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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 32 (2022), No. 1, 99-104

ANOTHER MENON-TYPE IDENTITY DERIVED FROM GROUP ACTIONS

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Abstract. In this short note, we give a new Menon-type identity involving the sum of element orders and the sum of cyclic subgroup orders of a finite abelian group. This is based on the weighted form of Burnside's lemma applied to the action of the power automorphism group.

1. Introduction

One of the most interesting arithmetical identities is due to P.K. Menon [4].

Menon's identity. *For every positive integer n we have*

$$\sum_{a\in\mathbb{Z}_n^*} \gcd(a-1,n)=\varphi(n)\,\tau(n),$$

where $\mathbb{Z}_n^*$ is the group of units of the ring $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$, $\gcd(,)$ represents the greatest common divisor, φ is the Euler's totient function and $\tau(n)$ is the number of divisors of n.

Keywords and phrases: Menon's identity, weighted Burnside's lemma, group action, sum of element orders, sum of cyclic subgroup orders, abelian group.
(2020) Mathematics Subject Classification: 11A25, 20D60, 20K01.

There are several approaches to Menon's identity and many generalisations. One of the methods used to prove Menon-type identities is based on the Burnside's Lemma concerning group actions (see e.g. [4, 5, 7, 9, 10, 12]). The starting point for our discussion is given by the paper [11] which uses a generalisation of this result, called the Weighted Form of Burnside's Lemma (see e.g. [2]).

Weighted Form of Burnside's Lemma. *Given a finite group G acting on a finite set X, we denote*

$$Fix(g) = \{x \in X : g \circ x = x\}, \forall\, g \in G.$$

Let R be a commutative ring containing the rationals and $w : X \longrightarrow R$ be a weight function that is constant on the distinct orbits O_{x_1}, ..., O_{x_r} of X. For every $i = 1, ..., r$, let $w(O_{x_i}) = w(x)$, where $x \in O_{x_i}$. Then

$$\sum_{i=1}^{r} w(O_{x_i}) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fix(g)} w(x). \tag{1}$$

Note that the Burnside's Lemma is obtained from (1) by taking the weight function $w(x) = 1$, $\forall\, x \in X$.

In what follows, let G be a finite abelian group of order n and

$$G = G_1 \times \cdots \times G_k$$

be the primary decomposition of G, where G_i is a p_i-group for all $i = 1, \ldots, k$. Then every G_i is of type

$$G_i = \mathbb{Z}_{p_i^{\alpha_{i1}}} \times \cdots \times \mathbb{Z}_{p_i^{\alpha_{ir_i}}},$$

where $1 \leq \alpha_{i1} \leq \cdots \leq \alpha_{ir_i}$. We will apply the weighted form of Burnside's lemma to the natural action of the power automorphism group $\mathrm{Pot}(G)$ on G. Recall that an automorphism f of G is called a *power automorphism* if $f(H) = H$, $\forall\, H \leq G$, and that the set $\mathrm{Pot}(G)$ of all power automorphisms of G is a subgroup of $\mathrm{Aut}(G)$. Also, it is well-known that every power automorphism of a finite abelian group is *universal*, i.e. there exists an integer m such that $f(x) = mx$ for all $x \in G$. The structure of $\mathrm{Pot}(G)$ is given by Theorem 1.5.6 in [6]:

$$\mathrm{Pot}(G) \cong \mathrm{Pot}(G_1) \times \cdots \times \mathrm{Pot}(G_k) \cong \mathrm{Aut}(\mathbb{Z}_{p_1^{\alpha_{1r_1}}}) \times \cdots \times \mathrm{Aut}(\mathbb{Z}_{p_k^{\alpha_{kr_k}}}).$$

We will also consider the functions

$$\psi(G) = \sum_{g \in G} o(g) \ \text{ and } \ \sigma(G) = \sum_{H \in C(G)} |H|,$$

where $o(g)$ is the order of $g \in G$ and $C(G)$ is the set of cyclic subgroups of G. [1]

Our main result is stated as follows.

Theorem 1. *Under the above notations, we have*

$$\prod_{i=1}^{k} \sum_{\substack{1 \le m_i \le p_i^{\alpha_{ir_i}} \\ p_i \nmid m_i}} \psi\left(\prod_{j=1}^{r_i} \mathbb{Z}_{\gcd(m_i-1, p_i^{\alpha_{ij}})}\right) = \varphi(\exp(G))\,\sigma(G). \tag{2}$$

Clearly, (2) gives yet another connection between the functions $\psi(G)$ and $\sigma(G)$ associated to a finite abelian group G. For $G = \mathbb{Z}_n$, Theorem 1 leads to the following corollary.

Corollary 2. *We have*

$$\prod_{i=1}^{k} \sum_{\substack{1 \le m_i \le p_i^{\alpha_{i1}} \\ p_i \nmid m_i}} \psi\left(\mathbb{Z}_{\gcd(m_i-1, p_i^{\alpha_{i1}})}\right) = \varphi(n)\,\sigma(n), \tag{3}$$

where $\sigma(n)$ is the sum of divisors of n.

Since the values of ψ on finite abelian groups G are well-known (see e.g. Theorem 1 of [8]), the equality (2) can be seen as a formula to compute $\sigma(G)$. We exemplify it in some particular cases.

Example 3.

a) For $G = \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, we obtain

$$\begin{aligned}
\sigma(G) = \sigma(12) &= \frac{1}{\varphi(12)} \left(\sum_{\substack{1 \le m_1 \le 4 \\ 2 \nmid m_1}} \psi\left(\mathbb{Z}_{\gcd(m_1-1,4)}\right)\right)\left(\sum_{\substack{1 \le m_2 \le 3 \\ 3 \nmid m_2}} \psi\left(\mathbb{Z}_{\gcd(m_2-1,3)}\right)\right) \\
&= \frac{1}{4}\left(\psi(\mathbb{Z}_4) + \psi(\mathbb{Z}_2)\right)\left(\psi(\mathbb{Z}_3) + \psi(\mathbb{Z}_1)\right) \\
&= \frac{1}{4}(11+3)(7+1) = \frac{1}{4} \cdot 14 \cdot 8 = 28.
\end{aligned}$$

[1] For more details concerning these functions, we refer the reader to [1] and [3], respectively.

b) For $G=\mathbb{Z}_2\times\mathbb{Z}_6\cong\mathbb{Z}_2^2\times\mathbb{Z}_3$, we obtain

$$\begin{aligned}\sigma(G) &= \frac{1}{\varphi(6)}\left(\sum_{\substack{1\le m_1\le 2\\ 2\nmid m_1}}\psi\left(\mathbb{Z}^2_{\gcd(m_1-1,2)}\right)\right)\left(\sum_{\substack{1\le m_2\le 3\\ 3\nmid m_2}}\psi\left(\mathbb{Z}_{\gcd(m_2-1,3)}\right)\right)\\ &= \frac{1}{2}\,\psi(\mathbb{Z}_2\times\mathbb{Z}_2)\left(\psi(\mathbb{Z}_3)+\psi(\mathbb{Z}_1)\right)\\ &= \frac{1}{2}\cdot 7\cdot 8 = 28.\end{aligned}$$

c) For $G=\mathbb{Z}_6\times\mathbb{Z}_{12}\cong\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_3^2$, we obtain

$$\begin{aligned}\sigma(G) &= \frac{1}{\varphi(12)}\left(\sum_{\substack{1\le m_1\le 4\\ 2\nmid m_1}}\psi\left(\mathbb{Z}_{\gcd(m_1-1,2)}\times\mathbb{Z}_{\gcd(m_1-1,4)}\right)\right)\left(\sum_{\substack{1\le m_2\le 3\\ 3\nmid m_2}}\psi\left(\mathbb{Z}^2_{\gcd(m_2-1,3)}\right)\right)\\ &= \frac{1}{4}\left(\psi(\mathbb{Z}_2\times\mathbb{Z}_4)+\psi(\mathbb{Z}_2^2)\right)\left(\psi(\mathbb{Z}_3^2)+\psi(\mathbb{Z}_1^2)\right)\\ &= \frac{1}{4}\,(23+7)\,(25+1) = \frac{1}{4}\cdot 30\cdot 26 = 195.\end{aligned}$$

2. Proof of Theorem 1

The natural action of $\mathrm{Pot}(G)$ on G is

$$f\circ a = f(a),\ \forall\,(f,a)\in\mathrm{Pot}(G)\times G.$$

By using the direct decompositions of $\mathrm{Pot}(G)$ and G in Section 1, it can be written as

$$(f_1,...,f_k)\circ(a_1,...,a_k)=(f_1(a_1),...,f_k(a_k)),\ \forall\,(f_i,a_i)\in\mathrm{Pot}(G_i)\times G_i,\ i=1,...,k.$$

In [10], we proved that two elements of G belong to the same orbit if and only if they generate the same cyclic subgroup. This shows that the weight function $w: G\longrightarrow\mathbb{R}$, $w(g)=o(g)$, $\forall\, g\in G$, is constant on the distinct orbits O_{g_1}, ..., O_{g_r} of G. Thus we can apply the Weighted Form of Burnside's Lemma.

First of all, we observe that $w(O_{g_i})=o(g_i)=|\langle g_i\rangle|$, $\forall\, i=1,...,k$, and therefore the left side of (1) is $\sigma(G)$. Also, by [10] we have

$$|\mathrm{Pot}(G)|=\varphi(\exp(G))$$

and

$$Fix(f_i)\cong\prod_{j=1}^{r_i}\mathbb{Z}_{\gcd(m_i-1,p_i^{\alpha_{ij}})}.$$

Consequently, the right side of (1) is

$$\frac{1}{|\mathrm{Pot}(G)|}\sum_{f\in\mathrm{Pot}(G)}\sum_{a\in Fix(f)} w(a) = \frac{1}{\varphi(\exp(G))}\sum_{f\in\mathrm{Pot}(G)}\sum_{a\in Fix(f)} o(a)$$

$$= \frac{1}{\varphi(\exp(G))}\sum_{f_1\in\mathrm{Pot}(G_1)}\cdots\sum_{f_k\in\mathrm{Pot}(G_k)}\sum_{a_1\in Fix(f_1)}\cdots\sum_{a_k\in Fix(f_k)} o(a_1)\cdots o(a_k)$$

$$= \frac{1}{\varphi(\exp(G))}\prod_{i=1}^{k}\left(\sum_{f_i\in\mathrm{Pot}(G_i)}\sum_{a_i\in Fix(f_i)} o(a_i)\right)$$

$$= \frac{1}{\varphi(\exp(G))}\prod_{i=1}^{k}\left(\sum_{f_i\in\mathrm{Pot}(G_i)} \psi\left(Fix(f_i)\right)\right)$$

$$= \frac{1}{\varphi(\exp(G))}\prod_{i=1}^{k}\sum_{\substack{1\le m_i\le p_i^{\alpha_{ir_i}}\\ p_i\nmid m_i}} \psi\left(\prod_{j=1}^{r_i}\mathbb{Z}_{\gcd(m_i-1,p_i^{\alpha_{ij}})}\right),$$

as desired. ■

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