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CONVERGENCE IN SOBOLEV SPACES OF SOLUTIONS FOR ELLIPTIC PROBLEMS ON VARYING DOMAINS

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Abstract. In this note we discuss results regarding the convergence in the sense of Mosco of a sequence of open sets. This concept of convergence of sets is a tool in the study of the convergence in Sobolev spaces of the solutions of an elliptic boundary value problem, as the domain is varying.

1. INTRODUCTION

We deal with solutions of elliptic equations on arbitrary open sets. Our question is what happens if we vary the open sets. If they converge in some sense to be made precise, do the solutions converge? The conditions on the convergence of the open sets will depend on the kind of convergence of the solutions we are looking for.

The aim of this article is to study necessary and sufficient conditions on sequences of domains $\Omega_n \subset \mathbb{R}^N$, where $N \geq 2$, such that weak solutions u_n of the elliptic boundary value problems:

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$$
\begin{aligned}
Au + \lambda u &= f_n \text{ in } \Omega_n \\
u &= 0 \text{ on } \partial\Omega_n
\end{aligned} \tag{1}
$$

converge as $n \to \infty$ to a solution u of the corresponding problem on a limit domain $\Omega \subset \mathbb{R}^N$, i.e.

$$
\begin{aligned}
Au + \lambda u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega.
\end{aligned} \tag{2}
$$

Here A is a second order elliptic operator and $\lambda \in \mathbb{R}$.

The following types of convergence are usually considered in this setting:

(1) the convergence in $L^{\infty}_{loc}\left(\mathbb{R}^N\right)$ and $L^{\infty}\left(\mathbb{R}^N\right)$, that is the local and global uniform convergence, respectively;
(2) the convergence in $H^1_0\left(\mathbb{R}^N\right)$ and $L^2\left(\mathbb{R}^N\right)$.

The motivation to look at such problems comes from variational inequalities, numerical analysis, control and optimisations, Γ-convergence and non-linear elliptic equations (see, for instance [3], [4], [5], [6], [8]).

We will consider basic convergence results proved by Biergert and Daners ([2]) and Daners ([3]). Denote by $R_{\Omega_n}(\lambda)$ and $R_{\Omega}(\lambda)$ the solutions of the problem (1) and (2), respectively. We shall consider the simplest case when $A = -\Delta$ and $f_n,\ f \in L^{\infty}\left(\mathbb{R}^N\right)$. It has been shown that $R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f$ in $L^{\infty}_{loc}\left(\mathbb{R}^N\right)$ and $L^{\infty}\left(\mathbb{R}^N\right)$ for all $f \in L^{\infty}\left(\mathbb{R}^N\right)$ and for all $\lambda > 0$ if and only if this is the case for $f \equiv 1$ and some $\lambda > 0$. We will also reemember that the global uniform convergence of $R_{\Omega_n}(\lambda) f \to R_{\Omega}(\lambda) f$ in $L^{\infty}\left(\mathbb{R}^N\right)$ implies that $R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda)$ in $\mathcal{L}(L^p\left(\mathbb{R}^N\right))$ for all $p \in [1, \infty)$.

The essential part of our considerations will be devoted to the study of the convergence in sense of Mosco of a sequence $(\Omega_n)_{n\geq 1}$ of domains. This convergence is a necessary and sufficient (under additional asumptions on uniform boundedness of resolvents) condition on the convergence of solutions $R_{\Omega_n}(\lambda) f$ of (1) to a solution of (2) in $H^1\left(\mathbb{R}^N\right)$ for all $f \in H^{-1}\left(\mathbb{R}^N\right)$ and $\lambda \in \bigcap_{n\in\mathbb{N}^*} \rho(-A_{\Omega_n}) \cap \rho(-A_{\Omega})$.

Applying the results of Kato ([5]) concerning upper semi-continuity of the spectrum under "small" perturbations and the convergence $R_{\Omega_n}(\lambda) \to R_{\Omega}(\lambda)$ in $\mathcal{L}(L^2\left(\mathbb{R}^N\right))$, one proves the continuity of every finite system of eigenvalues.

2. PRELIMINARIES

We assume that Ω_n, $n \geq 1$ and Ω are open (possibly unbounded and disconnected) sets in $\mathbb{R}^N$, $N \geq 2$. The Lebesgue measure of a set $S \subset \mathbb{R}^N$ is denoted by $|S|$. We denote by $H_0^1(\Omega)$ the closure in $H^1(\Omega)$ of the set $C_c^\infty(\Omega)$ of test functions (compactly supported smooth functions), under the norm

$$\|u\|_{H^1} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{1/2}.$$

Here $\|u\|_2$ denotes the norm in L^2.

Extending elements of $C_c^\infty(\Omega)$ by zero outside Ω, we may consider in a natural way $C_c^\infty(\Omega)$ as a subspace of $C_c^\infty\left(\mathbb{R}^N\right)$. Hence, taking closures we may identify $H_0^1(\Omega)$ with a closed subspace of $H_0^1\left(\mathbb{R}^N\right) = H^1\left(\mathbb{R}^N\right)$.

We will consider the operator A of the form:

$$Au := -\sum_{i=1}^{N} \partial_i \left(\left(\sum_{j=1}^{N} a_{i,j} \partial_j u\right) + a_i u\right), \tag{3}$$

with $a_{i,j}$, $a_i \in L_\infty\left(\mathbb{R}^N\right)$, for all $i, j = 1, ..., N$. We assume that there exists an ellipticity constant $\alpha > 0$ such that

$$\sum_{j=1}^{N}\sum_{i=1}^{N} a_{i,j}\xi_i\xi_j \geq \alpha |\xi|^2$$

for almost all $x \in \mathbb{R}^N$ and all $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. The simplest form of the operator A is given by the Laplace operator, and then A is an uniformly elliptic operator with an ellipticity constant $\alpha = 1$. We define a bounded bilinear form on $H^1\left(\mathbb{R}^N\right)$, and thus on $H_0^1(\Omega)$ for every open set $\Omega \subset \mathbb{R}^N$, associated with A by:

$$a(u, v) := \int_{\mathbb{R}^N} \left[\sum_{i=1}^{N} \left(\left(\sum_{j=1}^{N} a_{i,j} \partial_j u\right) + a_i u\right) \partial_i v\right] dx, \tag{4}$$

for any $u, v \in H^1\left(\mathbb{R}^N\right)$.

The adjoint form $a^{\#}(u, v) := a(v, u)$, for any $u, v \in H_0^1\left(\mathbb{R}^N\right)$, is the bilinear form associated with the formally adjoint operator of A denoted by $A^{\#}$ and given by

$$A^{\#} v := -\sum_{i=1}^{N} \partial_i \left(\left(\sum_{j=1}^{N} a_{i,j} \partial_j v\right) + a_i v\right) \tag{5}$$

If we denote by $A_\Omega^\# \in L\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ the operator induced by $a^\#(\cdot,\cdot)$ then $A'_\Omega = A_\Omega^\#$ and $\left(A_\Omega^\#\right)' = A_\Omega$. We recall that an operator and its dual have the same spectrum.

Let $u, v : \Omega \to \mathbb{R}$ two measurable functions. We set $\langle u, v\rangle := \int_\Omega uv dx$. By Riesz representation theorem we identify $L_2(\Omega)$ with its dual, therefore $H_0^1(\Omega) \hookrightarrow L_2(\Omega) \hookrightarrow H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the topological dual of $H_0^1(\Omega)$ equipped with the dual norm. The duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is also denoted by $\langle\cdot,\cdot\rangle$.

Given $f : H^{-1}(\Omega)$, u is called a weak solution of equation (2) if $u \in H_0^1(\Omega)$ and

$$a(u,v) + \lambda\langle u, v\rangle = \langle f, v\rangle, \text{ for all } v \in H_0^1(\Omega). \tag{6}$$

If we set $\lambda_0 := \frac{1}{2\alpha}\sum_{i=1}^{N}\|a_i\|_\infty^2$ then $\frac{\alpha}{2}\|u\|_{H^1(\mathbb{R}^N)}^2 \le a(u,u) + \lambda\|u\|_2^2$ for all $u \in H^1\left(\mathbb{R}^N\right)$ and $\lambda \ge \lambda_0$. Moreover,

$$\frac{\alpha}{2}\|u\|_{H_0^1(\Omega)}^2 \le a(u,u) + \langle\lambda u, u\rangle = \langle f, u\rangle \le \|u\|_{H_0^1(\Omega)}\|f\|_{H^{-1}(\Omega)}.$$

Divinding by $\|u\|_{H_0^1(\Omega)}$ we get $\|u\|_{H_0^1(\Omega)} \le \frac{2}{\alpha}\|f\|_{H^{-1}(\Omega)}$.

It is convenient to write (2) in an abstract form. For a bilinear bounded form $a(\cdot,\cdot)$ defined on $H_0^1(\Omega)$ there exists $A_\Omega \in L\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$ such that $a(u,v) = \langle A_\Omega u, v\rangle$, for any $u, v \in H_0^1(\Omega)$. Here we denoted by A_Ω the operator induced by A on Ω.

So, we call $u \in H_0^{-1}(\Omega)$ a weak solution for (2) if and only if u is a solution of $(\lambda + A_\Omega)u = f$, in $H^{-1}(\Omega)$. Sometimes is useful to consider A_Ω as an operator on $H^{-1}(\Omega)$ with the domain $H_0^1(\Omega)$.

We denote by $\rho(A_\Omega)$ the resolvent set and by $\sigma(A_\Omega)$ the spectrum of A_Ω. By the previous considerations and by the estimate of the solution $\|u\|_{H_0^1(\Omega)} \le \frac{2}{\alpha}\|f\|_{H^{-1}(\Omega)}$ we conclude that $[\lambda_0, \infty) \subset \rho(-A_\Omega)$, $\forall\Omega \subset \mathbb{R}^N$.

For varying domains we need a family of operators with domain and range independent of Ω_n and $\Omega \subset \mathbb{R}^N$. So, we denote by $i_\Omega \in L\left(H_0^1(\Omega), H^1\left(\mathbb{R}^N\right)\right)$ the operator extending functions in $H_0^1(\Omega_n)$ by 0 outside Ω and by $r_\Omega \in L\left(H^{-1}\left(\mathbb{R}^N\right), H^{-1}(\Omega)\right)$ the operator restricting functionals $f \in H^{-1}\left(\mathbb{R}^N\right)$ to $H_0^1(\Omega)$. Obviously

$$\langle f, i_{\Omega_n}(u)\rangle = \langle r_{\Omega_n}(f), u\rangle, \quad \text{whenever } u \in H_0^1(\Omega_n),\ f \in H^{-1}\left(\mathbb{R}^N\right)$$

so $i'_\Omega = r_\Omega$ and $r'_\Omega = i_\Omega$.

We set

(7) $R_n(\lambda) := i_{\Omega_n} \circ (\lambda + A_{\Omega_n})^{-1} \circ r_{\Omega_n}$ and $R(\lambda) := i_{\Omega} \circ (\lambda + A_{\Omega})^{-1} \circ r_{\Omega}$

whenever the operators are defined. Corresponding to the adjoint we set

(8)

$$R_n^{\#}(\lambda) := i_{\Omega_n} \circ \left(\lambda + A_{\Omega_n}^{\#}\right)^{-1} \circ r_{\Omega_n} \text{ and } R^{\#}(\lambda) := i_{\Omega} \circ \left(\lambda + A_{\Omega}^{\#}\right)^{-1} \circ r_{\Omega}.$$

We observe that $\left(R_n^{\#}(\lambda)\right)' = R_n(\lambda)$ and $\left(R^{\#}(\lambda)\right)' = R(\lambda)$. Here $R(\lambda)$ is a pseudoresolvent, that is, a family of operators satisfying the resolvent identity.

Theorem 1. *Suppose that $u_n \in H_0^1(\Omega_n)$ are weak solutions of (1) for all $n \in \mathbb{N}$. Suppose that for every $\varphi \in H_0^1(\Omega)$ there exists a sequence $\varphi_n \in H_0^1(\Omega_n)$, $n \geq 1$, such that $\varphi_n \to \varphi$ in $H^1\left(\mathbb{R}^N\right)$. Then every weak limit point of $(u_n)_{n\in\mathbb{N}}$ lying in $H_0^1(\Omega)$ is a weak solution of (2) for some $f \in H^{-1}(\Omega)$.*

The proof of the previous theorem can be found in ([3]).

3. NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE OF OPEN SETS

4. IN THE SENSE OF MOSCO

The results considered here have been applied to study the convergence of weak solutions of the elliptic boundary value problem (1) to a weak solution of the corresponding problem (2). We will discuss some necessary and sufficient conditions for convergence in sense of Mosco.

Definition 2. ***(the convergence in sense of Mosco)*** *Let Ω and Ω_n, $n \geq 1$, be open sets in $\mathbb{R}^N$. We say that the sequence $(\Omega_n)_{n\geq 1}$ converges to Ω in the sense of Mosco and we write $\Omega_n \to \Omega$ if the following conditions are satisfied:*

- **i.:** *The sequentially weak limit points in $H^1\left(\mathbb{R}^N\right)$ of every sequence $(u_n)_{n\geq 1}$, with $u_n \in H_0^1(\Omega_n)$ for all $n \geq 1$, are in $H_0^1(\Omega)$;*
- **ii.:** *For every $u \in H_0^1(\Omega)$ there exists a sequence $(u_n)_{n\geq 1}$, with $u_n \in H_0^1(\Omega_n)$ for all $n \geq 1$, such that $u_n \to u$ in $H^1\left(\mathbb{R}^N\right)$;*
- **:** *(Condition (ii) is equivalent to $H_0^1(\Omega_n) \to H_0^1(\Omega)$).*

One obviously necessary condition is that the support of the weak limit of every convergent subsequence of solutions of (1) is in $\overline{\Omega}$. We will characterize this by looking at the spectral bound of $(-\Delta)$ on bounded sets outside $\overline{\Omega}$.

We will write $S \subset\subset T$ if $\overline{S}$ is compact and contained in the interior of T.

Theorem 3. *Suppose that Ω_n, $n \geq 1$ and Ω are open sets in $\mathbb{R}^N$. Then the following assertions are equivalent:*

1. *The weak limit points af every sequence $u_n \in H_0^1(\Omega_n)$, $n \in \mathbb{N}$ in $H^1(\mathbb{R}^N)$ have support in $\overline{\Omega}$;*
2. *For all open sets $B \subset\subset \mathbb{R}^N \setminus \overline{\Omega}$, $\lim\limits_{n\to\infty} \lambda_1(\Omega_n \cap B) = \infty$;*
3. *There exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that the previous relation holds for all $B \in \mathcal{O}$.*

If condition (i) from Definition 2 is satisfied, then $\lim\limits_{n\to\infty} \lambda_1(\Omega_n \cap B) = \infty$ for all bounded sets $B \subset \mathbb{R}^N \setminus \overline{\Omega}$.

The proof and exemples can be find in [3], page 15. So far, we discussed necessary conditions on Ω_n outside $\overline{\Omega}$. In order to state more necessary conditions and also sufficient conditions for $\Omega_n \to \Omega$, we will need a type of variational Sobolev capacity.

Definition 4. *The capacity (more specific, the $(1,2)-$capacity) of a set $E \subset \mathbb{R}^N$ is given by*

$$cap(E) := \inf\left\{ \|u\|_{H^1}^2 : u \in H_0^1(B) \text{ and } u \geq 1 \text{ in a neighborhood of } E \right\}.$$

Now we can state a necessary condition on the part of Ω_n inside Ω.

Proposition 5. *Suppose that Ω_n, $n \geq 1$ and Ω are open sets in $\mathbb{R}^N$. Then condition (ii) from Definition 2 holds if and only if for every compact set $K \subset \Omega$ we have*

$$\lim_{n\to\infty} cap\left(K \cap \mathbb{R}^N \setminus \Omega_n\right) = 0.$$

We consider monotone approximations of an open set Ω by open sets from the inside and from the outside.

Proposition 6. *Suppose that Ω_n, $n \geq 1$ and Ω are open sets in $\mathbb{R}^N$ such that $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}^*$ and $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$. Then $\Omega_n \to \Omega$.*

This proposition is a tool to get results for non-smooth domains using the results on smooth domains. For approximations from the outside we need a weak regularity condition on the boundary of Ω, whose formulation requires some properties of functions in $H^1(\mathbb{R}^N)$.

Definition 7. *We say that an arbitrary set $S \subset \mathbb{R}^N$ is stable if $H_0^1(S) = H_0^1(S^\circ)$ where S° denotes the interior of S.*

Proposition 8. *A set $S \subset \mathbb{R}^N$ with non-empty interior is stable if one of the following conditions is satisfied:*

(1) *$\partial S \cap S$ has the segment property except possibly on a set capacity zero;*
(2) *all points in $\partial S \cap S$ except possibly a set of capacity zero are Wiener regular;*
(3) *for all $x \in \partial S \cap S$ except possibly a set of capacity zero $\lim\limits_{r \to 0} \frac{cap\left(\mathbb{R}^N \setminus S \cap B(x,r)\right)}{cap\left(\mathbb{R}^N \setminus S^\circ \cap B(x,r)\right)} > 0$ where $B(x,r)$ is the ball of radius r centred in x.*

The last condition is in fact necessary and sufficient for the stability of $\Omega \cup \Gamma$. Indications about the demonstration of the previous proposition can be also find in [3], page 18. Note that, if $\partial S \cap S$ is Lipschitz (or even smoother), then $\partial S \cap S$ satisfies the segment condition and $\partial S \cap S$ is therefore stable.

Proposition 9. *Suppose that $\Omega_n \supset \Omega$ for all $n \in \mathbb{N}$ and that $\bigcap\limits_{n \in \mathbb{N}} \Omega_n = \overline{\Omega}$. If $\overline{\Omega}$ is stable then $\Omega_n \to \Omega$.*

Proof. The condition (**i**) from the definition of the convergence in sense of Mosco holds. As $\bigcap\limits_{n \in \mathbb{N}} \Omega_n = \overline{\Omega}$, it follows that all weak limit points of $u_n \in H_0^1(\Omega)$ for $n \in \mathbb{N}$ have support in $\overline{\Omega}$. Hence by definition of stability all weak limit points are in $H_0^1(\Omega)$, as required in the condition (**i**) from definition *2*. It follows that $\Omega_n \to \Omega$. ∎

Theorem 10. *Suppose that Ω_n, $n \geq 1$ and Ω are open sets in $\mathbb{R}^N$ (not necessarily bounded). If the following three conditions are satisfied, then $\Omega_n \to \Omega$:*

(1) $\lim\limits_{r \to 0} cap\left(K \cap \mathbb{R}^N \setminus \Omega_n\right) = 0$ *for all compact sets $K \subset \Omega$;*
(2) *there exists an open covering $\mathcal{O}$ of $\mathbb{R}^N \setminus \overline{\Omega}$ such that $\lambda_1(U \cap \Omega_n) \to \infty$ as $n \longrightarrow \infty$ for all $U \in \mathcal{O}$,*
(3) $H_0^1(\Omega) = H_0^1(\Omega \cup \Gamma)$, *where* $\Gamma := \bigcap\limits_{n \in \mathbb{N}} \left(\overline{\bigcup\limits_{k \geq n} (\Omega_k \cap \partial\Omega)} \right) \subset \partial\Omega.$

References

[1] Arendt W., Daners D., **Uniform convergence for elliptic problems on varying domains, Mathematische Nachrichten**, 280 (2007), 28-49.

[2] Biegert M., Daners D., **Local and Global Uniform convergence for elliptic problems on varying domains**, Journal of Differential Equations, 223 (2006), 1-32.

[3] Daners D., **Dirichlet problems on varying domains**, Journal of Differential Equations 188 (2003), 591-624.

[4] Henrot A., **Continuity with respect to the domain for the Laplacian: a survey**, Control Cybernet.23(1994), 427-443.

[5] Kato T., **Perturbation Theory for Linear Operators**, second ed., Grundlehren der Mathematischen Wissenschaften, vol. 132, Springer, Berlin, 1976.

[6] Mosco U., **Convergence of convex sets and of solutions of variational inequalities**, Advances in Math. 3 (1969), 510-585.

[7] Pazy A., **Semigroups of linear operators and applications to partial differential equations**, Springer Verlag, New York, 1983.

[8] Stolmann P., **A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains**, Math. Z 219 (1995), 275-287.

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