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Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 32 (2022), No. 2, 13-28

FUZZY PRE-SEMI-CONTINUOUS FUNCTIONS AND FUZZY PRE-SEMI-IRRESOLUTE FUNCTIONS

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Abstract. This paper deals with a new type of fuzzy open-like set, viz., fuzzy pre-semiopen set, the class of which is strictly larger than that of fuzzy preopen sets and fuzzy s^*-open sets. Using this concept as a basic tool, here we introduce fuzzy pre-semiclosure operator which is an idempotent operator. Afterwards, we introduce the notion of fuzzy pre-semi-regular space in which the class of all fuzzy closed sets and that of all fuzzy pre-semiclosed sets coincide. Lastly, we introduce and characterize two types of functions, viz., fuzzy pre-semi-continuous and fuzzy pre-semi-irresolute functions, classes that are strictly larger than that of fuzzy almost s-continuous and fuzzy almost s^*-continuous functions, respectively, and establish some applications of these two classes of functions on fuzzy pre-semi-regular spaces.

Keywords and phrases: Fuzzy semiopen set, fuzzy pre-semiopen set, fuzzy pre-semi-regular space, fuzzy pre-semi-continuous function, fuzzy pre-semi-irresolute function.
(2020) Mathematics Subject Classification: 54A40, 03E72.

1. Introduction

After introducing fuzzy open set [4], several types of fuzzy open-like functions are introduced and studied. In this context we have to mention fuzzy regular open [1], fuzzy semiopen [1], fuzzy preopen [6], fuzzy s^*-open [3] set. Here we introduce fuzzy pre-semiopen set. In [4], fuzzy continuous function was introduced. Using fuzzy pre-semiopen set as a basic tool, here we introduce fuzzy pre-semi-continuous function, the class of which is strictly larger than that of fuzzy continuous function. Again fuzzy pre-semi-irresolute function is introduced here, the class of which is coarser than that of fuzzy pre-semi-continuous functions. Here we also introduce the notion of fuzzy pre-semi-regular space in which fuzzy pre-semi-continuous function and fuzzy pre-semi-irresolute function coincide.

2. Preliminaries

Throughout this paper, (X, τ) or simply by X we shall mean a fuzzy topological space. A fuzzy set A is a function from a non-empty set X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$ [8]. The support [8] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [8] of a fuzzy set A in an fts X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [8] while AqB means A is quasi-coincident (q-coincident, for short) [7] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not q B$ respectively. For a fuzzy set A, clA and $intA$ will stand for fuzzy closure [4] and fuzzy interior [4] respectively. A fuzzy set A in X is called a fuzzy neighbourhood (nbd, for short) [7] of a fuzzy point x_t if there exists a fuzzy open set G in X such that $x_t \in G \leq A$. If, in addition, A is fuzzy open, then A is called fuzzy open nbd of x_t. A fuzzy set A is said to be a fuzzy quasi neighbourhood (q-nbd for short) of a fuzzy point x_t in an fts X if there is a fuzzy open set U in X such that $x_t q U \leq A$. If, in addition, A is fuzzy open, then A is called a fuzzy open q-nbd of x_t [7].

A fuzzy set A in an fts (X, τ) is called fuzzy regular open [1] (resp., fuzzy semiopen [1], fuzzy preopen [6]) if $A = intclA$ (resp., $A \leq cl(intA)$, $A \leq int(clA)$). The complement of a fuzzy semiopen set is called fuzzy semiclosed [1]. The union (intersection) of all fuzzy semiopen (resp., fuzzy semiclosed) sets contained in (resp., containing) a fuzzy set A is called fuzzy semiinterior [1] (resp., fuzzy semiclosure [1]) of A, denoted by $sintA$ (resp., $sclA$). A fuzzy set A in an fts X is called a fuzzy semi neighbourhood (semi nbd, for short) [1] of a fuzzy point x_α in X if there exists a fuzzy semiopen set U in X such that $x_\alpha \leq U \leq A$. The collection of all fuzzy semiopen (resp., fuzzy semiclosed) sets in X is denoted by $FSO(X)$ (resp., $FSC(X)$) and that of fuzzy preopen (resp. fuzzy preclosed) sets is denoted by $FPO(X)$ (resp. $FPC(X)$).

3. Fuzzy Pre-Semiopen and Pre-Semiclosed Sets : Some Properties

In this section, we introduce fuzzy pre-semiopen and fuzzy pre-semiclosed sets. Also here we introduce fuzzy pre-semi closure operator which is an idempotent operator. Afterwards, it is shown that the class of all fuzzy pre-semiopen sets is strictly larger than that of fuzzy preopen [6] and fuzzy s^*-open [3] sets.

Definition 3.1. A fuzzy set A in an fts (X, τ) is called *fuzzy pre-semiopen* if $A \leq sint(clA)$. The complement of this set is called fuzzy pre-semiclosed set.

The collection of fuzzy pre-semiopen (resp., fuzzy pre-semiclosed) sets in (X, τ) is denoted by $FPSO(X)$ (resp., $FPSC(X)$).

The union (resp., intersection) of all fuzzy pre-semiopen (resp., fuzzy pre-semiclosed) sets contained in (containing) a fuzzy set A is called fuzzy pre-semi interior (resp., fuzzy pre-semi closure) of A, denoted by $psintA$ (resp., $psclA$).

Result 3.2. *Every union of two fuzzy pre-semiopen sets in an fts X is also fuzzy pre-semiopen.*

Proof. Let $A, B \in FPSO(X)$. Then $A \leq sint(clA), B \leq sint(clB)$. Now $sint(cl(A \bigvee B)) = sint(clA \bigvee clB) \geq sint(clA) \bigvee sint(clB) \geq A \bigvee B$.

Remark 3.3. The intersection of two fuzzy pre-semiopen sets need not be fuzzy pre-semiopen, as follows from the next example.

Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, B\}$ where $B \geq A$. Consider two fuzzy sets U, V defined by $U(a) = 0.5, U(b) = 0.7, V(a) = 0.6, V(b) = 0.3$. Then clearly $U, V \in FPSO(X)$. Let $W = U \bigwedge V$. Then $W(a) = 0.5, W(b) = 0.3$. Now $sint(clW) = sint(1_X \setminus A) = 0_X \not\geq W \Rightarrow W \notin FPSO(X)$.

Note 3.5. Clearly, every fuzzy preopen set is fuzzy pre-semiopen. The converse is false, as follows from the next example.

Example 3.6. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Consider the fuzzy set B defined by $B(a) = B(b) = 0.5$. Then $sint(clB) = 1_X \setminus A \geq B \Rightarrow B \in FPSO(X)$. But $int(clB) = A \not\geq B \Rightarrow B$ is not fuzzy preopen set in (X, τ).

Next we recall the following definition from [3] for ready references.

Definition 3.7 [3]. A fuzzy set A in an fts X is called fuzzy s^*-open if $A \leq int(sclA)$.

Remark 3.8. It is clear from definitions that fuzzy s^*-open set is fuzzy pre-semiopen. But converse need not be true follows from the following example.

Example 3.9. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55$. Then (X, τ) is an fts. Here $FSO(X) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus B, B \leq V \leq 1_X \setminus A$ and $FSC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $B \leq 1_X \setminus U \leq 1_X \setminus A, A \leq 1_X \setminus V \leq 1_X \setminus B$. Consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.43$. Now $sint(clC) = sint(1_X \setminus B) = 1_X \setminus B \geq C \Rightarrow C \in FPSO(X)$. But $int(sclC) = intC = A \not\geq C \Rightarrow C$ is not fuzzy s^*-open set in X.

Definition 3.10. A fuzzy set A in an fts (X, τ) is called *fuzzy pre-semi neighbourhood* (fuzzy pre-semi nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre-semiopen set U in X such that $x_\alpha \leq U \leq A$. If, in addition, A is fuzzy pre-semiopen, then A is called fuzzy pre-semiopen nbd of x_α.

Definition 3.11. A fuzzy set A in an fts (X, τ) is called *fuzzy pre-semi quasi neighbourhood* (fuzzy pre-semi q-nbd, for short) of a fuzzy point x_α if there exists a fuzzy pre-semiopen set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy pre-semiopen, then A is called fuzzy pre-semiopen q-nbd of x_α.

Remark 3.12. It is clear from definitions that fuzzy nbd (resp., fuzzy q-nbd) of a fuzzy point is a fuzzy pre-semi nbd (resp., fuzzy pre-semi q-nbd) in an fts. But converse may not be true follows from the following example.

Example 3.13. Let $X = \{a, b\}$, $\tau = \{0_X, 1_X\}$. Then (X, τ) is an fts. Now consider the fuzzy point $a_{0.3}$ and the fuzzy set A defined by $A(a) = A(b) = 0.8$. Since every fuzzy set in (X, τ) is fuzzy pre-semiopen in (X, τ), clearly A is fuzzy pre-semi nbd (resp., fuzzy pre-semi q-nbd) of $a_{0.3}$. But A is not a fuzzy nbd (resp., fuzzy q-nbd) of $a_{0.3}$ as there is no fuzzy open set U in X such that $a_{0.3} \leq U \leq A$ (resp., $a_{0.3}qU \leq A$).

Theorem 3.14. *For any fuzzy set A in an fts (X, τ), $x_\alpha \in psclA$ if and only if UqA* for *every fuzzy pre-semiopen q-nbd U of x_α .*

Proof. Let $x_\alpha \in psclA$ for any fuzzy set A in an fts (X, τ). Let $U \in FPSO(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq 1_X \setminus U \in FPSC(X)$. Then by definition, $A \not\leq 1_X \setminus U \Rightarrow$ there exists $y \in X$ such that $A(y) > 1 - U(y) \Rightarrow A(y) + U(y) > 1 \Rightarrow UqA$.
Conversely, let the given condition hold. Let $U \in FPSC(X)$ with $A \leq U$ ... (1). We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha \Rightarrow x_\alpha q(1_X \setminus U)$ where $1_X \setminus U \in FPSO(X)$. By hypothesis, $(1_X \setminus U)qA \Rightarrow$ there exists $y \in X$ such that $1 - U(y) + A(y) > 1 \Rightarrow A(y) > U(y)$, contradicts (1).

Theorem 3.15. *$pscl(psclA) = psclA$ for any fuzzy set A in an fts (X, τ).*

Proof. Let $A \in I^X$. Then $A \leq psclA \Rightarrow psclA \leq pscl(psclA)$ ... (1).
Conversely, let $x_\alpha \in pscl(psclA)$. If possible, let $x_\alpha \notin psclA$. Then there exists $U \in FPSO(X)$,

$$x_\alpha qU, U \not{q} A ...(2)$$

But as $x_\alpha \in pscl(psclA)$, $Uq(psclA) \Rightarrow$ there exists $y \in X$ such that $U(y) + (psclA)(y) > 1 \Rightarrow U(y) + t > 1$ where $t = (psclA)(y)$. Then $y_t \in psclA$ and $y_t qU$ where $U \in FPSO(X)$. Then by definition, UqA, contradicts (2). So

$$pscl(psclA) \leq psclA ...(3)$$

Combining (1) and (3), we get the result.

4. Fuzzy Pre-Semi-Continuous Function : Some Characterizations

In this section we introduce and characterize fuzzy pre-semi-continuous function, the class of which is strictly larger than that of fuzzy continuous function [4] and fuzzy almost s-continuous function [3].

Definition 4.1. A function $f : X \to Y$ is said to be *fuzzy pre-semi-continuous* if for each fuzzy point x_α in X and every fuzzy nbd V of $f(x_\alpha)$ in Y, $cl(f^{-1}(V))$ is a fuzzy seminbd of x_α in X.

Theorem 4.2. *For a function $f : X \to Y$, the following statements are equivalent :*
(a) f is fuzzy pre-semi-continuous,

(b) $f^{-1}(B) \leq sint(cl(f^{-1}(B)))$*, for every fuzzy open set* B *of* Y*,*
(c) $f(sclA) \leq cl(f(A))$*, for every fuzzy open set* A *in* X*.*

Proof (a) $\Rightarrow$ (b). Let B be any fuzzy open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \in B \Rightarrow B$ is a fuzzy nbd of $f(x_\alpha)$ in Y. By (a), $cl(f^{-1}(B))$ is a fuzzy seminbd of x_α in X. So $x_\alpha \in sint(cl(f^{-1}(B)))$. Since x_α is taken arbitrarily, $f^{-1}(B) \leq sint(cl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \in f^{-1}(B) \leq sint(cl(f^{-1}(B)))$ (by (b)) $\leq cl(f^{-1}(B))$. So $cl(f^{-1}(B))$ is a fuzzy seminbd of x_α in X.
(b) $\Rightarrow$ (c). Let A be a fuzzy open set in X. Then $1_Y \setminus cl(f(A))$ is a fuzzy open set in Y. By (b), $f^{-1}(1_Y \setminus cl(f(A))) \leq sint(cl(f^{-1}(1_Y \setminus cl(f(A))))) = sint(cl(1_X \setminus f^{-1}(cl(f(A))))) \leq sint(cl(1_X \setminus f^{-1}(f(A)))) \leq sint(cl(1_X \setminus A)) = sint(1_X \setminus A) = 1_X \setminus sclA$. Then $sclA \leq 1_X \setminus f^{-1}(1_Y \setminus cl(f(A))) = f^{-1}(cl(f(A)))$. So $f(sclA) \leq cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy open set in Y. Then $int(f^{-1}(1_Y \setminus B))$ is a fuzzy open set in X. By (c), $f(scl(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(int(f^{-1}(1_Y \setminus B)))) \leq cl(f(f^{-1}(1_Y \setminus B))) \leq cl(1_Y \setminus B) = 1_Y \setminus B \Rightarrow B \leq 1_Y \setminus f(scl(int(f^{-1}(1_Y \setminus B))))$. Then $f^{-1}(B) \leq f^{-1}(1_Y \setminus f(scl(int(f^{-1}(1_Y \setminus B))))) = 1_X \setminus f^{-1}(f(scl(int(f^{-1}(1_Y \setminus B))))) \leq 1_X \setminus scl(int(f^{-1}(1_Y \setminus B))) = 1_X \setminus scl(int(1_X \setminus f^{-1}(B))) = sint(cl(f^{-1}(B)))$.

Note 4.3. It is clear from Theorem 4.2 that the inverse image under fuzzy pre-semi-continuous function of any fuzzy open set is fuzzy pre-semiopen.

Theorem 4.4. *For a function* $f : X \to Y$*, the following statements are equivalent :*
(a) f *is fuzzy pre-semi-continuous,*
(b) $f^{-1}(B) \leq sint(cl(f^{-1}(B)))$*, for every fuzzy open set* B *of* Y*,*
(c) for each fuzzy point x_α *in* X *and each fuzzy open nbd* V *of* $f(x_\alpha)$ *in* Y*, there exists* $U \in FPSO(X)$ *containing* x_α *such that* $f(U) \leq V$*,*
(d) $f^{-1}(F) \in FPSC(X)$*, for all fuzzy closed sets* F *in* Y*,*
(e) for each fuzzy point x_α *in* X*, the inverse image under* f *of every fuzzy nbd of* $f(x_\alpha)$ *is a fuzzy pre-semi nbd of* x_α *in* X*,*
(f) $f(psclA) \leq cl(f(A))$*, for every fuzzy open set* A *in* X*,*
(g) $pscl(f^{-1}(B)) \leq f^{-1}(clB)$*, for every fuzzy open set* B *in* Y*,*
(h) $f^{-1}(intB) \leq psint(f^{-1}(B))$*, for every fuzzy open set* B *in* Y*,*
(i) $f^{-1}(V) \in FPSO(X)$*, for every basic open fuzzy set* V *in* Y *.*

Proof (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open nbd of $f(x_\alpha)$ in Y. By (b), $f^{-1}(V) \leq sint(cl(f^{-1}(V)))$ ... (1). Now

$f(x_\alpha) \in V \Rightarrow x_\alpha \in f^{-1}(V)$ ($= U$, say). Then $x_\alpha \in U$ and by (1), $U(= f^{-1}(V)) \in FPSO(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and let $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (c), there exists $U \in FPSO(X)$ containing x_α such that $f(U) \leq V$. Then $x_\alpha \in U \leq f^{-1}(V)$. Now $U \leq sint(clU)$. Then $U \leq sint(clU) \leq sint(cl(f^{-1}(V))) \Rightarrow x_\alpha \in U \leq sint(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq sint(cl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy nbd of $f(x_\alpha)$ in Y. Then there exists a fuzzy open set V in Y such that $f(x_\alpha) \in V \leq W \Rightarrow V$ is a fuzzy open nbd of $f(x_\alpha)$ in Y. Then by (b), $f^{-1}(V) \in FPSO(X)$ and $x_\alpha \in f^{-1}(V) \leq f^{-1}(W) \Rightarrow f^{-1}(W)$ is a fuzzy pre-semi nbd of x_α in X.
(e) $\Rightarrow$ (b). Let V be a fuzzy open set in Y and $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V$. Then V is a fuzzy open nbd of $f(x_\alpha)$ in Y. By (e), there exists $U \in FPSO(X)$ containing x_α such that $U \leq f^{-1}(V) \Rightarrow x_\alpha \in U \leq sint(clU) \leq sint(cl(f^{-1}(V)))$. Since x_α is taken arbitrarily, $f^{-1}(V) \leq sint(cl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $cl(f(A))$ is a fuzzy closed set in Y. By (d), $f^{-1}(cl(f(A))) \in FPSC(X)$ containing A. Therefore, $psclA \leq pscl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(psclA) \leq cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), $f(pscl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB = B \Rightarrow pscl(f^{-1}(B)) \leq f^{-1}(B) \Rightarrow f^{-1}(B) \in FPSC(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f), $f(pscl(f^{-1}(B))) \leq cl(f(f^{-1}(B))) \leq clB \Rightarrow pscl(f^{-1}(B)) \leq f^{-1}(clB)$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $pscl(f^{-1}(f(A))) \leq f^{-1}(cl(f(A))) \Rightarrow psclA \leq f^{-1}(cl(f(A))) \Rightarrow f(psclA) \leq cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $intB$ is a fuzzy open set in Y. By (b), $f^{-1}(intB) \leq sint(cl(f^{-1}(intB))) \Rightarrow f^{-1}(intB) \in FPSO(X) \Rightarrow f^{-1}(intB) = psint(f^{-1}(intB)) \leq psint(f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy open set in Y. Then $f^{-1}(A) = f^{-1}(intA) \leq psint(f^{-1}(A))$ (by (h)) $\Rightarrow f^{-1}(A) \in FPSO(X)$.
(b) $\Rightarrow$ (i). Obvious.
(i) $\Rightarrow$ (b). Let W be any fuzzy open set in Y. Then there exists a collection $\{W_\alpha : \alpha \in \Lambda\}$ of fuzzy basic open sets in Y such that $W = \bigvee_{\alpha \in \Lambda} W_\alpha$. Now $f^{-1}(W) = f^{-1}(\bigvee_{\alpha \in \Lambda} W_\alpha) = \bigvee_{\alpha \in \Lambda} f^{-1}(W_\alpha) \in FPSO(X)$

(by (i) and by Result 3.2). Hence (b) follows.

Theorem 4.5. *A function $f : X \to Y$ is fuzzy pre-semi-continuous if and only if for each fuzzy point x_α in X and each fuzzy open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-semi q-nbd W in X such that $f(W) \leq V$.*

Proof. Let f be fuzzy pre-semi-continuous function and x_α be a fuzzy point in X and V be a fuzzy open q-nbd of $f(x_\alpha)$ in Y. Then $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1 \Rightarrow V(y) > 1 - \alpha \Rightarrow V(y) > \beta > 1 - \alpha$, for some real number β. Then V is a fuzzy open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in FPSO(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - \alpha \Rightarrow x_\alpha qW$ and $f(W) \leq V$.
Conversely, let the given condition hold and let V be a fuzzy open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, $1/n \leq 1/m \Rightarrow 1 + 1/n \leq 1 + 1/m \Rightarrow \alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N} \Rightarrow 0 < \alpha_n \leq 1$ so that $V(y)+\alpha_n > 1 \Rightarrow y_{\alpha_n}qV \Rightarrow V$ is a fuzzy open q-nbd of y_{α_n}. By the given condition, there exists $U_n^x \in FPSO(X)$ such that $x_{\alpha_n}qU_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in FPSO(X)$ (by Result 3.2) and $f(U^x) \leq V$. Again $n \geq m \Rightarrow U_n^x(x) + \alpha_n > 1 \Rightarrow U_n^x(x) + 1 + 1/n - W(x) > 1 \Rightarrow U_n^x(x) > W(x) - 1/n \Rightarrow U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0 \Rightarrow W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee\limits_{x \in W_0} U^x = U$ (say) ... (1) and $f(U^x) \leq V$, for all $x \in W_0 \Rightarrow f(U) \leq V \Rightarrow U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ ... (2). By (1) and (2), $U = W = f^{-1}(V) \Rightarrow f^{-1}(V) \in FPSO(X)$. Hence by Theorem 4.2, f is fuzzy pre-semi-continuous function.

Note 4.6. Since fuzzy regular open set is fuzzy open, by Note 4.3, we can easily say that the inverse image of fuzzy regular open set under fuzzy pre-semi-continuous function is fuzzy pre-semiopen.

Remark 4.7 (i) The inverse image of a fuzzy semiopen (resp., fuzzy preopen) set under fuzzy pre-semi-continuous function need not be so follows from the following example.
(ii) Composition of two fuzzy pre-semi-continuous functions need not be so, follows from the following example.

Example 4.8. Let $X = \{a\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.45, B(a) = 0.48, C(a) = 0.52$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in \tau_2, i^{-1}(C) = C = sint_{\tau_1}(cl_{\tau_1}C) = 1_X \setminus B \in FPSO(X, \tau_1) \Rightarrow$ i is fuzzy pre-semi-continuous function. Consider the fuzzy set D defined by $D(a) = 0.53$. Then $D \in FSO(X, \tau_2)$ as well as $D \in FPO(X, \tau_2)$. Now $i^{-1}(D) = D$. Then $sint_{\tau_1}(cl_{\tau_1}D) = sint_{\tau_1}(1_X \setminus A) = 1_X \setminus B \not\geq D \Rightarrow D \notin FPSO(X, \tau_1)$.

Example 4.9. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly i_1 and i_2 are fuzzy pre-semi-continuous functions. But $B \in \tau_3$, $(i_2 \circ i_1)^{-1}(B) = B \not\leq sint_{\tau_1}(cl_{\tau_1}B) = 0_X \Rightarrow i_2 \circ i_1$ is not a fuzzy pre-semi-continuous function.

Lemma 4.10 [2]. *Let Z, X, Y be fts's and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1, U_2 are fuzzy sets in Z, X, Y respectively such that $f(B) \leq U_1 \times U_2$, then $f_1(B) \leq U_1$ and $f_2(B) \leq U_2$.*

Theorem 4.11. *Let Z, X, Y be fts's. For any functions $f_1 : Z \to X, f_2 : Z \to Y$, if $f : Z \to X \times Y$, defined by $f(x) = (f_1(x), f_2(x))$, for all $x \in Z$, is fuzzy pre-semi-continuous function, so are f_1 and f_2.*

Proof. Let U_1 be any fuzzy open q-nbd of $f_1(x_\alpha)$ in X for any fuzzy point x_α in Z. Then $U_1 \times 1_Y$ is a fuzzy open q-nbd of $f(x_\alpha)$, i.e., $(f(x))_\alpha$ in $X \times Y$. Since f is fuzzy pre-semi-continuous, there exists $V \in FPSO(Z)$ with $x_\alpha q V$ such that $f(V) \leq U_1 \times 1_Y$. By Lemma 4.10, $f_1(V) \leq U_1$, $f_2(V) \leq 1_Y$. Consequently, f_1 is fuzzy pre-semi-continuous.

Similarly, f_2 is fuzzy pre-semi-continuous.

Lemma 4.12 [1]. *Let X, Y be fts's and let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then if A, B are fuzzy sets in X and Y respectively, $g^{-1}(A \times B) = A \bigcap f^{-1}(B)$.*

Theorem 4.13. *Let $f : X \to Y$ be a function from an fts X to an fts Y and $g : X \to X \times Y$ be the graph function of f. If g is a fuzzy pre-semi-continuous function, then f is also fuzzy pre-semi-continuous.*

Proof. Let g be fuzzy pre-semi-continuous function and B be a fuzzy set in Y. Then by Lemma 4.12, $f^{-1}(B) = 1_X \bigwedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now if B is fuzzy open in Y, then $1_X \times B$ is fuzzy open in $X \times Y$. Again, $g^{-1}(1_X \times B) = f^{-1}(B) \in FPSO(X)$ as g is fuzzy pre-semi-continuous function. Hence f is fuzzy pre-semi-continuous.

Let us now recall the following definition from [4] for ready references.

Definition 4.14 [4]. A function $f : X \to Y$ is called *fuzzy continuous* if the inverse image of every fuzzy open set in Y is fuzzy open set in X.

Note 4.15. It is clear from definitions that every fuzzy continuous function is fuzzy pre-semi-continuous. But the converse is not necessarily true, as follows from the next example.

Example 4.16. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Clearly i is not fuzzy continuous function. Now every fuzzy set in (X, τ_1) is fuzzy pre-semiopen set in (X, τ_1) and so i is fuzzy pre-semi-continuous function.

Let us now recall the following definition from [3] for ready references.

Definition 4.17 [3]. A fuzzy set A in an fts X is called *fuzzy s^*-open* if $A \leq int(sclA)$.

Definition 4.18 [3]. A function $f : X \to Y$ is called *fuzzy almost s-continuous* if $f^{-1}(B)$ is fuzzy s^*-open set in X for every fuzzy open set B in Y.

Remark 4.19. It is clear from definitions that every fuzzy almost s-continuous function is fuzzy pre-semi-continuous. The converse is false, as it is seen from the following example.

Example 4.20. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55, C(a) = 0.5, C(b) = 0.43$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in \tau_2, i^{-1}(C) = C \leq sint_{\tau_1}(cl_{\tau_1}C) = sint_{\tau_1}(1_X \backslash B) = 1_X \backslash B \Rightarrow C \in FPSO(X, \tau_1) \Rightarrow i$ is fuzzy pre-semi-continuous function. But $int_{\tau_1}(scl_{\tau_1}C) = int_{\tau_1}C = A \not\geq C \Rightarrow C$ is not a fuzzy s^*-open set in $X \Rightarrow i$ is not a fuzzy almost s-continuous function.

5. Fuzzy Pre-Semi-Irresolute Function: Some Properties

In this section we introduce a new type of function, viz., fuzzy pre-semi-irresolute functions, the class of which is coarser than that of fuzzy pre-semi-continuous functions.

Definition 5.1. A function $f : X \to Y$ is called *fuzzy pre-semi-irresolute* if the inverse image of every fuzzy pre-semiopen set in Y is fuzzy pre-semiopen in X.

Theorem 5.2. *For a function $f : X \to Y$, the following statements are equivalent :*
(a) f is fuzzy pre-semi-irresolute,
(b) for each fuzzy point x_α in X and each fuzzy pre-semiopen nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-semiopen nbd U of x_α in X and $f(U) \leq V$,
(c) $f^{-1}(F) \in FPSC(X)$, for all $F \in FPSC(Y)$,
(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy pre-semiopen nbd of $f(x_\alpha)$ is a fuzzy pre-semiopen nbd of x_α in X,
(e) $f(psclA) \leq pscl(f(A))$, for all $A \in I^X$,
(f) $pscl(f^{-1}(B)) \leq f^{-1}(psclB)$, for all $B \in I^Y$,
(g) $f^{-1}(psintB) \leq psint(f^{-1}(B))$, for all $B \in I^Y$.

Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function $f : X \to Y$ is fuzzy pre-semi-irresolute if and only if for each fuzzy point x_α in X and corresponding to any fuzzy pre-semiopen q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy pre-semiopen q-nbd W of x_α in X such that $f(W) \leq V$.*

Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Note 5.4. Composition of two fuzzy pre-semi-irresolute functions is also so.

Theorem 5.5. *If $f : X \to Y$ is fuzzy pre-semi-irresolute and $g : Y \to Z$ is fuzzy pre-semi-continuous (resp., fuzzy continuous), then $g \circ f : X \to Z$ is fuzzy pre-semi-continuous.*

Proof. Obvious.

Remark 5.6. Every fuzzy pre-semi-irresolute function is fuzzy pre-semi-continuous, but the converse is not true, in general, follows from the following example.

Example 5.7. There exists a fuzzy pre-semi-continuous function which is not fuzzy pre-semi-irresolute.

Let $X = \{a, b\}$, $\tau = \{0_X, 1_X, A\}$, $\tau_1 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) and (X, τ_1) are fts's. Consider the identity function $i : (X, \tau) \to (X, \tau_1)$. Clearly i is fuzzy pre-semi-continuous function. Now every fuzzy set in (X, τ_1) is fuzzy pre-semiopen set in (X, τ_1). Consider the fuzzy set B defined by $B(a) = B(b) = 0.4$. Then $B \in FPSO(X, \tau_1)$. Now $i^{-1}(B) = B \not\leq$

$sint_\tau(cl_\tau B) = 0_X \Rightarrow B \notin FPSO(X,\tau) \Rightarrow i$ is not fuzzy pre-semi-irresolute function.

6. Fuzzy Pre-Semi-Regular Space

In this section fuzzy pre-semi-regular space is introduced in which space fuzzy closed set and fuzzy pre-semiclosed set coincide.

Definition 6.1. An fts (X,τ) is said to be a *fuzzy pre-semi-regular space* if for each fuzzy pre-semiclosed set F in X and each fuzzy point x_α in X with $x_\alpha \notin F$, there exist a fuzzy open set U in X and a fuzzy pre-semiopen set V in X such that $x_\alpha qU$, $F \leq V$ and $U \not{q} V$.

Theorem 6.2. *For an fts (X,τ), the following statements are equivalent:*
(a) X is fuzzy pre-semi regular,
(b) for each fuzzy point x_α in X and each fuzzy pre-semiopen set U in X with $x_\alpha qU$, there exists a fuzzy open set V in X such that $x_\alpha qV \leq psclV \leq U$,
(c) for each fuzzy pre-semiclosed set F in X, $\bigwedge\{clV : F \leq V, V \in FPSO(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy pre-semiopen set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $psclV \leq U$.

Proof (a)$\Rightarrow$(b). Let x_α be a fuzzy point in X and U, a fuzzy pre-semiopen set in X with $x_\alpha qU$. Then $x_\alpha \notin 1_X \setminus U \in FPSC(X)$. By (a), there exist a fuzzy open set V and a fuzzy pre-semiopen set W in X such that $x_\alpha qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_\alpha qV \leq 1_X \setminus W \leq U \Rightarrow x_\alpha qV \leq psclV \leq pscl(1_X \setminus W) = 1_X \setminus W \leq U$.
(b)$\Rightarrow$(a). Let F be a fuzzy pre-semiclosed set in X and x_α be a fuzzy point in X with $x_\alpha \notin F$. Then $x_\alpha q(1_X \setminus F) \in FPSO(X)$. By (b), there exists a fuzzy open set V in X such that $x_\alpha qV \leq psclV \leq 1_X \setminus F$. Put $U = 1_X \setminus psclV$. Then $U \in FPSO(X)$ and $x_\alpha qV$, $F \leq U$ and $U \not{q} V$.
(b)$\Rightarrow$(c). Let F be fuzzy pre-semiclosed set in X. Then $F \leq \bigwedge\{clV : F \leq V, V \in FPSO(X)\}$.
Conversely, let $x_\alpha \not\leq F \in FPSC(X)$. Then $F(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FPSO(X)$. By (b), there exists a fuzzy open set U in X such that $x_\alpha qU \leq psclU \leq 1_X \setminus F$. Put $V = 1_X \setminus psclU$. Then $F \leq V$ and $U \not{q} V \Rightarrow x_\alpha \notin clV \Rightarrow \bigwedge\{clV : F \leq V, V \in FPSO(X)\} \leq F$.
(c)$\Rightarrow$(b). Let V be any fuzzy pre-semiopen set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x) + \alpha > 1 \Rightarrow x_\alpha \not\leq (1_X \setminus V)$ where $1_X \setminus V \in FPSC(X)$. By (c), there exists $G \in FPSO(X)$ such

that $1_X \setminus V \leq G$ and $x_\alpha \notin clG$. Then there exists a fuzzy open set U in X with $x_\alpha qU$, $U\not{q}G \Rightarrow U \leq 1_X \setminus G \leq V \Rightarrow x_\alpha qU \leq psclU \leq pscl(1_X \setminus G) = 1_X \setminus G \leq V$.

(c)$\Rightarrow$(d). Let G be any fuzzy set in X and U be any fuzzy pre-semiopen set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_\alpha qU \Rightarrow x_\alpha \nleq 1_X \setminus U$ where $1_X \setminus U \in FPSC(X)$. By (c), there exists $W \in FPSO(X)$ such that $1_X \setminus U \leq W$ and $x_\alpha \notin clW \Rightarrow (clW)(x) < \alpha \Rightarrow x_\alpha q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x) + \alpha > 1 \Rightarrow V(x)+G(x) > 1 \Rightarrow VqG$ and $psclV = pscl(1_X \setminus clW) \leq pscl(1_X \setminus W) = 1_X \setminus W \leq U$.

(d)$\Rightarrow$(b). Obvious.

Note 6.3. It is clear from Theorem 6.2 that in a fuzzy pre-semi-regular space, every fuzzy pre-semiclosed set is fuzzy closed and hence every fuzzy pre-semiopen set is fuzzy open. As a result, in a fuzzy pre-semi-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and the collection of fuzzy pre-semiclosed (resp., fuzzy pre-semiopen) sets coincide.

Theorem 6.4. *If $f : X \to Y$ is fuzzy pre-semi-continuous function and Y is fuzzy pre-semi-regular space, then f is a fuzzy pre-semi-irresolute function.*

Proof. Let x_α be a fuzzy point in X and V be any fuzzy pre-semiopen q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy pre-semi-regular space. By Theorem 6.2 (a)$\Rightarrow$(b), there exists a fuzzy open set W in Y such that $f(x_\alpha)qW \leq psclW \leq V$. Since f is fuzzy pre-semi-continuous function, by Theorem 4.5, there exists $U \in FPSO(X)$ with $x_\alpha qU$ and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy pre-semi-irresolute function.

Let us now recall following definitions from [4, 5] for ready references.

Definition 6.5 [4]. A collection $\mathcal{U}$ of fuzzy sets in an fts X is said to be a *fuzzy cover* of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called a fuzzy open cover of X.

Definition 6.6 [4]. A fuzzy cover $\mathcal{U}$ of an fts X is said to have a *finite subcover* $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.

Definition 6.7 [5]. An fts (X, τ) is said to be *fuzzy almost compact* if every fuzzy open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{clU : U \in \mathcal{U}_0\}$ is again a fuzzy cover of X.

Theorem 6.8. *If $f : X \to Y$ is a fuzzy pre-semi-continuous, surjective function and X is a fuzzy pre-semi-regular and almost compact* *space, then Y is a fuzzy almost compact space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy open cover of Y. Then as f is fuzzy pre-semi-continuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy pre-semiopen and hence fuzzy open cover of X as X is fuzzy pre-semi-regular space (by Note 6.3). Since X is fuzzy almost compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy pre-semi-regular, by Note 6.3, $clA = psclA$ for all $A \in I^X$ and so $1_X = \bigcup_{i=1}^{n} pscl(f^{-1}(U_i)) \Rightarrow 1_Y = f(\bigcup_{i=1}^{n} pscl(f^{-1}(U_i))) = \bigcup_{i=1}^{n} f(pscl(f^{-1}(U_i))) \leq \bigcup_{i=1}^{n} cl(f(f^{-1}(U_i)))$ (by Theorem 4.4 (a)$\Rightarrow$(f)) $\leq \bigcup_{i=1}^{n} cl(U_i) \Rightarrow \bigcup_{i=1}^{n} cl(U_i) = 1_Y \Rightarrow Y$ is fuzzy almost compact space.

Let us now recall the following definition from [3] for ready references.

Definition 6.9 [3]. A function $f : X \to Y$ is called *fuzzy almost s^*-continuous function* if the inverse image of every fuzzy s^*-open set in Y is fuzzy s^*-open in X.

Remark 6.10. It is clear from definitions that every fuzzy almost s^*-continuous function is fuzzy pre-semi-irresolute. The converse is false, as follows from the following example.

Example 6.11. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.6, B(a) = B(b) = 0.2$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now every fuzzy set in (X, τ_1) is fuzzy pre-semiopen set in (X, τ_1) and every fuzzy set in (X, τ_2) is fuzzy pre-semiopen as well as fuzzy s^*-open. Now consider the fuzzy set C defined by $C(a) = 0.5, C(b) = 0.2$. Then $C \in FPSO(X, \tau_2)$ as well as C is fuzzy s^*-open set in (X, τ_2). Now $i^{-1}(C) = C \leq sint_{\tau_1}(cl_{\tau_1}C) = 1_X \setminus A \Rightarrow C \in FPSO(X, \tau_1)$. But $C \notin int_{\tau_1}(scl_{\tau_1}C) = B \Rightarrow C$ is not fuzzy s^*-open set in $(X, \tau_1) \Rightarrow i$ is fuzzy pre-semi-irresolute function but not fuzzy almost s^*-continuous function.

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