{x \in X : F \text{ is not lower } (\tau, m)\text{-continuous at } x \in X\}$.

Theorem 3.5. [25] *For a multifunction $F : (X, \tau) \to (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following equalities hold:*
$D^+_{\tau m}(F) = \bigcup_{G \in m_Y} \{F^+(G) - \mathrm{Int}(F^+(G))\}$

$= \bigcup_{B\in P(Y)}\{F^+(\mathrm{mInt}(B)) - \mathrm{Int}(F^+(B))\}$
$= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{mCl}(B))\}$
$= \bigcup_{H\in\mathcal{F}}\{\mathrm{Cl}(F^-(H)) - F^-(H)\}$, *where*
$P(Y)$ *is the family of all subsets of* Y,
$\mathcal{F}$ *is the family of all* m_Y*-closed sets of* (Y, m_Y).

Theorem 3.6. [25] *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$, *where* m_Y *has property* $\mathcal{B}$, *the following equalities hold:*
$D^-_{\tau m}(F) = \bigcup_{G\in m_Y}\{F^-(G) - \mathrm{Int}(F^-(G))\}$
$= \bigcup_{B\in P(Y)}\{F^-(\mathrm{mInt}(B)) - \mathrm{Int}(F^-(B))\}$
$= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(F^+(B)) - F^+(\mathrm{mCl}(B))\}$
$= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - F^+(\mathrm{mCl}(F(A)))\}$
$= \bigcup_{H\in\mathcal{F}}\{\mathrm{Cl}(F^+(H)) - F^+(H)\}$, *where*
$P(X)$ *is the family of all subsets of* X,
$P(Y)$ *is the family of all subsets of* Y,
$\mathcal{F}$ *is the family of all* m*-closed sets of* (Y, m_Y).

Let (X, τ) be a topological space and (Y, m_Y) an m-space. For a function $f : (X, \tau) \to (Y, m_Y)$, we define $D_{\tau m}(f)$ as follows:

$$D_{\tau m}(f) = \{x \in X : f \text{ is not } (\tau, m)\text{-continuous at } x\}.$$

Corollary 3.3. [25] *For a function* $f : (X, \tau) \to (Y, m_Y)$, *where* m_Y *has property* $\mathcal{B}$, *the following equalities hold:*
$D_{\tau m}(f) = \bigcup_{G\in m_Y}\{f^{-1}(G) - \mathrm{Int}(f^{-1}(G))\}$
$= \bigcup_{B\in P(Y)}\{f^{-1}(\mathrm{mInt}(B)) - \mathrm{Int}(f^{-1}(B))\}$
$= \bigcup_{B\in P(Y)}\{\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{mCl}(B))\}$
$= \bigcup_{A\in P(X)}\{\mathrm{Cl}(A) - f^{-1}(\mathrm{mCl}(f(A)))\}$
$= \bigcup_{H\in\mathcal{F}}\{\mathrm{Cl}(f^{-1}(H)) - f^{-1}(H)\}$, *where*
$P(X)$ *is the family of all subsets of* X,
$P(Y)$ *is the family of all subsets of* Y,
$\mathcal{F}$ *is the family of all* m*-closed sets of* (Y, m_Y).

Theorem 3.7. [25] *For a multifunction* $F : (X, \tau) \to (Y, m_Y)$, $D^+_{\tau m}(F)$ *(resp.* $D^-_{\tau m}(F)$*) is identical with the union of the frontiers of the upper (resp. lower) inverse images of* m*-open sets of* Y *containing (resp. meeting) F(x).*

Definition 3.2. Let S be a subset of an m-space (Y, m_Y). A point $y \in Y$ is called an *$m\theta$-adherent* point of S [31] if $\mathrm{mCl}(V) \cap S \neq \emptyset$ for every m_Y-open set V containing y.

The set of all $m\theta$-adherent points of S is called the *$m\theta$-closure* of S and is denoted by $\mathrm{mCl}\theta(S)$. If $S = \mathrm{mCl}\theta(S)$, then S is said to be *$m\theta$-closed* [31]. The complement of an $m\theta$-closed set is said to be *$m\theta$-open*.

Definition 3.3. An m-space (Y, m_Y) is said to be *m-regular* [31] if for each m_Y-closed set F and each $y \notin F$, there exist disjoint m_Y-open sets U and V such that $y \in U$ and $F \subset V$.

Theorem 3.8. [25] *Let (Y, m_Y) be an m-regular space, where m_Y has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) F is upper (τ, m)-continuous;
(2) $F^-(\mathrm{mCl}\theta(B))$ is closed in X for every subset B of Y;
(3) $F^-(K)$ is closed in X for every $m\theta$-closed set K of Y;
(4) $F^+(V)$ is open in X for every $m\theta$-open set V of Y.

Theorem 3.9. [25] *Let (Y, m_Y) be an m-regular space, where m_Y has property $\mathcal{B}$. Then, for a multifunction $F : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) F is lower (τ, m)-continuous;
(2) $F^+(\mathrm{mCl}\theta(B))$ is closed in X for every subset B of Y;
(3) $F^+(K)$ is closed in X for every $m\theta$-closed set K of Y;
(4) $F^-(V)$ is open in X for every $m\theta$-open set V of Y.

Corollary 3.4. *Let (Y, m_Y) be an m-regular space, where m_Y has property $\mathcal{B}$. Then, for a function $f : (X, \tau) \to (Y, m_Y)$, the following properties are equivalent:*
(1) f is (τ, m)-continuous;
(2) $f^{-1}(\mathrm{mCl}\theta(B))$ is closed in X for every subset B of Y;
(3) $f^{-1}(K)$ is closed in X for every $m\theta$-closed set K of Y;
(4) $f^{-1}(V)$ is open in X for every $m\theta$-open set V of Y.

Let (Y, m_Y) be an m-space. By $\mathrm{mCl}(F) : (X, \tau) \to (Y, m_Y)$, we denote a multifunction defined by $\mathrm{mCl}(F)(x) = \mathrm{mCl}(F(x))$ for each $x \in X$.

Theorem 3.10. [25] *A multifunction $F : (X, \tau) \to (Y, m_Y)$ is lower (τ, m)-continuous if and only if $\mathrm{mCl}(F) : (X, \tau) \to (Y, m_Y)$ is lower (τ, m)-continuous.*

4. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals has been introduced in [16] and [32] and further investigated in [14]

Definition 4.1. A nonempty collection I of subsets of a set X is called an *ideal* on X [16], [32] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [14]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 4.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ *implies* $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ *and* $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 4.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *I-open* [15] if $A \subset \mathrm{Int}(A^\star)$,
(2) *α-I-open* [12] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(3) *semi-I-open* [12] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(4) *pre-I-open* [9] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(5) *b-I-open* [8] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(6) *β-I-open* [13] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(7) *weakly semi-I-open* [10] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *weakly b-I-open* [20] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(9) *strongly β-I-open* [11] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

The family of all I-open (resp. α-I-open, semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\mathrm{IO}(X)$ (resp. $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$).

Definition 4.3. By $\mathrm{mIO}(X)$, we denote each one of the families $\tau^\star$, $\mathrm{IO}(X)$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, and $\mathrm{S}\beta\mathrm{IO}(X)$.

Lemma 4.2. *Let (X, τ, I) be an ideal topological space. Then* $\mathrm{mIO}(X)$ *is a minimal structure and has property $\mathcal{B}$.*

Proof. The proof follows by Lemmas 2.1 and 4.1.

Remark 4.1. It is shown in Theorem 3.4 of [13] (resp. Theorem 10 of [9], Theorem 21 of [1], Theorem 2.7 of [10], Theorem 3 of [11]) that $\mathrm{mIO}(X)$ has properety $\mathcal{B}$.

Definition 4.4. Let (X, τ, I) be an ideal topological space. For a subset A of X, $\mathrm{mCl_I}(A)$ and $\mathrm{mInt_I}(A)$ as follows:
(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO(X)}\}$,
(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO(X)}\}$.

Let (X, τ, I) be an ideal topological space and $\mathrm{mIO}(X)$ the m-structure on X. If $\mathrm{mIO}(X) = \tau^\star$ (resp. $\mathrm{IO}(X)$, $\alpha\mathrm{IO}(X)$, $\mathrm{SIO}(X)$, $\mathrm{PIO}(X)$, $\mathrm{BIO}(X)$, $\beta\mathrm{IO}(X)$, $\mathrm{WSIO}(X)$, $\mathrm{WBIO}(X)$, $\mathrm{S}\beta\mathrm{IO}(X)$), then we have
(1) $\mathrm{mCl_I}(A) = \mathrm{Cl}^\star(A)$ (resp. $\mathrm{Cl_I}(A)$, $\alpha\mathrm{Cl_I}(A)$, $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$),
(2) $\mathrm{mInt_I}(A) = \mathrm{Int}^\star(A)$ (resp. $\mathrm{Int_I}(A)$, $\alpha\mathrm{Int_I}(A)$ (resp. $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$).

5. Upper and lower (τ, m)-J-continuous multifunctions

Definition 5.1. A multifunction $F : (X, \tau) \to (Y, \sigma, J)$ is said to be
(1) *upper* (τ, m)*-J-continuous* at $x \in X$ if for each $V \in mJO(Y)$ containing $F(x)$, there exists $U \in \tau$ containing x such that $F(U) \subset V$,
(2) *lower* (τ, m)*-J-continuous* at $x \in X$ if for each $V \in mJO(Y)$ such that $F(x) \cap V \neq \emptyset$, there exists $U \in \tau$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$,
(3) *upper/lower* (τ, m)*-J-continuous* if F has this property at each $x \in X$.

Since $\mathrm{mJO}(Y)$ has property $\mathcal{B}$, by Definition 3.1 and Theorems 3.1-3.10, we obtain the following theorems

Theorem 5.1. *For a multifunction $F : (X, \tau) \to (Y, \sigma, J)$, the following properties are equivalent:*
(1) F is upper (τ, m)-J-continuous at $x \in X$;
(2) $x \in \mathrm{Int}(F^+(V))$ for each $V \in mJO(Y)$ containing F(x);
(3) $x \in F^-(\mathrm{mCl_J}(B))$ for every subset B of Y such that $x \in \mathrm{Cl}(F^-(B))$;
(4) $x \in \mathrm{Int}(F^+(B))$ for every subset B of Y such that $x \in F^+(\mathrm{mInt_J}(B))$.

Theorem 5.2. *For a multifunction $F : (X, \tau) \to (Y, \sigma, J)$, the following properties are equivalent:*

(1) F is lower (τ, m)*-J-continuous at* $x \in X$*;*
(2) $x \in \mathrm{Int}(F^-(V))$ *for each* $V \in mJO(Y)$ *meeting F(x);*
(3) $x \in F^+(\mathrm{mCl_J}(B))$ *for every subset B of Y such that* $x \in \mathrm{Cl}(F^+(B))$*;*
(4) $x \in \mathrm{Int}(F^-(B))$ *for every subset B of Y such that* $x \in F^-(\mathrm{mInt_J}(B))$*;*
(5) $x \in F^-(\mathrm{mCl_J}(F(A)))$ *for every subset A of X such that* $x \in \mathrm{Cl}(A))$.

Corollary 5.1. *For a function* $f : (X, \tau) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) f is (τ, m)*-continuous at* $x \in X$*;*
(2) $x \in \mathrm{Int}(f^{-1}(V))$ *for each mJ-open set V of Y containing f(x);*
(3) $x \in f^{-1}(\mathrm{mCl_J}(B))$ *for every subset B of Y such that* $x \in \mathrm{Cl}(f^{-1}(B))$*;*
(4) $x \in \mathrm{Int}(f^{-1}(B))$ *for every subset B of Y such that* $x \in f^{-1}(\mathrm{mInt_J}(B))$*,*
(5) $x \in f^{-1}(\mathrm{mCl_J}(f(A)))$ *for every subset A of X such that* $x \in \mathrm{Cl}(A))$.

Theorem 5.3. *For a multifunction* $F : (X, \tau) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) F is upper (τ, m)*-J-continuous;*
(2) $F^+(V)$ *is open for every* $V \in mJO(Y)$*;*
(3) $F^-(K)$ *is closed for every mJ-closed set K of Y;*
(4) $\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{mCl_J}(B))$ *for every subset B of Y;*
(5) $F^+(\mathrm{mInt_J}(B)) \subset \mathrm{Int}(F^+(B))$ *for every subset B of Y.*

Theorem 5.4. *For a multifunction* $F : (X, \tau) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) F is lower (τ, m)*-J-continuous;*
(2) $F^-(V)$ *is open for every* $V \in mJO(Y)$*;*
(3) $F^+(K)$ *is closed for every mJ-closed set K of Y;*
(4) $\mathrm{Cl}(F^+(B)) \subset F^+(\mathrm{mCl_J}(B))$ *for every subset B of Y;*
(5) $F(\mathrm{Cl}(A)) \subset \mathrm{mCl_J}(F(A))$ *for every subset A of X;*
(6) $F^-(\mathrm{mInt_J}(B)) \subset \mathrm{Int}(F^-(B))$ *for every subset B of Y.*

Corollary 5.2. *For a function* $f : (X, \tau) \to (Y, \sigma, J)$*, the following properties are equivalent:*
(1) f is (τ, m)*-J-continuous;*
(2) $f^{-1}(V)$ *is open for every* $V \in mJO(Y)$*;*
(3) $f^{-1}(K)$ *is closed for every mJ-closed set K of Y;*
(4) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{mCl_J}(B))$ *for every subset B of Y;*

(5) $f(\mathrm{Cl}(A)) \subset \mathrm{mCl_J}(f(A))$ *for every subset* A *of* X*;*
(6) $f^{-1}(\mathrm{mInt_J}(B)) \subset \mathrm{Int}(f^{-1}(B))$ *for every subset* B *of* Y*.*

For a multifunction $F : (X, \tau) \to (Y, \sigma, J)$, we define $D^+_{mJ}(F)$ and $D^-_{mJ}(F)$ as follows:

$D^+_{mJ}(F) = \{x \in X : F$ is not upper (τ, m)-J-continuous at $x \in X\}$,
$D^-_{mJ}(F) = \{x \in X : F$ is not lower (τ, m)-J-continuous at $x \in X\}$.

Theorem 5.5. *For a multifunction* $F : (X, \tau) \to (Y, \sigma, J)$*, the following equalities hold:*

$$\begin{aligned} D^+_{mJ}(F) &= \bigcup_{G \in mJO(Y)} \{F^+(G) - \mathrm{Int}(F^+(G))\} \\ &= \bigcup_{B \in P(Y)} \{F^+(\mathrm{mInt_J}(B)) - \mathrm{Int}(F^+(B))\} \\ &= \bigcup_{B \in P(Y)} \{\mathrm{Cl}(F^-(B)) - F^-(\mathrm{mCl_J}(B))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\mathrm{Cl}(F^-(H)) - F^-(H)\}, \text{ where} \end{aligned}$$

$P(Y)$ *is the family of all subsets of* Y*,*
$\mathcal{F}$ *is the family of all* mJ*-closed sets of* (Y, σ, J)*.*

Theorem 5.6. *For a multifunction* $F : (X, \tau) \to (Y, \sigma, J)$*, the following equalities hold:*

$$\begin{aligned} D^-_{mJ}(F) &= \bigcup_{G \in mJO(Y)} \{F^-(G) - \mathrm{Int}(F^-(G))\} \\ &= \bigcup_{B \in P(Y)} \{F^-(\mathrm{mInt_J}(B)) - \mathrm{Int}(F^-(B))\} \\ &= \bigcup_{B \in P(Y)} \{\mathrm{Cl}(F^+(B)) - F^+(\mathrm{mCl_J}(B))\} \\ &= \bigcup_{A \in P(X)} \{\mathrm{Cl}(A) - F^+(\mathrm{mCl_J}(F(A)))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\mathrm{Cl}(F^+(H)) - F^+(H)\}, \text{ where} \end{aligned}$$

$P(X)$ *is the family of all subsets of* X*,*
$P(Y)$ *is the family of all subsets of* Y*,*
$\mathcal{F}$ *is the family of all* mJ*-closed sets of* (Y, σ, J)*.*

Let (X, τ) be a topological space and (Y, σ, J) an ideal topological space. For a function $f : (X, \tau) \to (Y, \sigma, J)$, we define $D_{mJ}(f)$ as follows:

$$D_{mJ}(f) = \{x \in X : f \text{ is not } (\tau, m)\text{-}J\text{-continuous at } x\}.$$

Corollary 5.3. *For a function* $f : (X, \tau) \to (Y, \sigma, J)$*, the following equalities hold:*

$$\begin{aligned} D_{mJ}(f) &= \bigcup_{G \in mJO(Y)} \{f^{-1}(G) - \mathrm{Int}(f^{-1}(G))\} \\ &= \bigcup_{B \in P(Y)} \{f^{-1}(\mathrm{mInt_J}(B)) - \mathrm{Int}(f^{-1}(B))\} \\ &= \bigcup_{B \in P(Y)} \{\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{mCl_J}(B))\} \\ &= \bigcup_{A \in P(X)} \{\mathrm{Cl}(A) - f^{-1}(\mathrm{mCl_J}(f(A)))\} \\ &= \bigcup_{H \in \mathcal{F}} \{\mathrm{Cl}(f^{-1}(H)) - f^{-1}(H)\}, \text{ where} \end{aligned}$$

$P(X)$ *is the family of all subsets of* X*,*

$P(Y)$ is the family of all subsets of Y,
$\mathcal{F}$ is the family of all mJ-closed sets of (Y, σ, J).

Theorem 5.7. *For a multifunction $F : (X, \tau) \to (Y, \sigma, J)$, $D^+_{mJ}(F)$ (resp. $D^-_{mJ}(F)$) is identical with the union of the frontiers of the upper (resp. lower) inverse images of mJ-open sets of Y containing (resp. meeting) F(x).*

Definition 5.2. Let S be a subset of an m-space $(Y, mJO(Y))$. A point $y \in Y$ is called an $(mJ)\theta$*-adherent* point of S if $\mathrm{mCl_J}(V) \cap S \neq \emptyset$ for every mJ-open set V containing y.

The set of all $(mJ)\theta$-adherent points of S is called the $(mJ)\theta$*-closure* of S and is denoted by $\mathrm{(mJ)Cl}\theta(S)$. If $S = \mathrm{(mJ)Cl}\theta(S)$, then S is said to be $(mJ)\theta$*-closed*. The complement of an $(mJ)\theta$-closed set is said to be $(mJ)\theta$*-open*.

Definition 5.3. An m-space $(Y, mJO(Y))$ is said to be *mJ-regular* if for each mJ-closed set F and each $y \notin F$, there exist disjoint mJ-open sets U and V such that $y \in U$ and $F \subset V$.

Theorem 5.8. *Let $F : (X, \tau) \to (Y, \sigma, J)$ be a multifunction and $(Y, mJO(Y))$ be an mJ-regular space. Then, the following properties are equivalent:*
(1) F is upper (τ, m)-J-continuous;
(2) $F^-(\mathrm{(mJ)Cl}\theta(B))$ is closed in X for every subset B of Y;
(3) $F^-(K)$ is closed in X for every $(mJ)\theta$-closed set K of Y;
(4) $F^+(V)$ is open in X for every $(mJ)\theta$-open set V of Y.

Theorem 5.9. *Let $F : (X, \tau) \to (Y, \sigma, J)$ be a multifunction and $(Y, mJO(Y))$ be an mJ-regular space. Then, the following properties are equivalent:*
(1) F is lower (τ, m)-J-continuous;
(2) $F^+(\mathrm{(mJ)Cl}\theta(B))$ is closed in X for every subset B of Y;
(3) $F^+(K)$ is closed in X for every $(mJ)\theta$-closed set K of Y;
(4) $F^-(V)$ is open in X for every $(mJ)\theta$-open set V of Y.

Corollary 5.4. *Let $f : (X, \tau) \to (Y, \sigma, J)$ be a function and $(Y, mJO(Y))$ be an mJ-regular space. Then, the following properties are equivalent:*
(1) f is (τ, m)-J-continuous;
(2) $f^{-1}(\mathrm{(mJ)Cl}\theta(B))$ is closed in X for every subset B of Y;
(3) $f^{-1}(K)$ is closed in X for every $(mJ)\theta$-closed set K of Y;
(4) $f^{-1}(V)$ is open in X for every $(mJ)\theta$-open set V of Y.

Let (Y, σ, J) be an ideal topological space. By $\mathrm{mCl_J}F : (X, \tau) \to (Y, \sigma, J)$, we denote a multifunction defined by $\mathrm{mCl_J}(F)(x) = \mathrm{mCl_J}(F(x))$ for each $x \in X$.

Theorem 5.10. *A multifunction $F : (X, \tau) \to (Y, \sigma, J)$ is lower (τ, m)-J-continuous if and only if $\mathrm{mCl_J}(F) : (X, \tau) \to (Y, \sigma, J)$ is lower (τ, m)-J-continuous.*

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