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GENERALIZATIONS OF CLOSED FUNCTIONS IN SPACES WITH MINIMAL STRUCTURES

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Abstract. Noiri and Popa introduced and studied the notion of $M-$closed function between spaces with minimal structures, developing a unified theory of modifications of closedness such as $\alpha-$closedness, semi-closedness, preclosedness and $\beta-$closedness. Using a new notion, that of almost M- closed function, we extend the characterizations of $M-$closed functions proved by Popa and Noiri to spaces endowed with minimal structures not necessarily closed under arbitrary unions. These minimal structures are useful beyond General Topology. For bijections between spaces with minimal structure it turns out that almost M- closedness is equivalent to almost $M-$openess, both being equivalent to the $M-$continuity of the inverse function. Our main result generalizes a well-known theorem of Long and Herrington showing that $\theta-$open sets in topological spaces are preserved by every function that is both open and closed.

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1. Introduction

The notion of topological space has been generalized in many ways, through closure spaces [5], minimal structures [31], generalized topologies [6], Pervin relations [39]. The algebraic nature of generalizations of topological notions has been recently analyzed in [19]. During the last two decades such generalizations turned out to be useful tools for various applications in Theoretical Computer Science [38], [4], Chemistry and Biology [3], [41], Learning theory [8].

In 2000, Popa and Noiri initiated the study of functions between spaces endowed with minimal structures [31]. A minimal structure on a set X is a collection of subsets of X, that contains the empty set and X, so only the first axiom of a topological space is retained. Note that a minimal structure on X is closed under arbitrary unions if and only if it is a generalized topology containing X. We recall that, in topological spaces, each of the following classes of sets forms a minimal structures which plays an important role in the study of various forms of generalized continuity: semi-open sets [9], preopen sets [12], semi-preopen sets [2], $\alpha-$open sets [21], $\beta-$open sets [1], $\delta-$open sets [42], $\theta-$open sets [42]. Through minimal structure spaces, Popa and Noiri obtained unified theories for generalized forms of continuous functions [31], [32], [33], [34], [35], [36], [24], [25], contra-continuous functions [23], open functions [28], [29], [30], closed functions [26], [27]. Introducing and studying the notion of $M-$closed function between spaces with minimal structures in [27], Noiri and Popa developed a unified theory of modifications of closedness such as $\alpha-$closedness [13], semi-closedness [22], preclosedness [12] and $\beta-$closedness [1].

In this paper we extend the study of M-closed functions introduced by Noiri and Popa [27], by introducing some more generalizations of closed functions, namely the notions of almost $M-$closed function, weakly $M-$closed function and strongly $M-$closed function. These notions coincide in spaces with minimal structures closed under arbitrary unions, but differ in general.

Using the notion of almost M- closed function, we prove some refinements of the characterizations of $M-$closed functions obtained by Popa and Noiri in [27], removing the assumptions that one or both of the minimal structures involved are closed under arbitrary unions. For bijections between spaces with minimal structures it turns out that almost M- closedness is equivalent to almost $M-$openess, both being equivalent to the $M-$continuity of the inverse function. Our main result generalizes a well-known theorem of Long and Herrington

[10], which shows that $\theta-$open sets in topological spaces are preserved by every function that is both open and closed.

2. Preliminaries

A function u from the power set $\mathcal{P}(X)$ of a non-empty set X into itself is called a *generalized closure operator on* X (GCO, for short) and the pair (X, u) is said to be a *generalized closure space* (GCS, for short).

Definition 1. [40] *A GCO* $u : \mathcal{P}(X) \to \mathcal{P}(X)$ *is called grounded if* $u(\emptyset) = \emptyset$*, isotone if* $A \subset B \subset X$ *implies* $u(A) \subset u(B)$*, extensive if* $A \subset u(A)$ *for every* $A \subset X$*, contractive if* $A \subset u(A)$ *for every* $A \subset X$*, idempotent if* $u(u(A)) = u(A)$ *for every* $A \subset X$*, sublinear if* $u(A \cup B) \subset u(A) \cup u(B)$ *for every* $A, B \subset X$*.*

A generalized closure operator is called *Čech closure operator* if it is grounded, extensive, isotone and sublinear, respectively *Kuratowski closure operator* if it is an idempotent Čech closure operator [37].

To every GCO $u : \mathcal{P}(X) \to \mathcal{P}(X)$ one associates the $u-$*interior operator*, $u - Int : \mathcal{P}(X) \to \mathcal{P}(X)$, defined as the dual of u, by $u - Int(A) = X \setminus u(X \setminus A)$ for all $A \subset X$. Note that u is the dual of the GCO $u - Int$. In addition, u is isotone if and only if $u - Int$ is isotone, u is idempotent if and only if $u - Int$ is idempotent, while u is grounded if and only if $u - Int(X) = X$.

Definition 2 ([31], [32])**.** *A family* $m_X \subset \mathcal{P}(X)$ *is called a minimal structure (shortly, $m-$structure) on* X *if* $\emptyset \in m_X$ *and* $X \in m_X$*.*

The couple (X, m_X) is called a space with minimal structure or an $m-$space, for short. The members of m_X are called $m-$open sets and their complements form the class of $m-$closed sets.

Definition 3 ([31], [32])**.** *A minimal structure* m_X *is said to have property* $(\mathcal{B})$ *if it is closed under arbitrary unions. A minimal structure* m_X *is said to have property* $(\mathcal{I})$ *if it is closed under intersection.*

The class of minimal structures on X having property $(\mathcal{B})$ coincides with the class of generalized topologies on X, in the sense of Császár [6], that contain X.

For every family of sets $\mathcal{F}$ we will denote by $\mathcal{U}(\mathcal{F})$ the family of all unions of sets that belong to $\mathcal{F}$. Note that $\mathcal{F} \subset \mathcal{U}(\mathcal{F})$ always. A family $\mathcal{F}$ is closed under arbitrary unions if and only if $\mathcal{U}(\mathcal{F}) \subset \mathcal{F}$.

In particular, a minimal structure m_X has property $(\mathcal{B})$ if and only if $\mathcal{U}(m_X) = m_X$.

Spaces with minimal structure are generalized closure spaces [15], [16]. The fundamental generalized closure operators associated to a minimal structure have been introduced by Maki in [11].

Definition 4. *Let $m_X \subset \mathcal{P}(X)$ be a minimal structure. For each subset $A \subset X$ the m_X-closure of A and the m_X-interior of A are defined as follows:*

$$\begin{aligned} m_X - Cl(A) \ &: \ = \cap\{F : A \subset F \text{ and } X \setminus F \in m_X\}. \\ m_X - Int(A) \ &: \ = \cup\{U : U \subset A \text{ and } U \in m_X\} \end{aligned}$$

It is known that $x \in m_X - Cl(A)$ if and only if $D \cap A \neq \emptyset$ for every $D \in m_X$ containing x, respectively $x \in m_X - Int(A)$ if and only if there exists $D \in m_X$ containing x such that $D \subset A$.

Maki [11] proved that GCO's $m_X - Cl$ and $m_X - Int$ are dual to each other and are both grounded, isotone and idempotent, also that $m_X - Cl$ is extensive, while $m_X - Int$ is contractive. Unlike the standard closure operator Cl of a topological space, the GCO $m_X - Cl$ need not be sublinear, even if m_X has property $(\mathcal{B})$, see [18, Example 7].

The fixed points of the GCO's $m_X - Cl$ and $m_X - Int$ play an important role in the study of spaces with minimal structure, as generalizations of $m-$closed sets, respectively of m-open sets.

Definition 5. *Let (X, m_X) be an $m-$space. A set $A \subset X$ is said to be almost $m-$open if $m_X - Int(A) = A$.*

A set $B \subset X$ is said to be almost $m-$closed if $m_X - Cl(B) = B$.

We recall some properties of almost $m-$open sets and of almost $m-$closed sets.

Lemma 6. *Let (X, m_X) be an $m-$space.*

(1) A set $A \subset X$ is almost $m-$open if and only if its complement $X \setminus A$ is almost $m-$closed.

(2) A set $A \subset X$ is almost $m-$open if and only if A is the union of a family of $m-$open sets. The family of almost $m-$open sets is $\mathcal{U}(m_X)$.

(3) A set $B \subset X$ is almost $m-$closed if and only if B is the intersection of a family of $m-$closed sets.

(4) Every $m-$ open set is almost $m-$open. The converse holds if and only if m_X has property $(\mathcal{B})$.

(5) Every $m-$ closed set is almost $m-$closed. The converse holds if and only if m_X has property $(\mathcal{B})$.

Proof. (1) Since the GCO's $m_X - Cl$ and $m_X - Int$ are dual to each other, we have $m_X - Int(A) = A$ if and only if $m_X - Cl(X \setminus A) = X \setminus A$.

(2) Recall that $m_X - Int(A)$ is, by definition, the union of all $m-$open sets included in A. Therefore, necessity is immediate. To prove sufficiency, write $A = \bigcup\limits_{i \in I} U_i$ with $U_i \in m_X$ for each $i \in I$, then notice that $U_i \subset A$ for each $i \in I$, hence $A \subset m_X - Int(A)$, therefore $A = m_X - Int(A)$.

(3) By (1), $B \subset X$ is almost $m-$closed if and only if $X \setminus B$ is almost $m-$open, that by (2) is equivalent to the existence of a family $\{V_i : i \in I\} \subset m_X$ with $\bigcup\limits_{i \in I} V_i = X \setminus B$. If $B \subset X$ is almost $m-$closed, then $B = \bigcap\limits_{i \in I} (X \setminus V_i)$ is an intersection of $m-$closed sets. The converse is analogous.

(4) Since $\mathcal{F} \subset \mathcal{U}(\mathcal{F})$ for every family $\mathcal{F}$ of sets, the first claim follows by (2). We also may use the well-known fact [11] that $m_X - Int(A) = A$ whenever $A \in m_X$. For the converse, note that $\mathcal{U}(m_X) \subset m_X$ if and only if m_X has property $(\mathcal{B})$.

(5) Obvious by (4) and (1). ∎

3. Generalizations of closed functions in spaces with minimal structure

In [27] Noiri and Popa introduced studied $M-$closed functions between spaces with minimal structure, so developing a unified theory of modifications of closedness such as $\alpha-$closedness, semiclosedness, preclosedness and $\beta-$closedness.

Definition 7 ([27] , Definition 3.4)**.** *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M-$*closed *if the image $f(F)$ of an arbitrary $m-$closed set F in (X, m_X) is $m-$closed in (Y, m_Y).*

If m_X and m_Y are topologies, the notion of $M-$closed function coincides with the classical notion of closed function. Inspired by well-known characterizations of closed functions, Noiri and Popa proved several characterizations of $M-$closed functions, under the assumption that one or both of the minimal structures involved are closed under arbitrary unions. However, the study of minimal structures that are not necessarily closed under arbitrary unions is important beyond General Topology [8].

In the following we will introduce some natural modifications of $M-$closedness, namely the notions of almost $M-$closed function, weakly $M-$closed function and strongly $M-$closed function. Almost $M-$closedness and weakly $M-$closedness will be characterized in spaces with minimal structures that are not required to be closed under arbitrary unions, which allows us the extension of applicability of the unified theory of closed functions.

Definition 8. *A function $f:(X,m_X)\to(Y,m_Y)$ is said to be:*

(1) almost $M-$closed *if the image $f(F)$ of an arbitrary almost m_X-closed set $F\subset X$ is almost m_Y-closed.*

(2) weakly $M-$closed *if the image $f(F)$ of an arbitrary m_X-closed set $F\subset X$ is almost m_Y-closed.*

(3) strongly $M-$closed *if the image $f(F)$ of an arbitrary almost m_X-closed set $F\subset X$ is m_Y-closed.*

Every $m-$closed set is also almost $m-$closed, the converse being true if and only if the corresponding $m-$structure is closed under arbitrary unions. This shows that the following implications hold.

$$\begin{array}{ccc}
(3)\ f \text{ strongly } M-\text{closed} & \Longrightarrow & (0)\ f\ \ M-\text{closed} \\
\Downarrow & & \Downarrow \\
(1)\ f \text{ almost } M-\text{closed} & \Longrightarrow & (2)\ f \text{ weakly } M-\text{closed}
\end{array}$$

Lemma 9. *If $f:(X,m_X)\to(Y,m_Y)$ is $M-$closed and injective, then f is almost $M-$closed.*

Proof. Let $f:(X,m_X)\to(Y,m_Y)$ be $M-$closed and injective. Let $F\subset X$ be almost $m-$closed. We prove that $f(F)\subset Y$ is almost $m-$closed. By Lemma 6 (3), there exists a family of $m-$closed sets $\{F_i : i\in I\}$ such that $F=\bigcap\limits_{i\in I}F_i$. Since f is injective, $f(F)=\bigcap\limits_{i\in I}f(F_i)$. But $f(F_i)$ is $m-$closed for every $i\in I$, hence $f(F)$ is almost $m-$closed by Lemma 6 (3). ∎

If m_X is closed under arbitrary unions (in other words, if m_X has property $(\mathcal{B})$), then every $M-$closed function is strongly $M-$closed ((0) $\Rightarrow$ (3) holds), hence it is almost $M-$closed. If m_Y is closed to arbitrary unions, then every weakly $M-$closed function is $M-$closed, therefore every almost $M-$closed function is $M-$closed.

In general, neither of the above implications is reversible and the notions of $M-$closed function and almost $M-$closed function are independent, as the following examples show.

Example 10. *If m_X is not closed under arbitrary unions, then it is possible to find a function $f:(X,m_X)\to(Y,m_Y)$ that is $M-$closed without being almost $M-$closed (in particular, neither of (0) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) holds).*

Let $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2\}$. Consider $m_X=\{\emptyset,\{x_1\},\{x_2\},\{x_3\},X\}$ and $m_Y=\{\emptyset,\{y_2\},Y\}$.

The family of $m-$closed subsets of X is $\mathcal{F}_X=\{\emptyset,\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_3\},X\}$ and the family of almost $m-$closed subsets of X is $\mathcal{P}(X)$. The family of $m-$closed subsets of Y is $\mathcal{F}_Y=\{\emptyset,\{y_1\},Y\}$ and the family of almost $m-$closed subsets of Y is $\mathcal{P}(Y)\setminus\{\{y_2\}\}$. Define $f:(X,m_X)\to(Y,m_Y)$ by $f(x_1)=f(x_2)=y_1$ and $f(x_3)=y_2$. Then f is $M-$closed, as $f(\emptyset)=\emptyset$, $f(\{x_1,x_2\})=\{y_1\}$ and $f(\{x_1,x_3\})=f(X)=Y$. We see that $\{x_3\}$ is almost $m-$closed in (X,m_X), but $f(\{x_3\})=\{y_2\}$ is not almost $m-$closed in (Y,m_Y).

Example 11. *If m_Y is not closed under arbitrary unions, then it is possible to find a function $f:(X,m_X)\to(Y,m_Y)$ that is almost $M-$closed and is not $M-$closed (in particular, neither of (2) $\Rightarrow$ (0) and (1) $\Rightarrow$ (3) holds).*

Let $Y=\{y_1,y_2,y_3\}$. Considering $m_Y=\{\emptyset,\{y_1\},\{y_2\},\{y_3\},Y\}$, we seethat the family of $m-$closed subsets of Y is $\mathcal{F}_Y=\{\emptyset,\{y_1,y_2\},\{y_2,y_3\},\{y_1,y_3\},Y\}$. Since the family of almost $m-$closed subsets of Y is $\mathcal{P}(Y)$, for every space with minimal structure (X,m_X), every function $f:(X,m_X)\to(Y,m_Y)$ is almost $M-$closed. On the other hand, a constant function $f:(X,m_X)\to(Y,m_Y)$ cannot be $M-$closed.

Remark 12. *From Definition 8 it follows that the composition $g\circ f$ of two functions $f:(X,m_X)\to(Y,m_Y)$ and $g:(Y,m_Y)\to(Z,m_Z)$ is:*

(i) $M-$closed if f and g are $M-$closed and also if f is weakly $M-$closed and g is strongly $M-$closed;

(ii) almost $M-$closed if f and g are almost $M-$closed and also if f is strongly $M-$closed and g is weakly $M-$closed;

(iii) weakly $M-$closed if f is $M-$closed and g is weakly $M-$closed, as well as if f is weakly $M-$closed and g is almost $M-$closed;

(iv) strongly $M-$closed if f is almost $M-$closed and g is strongly $M-$closed, as well as if f is strongly closed and g is $M-$closed.

Our next goal is to provide characterizations of almost $M-$closed functions.

First we recall some set-theoretic properties regarding inverse images under a function.

Lemma 13. *Let $f : X \to Y$ be a function. Then for every $A \subset X$, we have $\{y \in Y : f^{-1}(y) \subset A\} = Y \setminus f(X \setminus A)$ and $f^{-1}(Y \setminus f(X \setminus A)) \subset A$.*

Theorem 14. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is almost $M-$closed;

(2) For each $B \subset Y$ and every $U \in \mathcal{U}(m_X)$ with $f^{-1}(B) \subset U$, there exists $V \in \mathcal{U}(m_Y)$ such that $B \subset V$ and $f^{-1}(V) \subset U$;

(3) For each $y \in Y$ and every $U \in \mathcal{U}(m_X)$ with $f^{-1}(y) \subset U$, there exists $V \in \mathcal{U}(m_Y)$ such that $y \in V$ and $f^{-1}(V) \subset U$.

Proof. (1) $\Rightarrow$ (2): Let $B \subset Y$ and $U \in \mathcal{U}(m_X)$ with $f^{-1}(B) \subset U$. Consider $V := Y \setminus f(X \setminus U)$. Then $f^{-1}(V) \subset U$ and $B \subset V$, by Lemma 13. Since $U \in \mathcal{U}(m_X)$, the set $X \setminus U$ is almost m_X-closed. By our assumption (1), it follows that $f(X \setminus U)$ is almost m_Y-closed. Then V is almost m_Y-open, i.e. $V \in \mathcal{U}(m_Y)$.

(2) $\Rightarrow$ (1): Let $A \subset X$ be almost $m-$closed. We have to prove that $f(A)$ is almost $m-$closed. Note that $X \setminus A \in \mathcal{U}(m_X)$ and, by Lemma 13, $f^{-1}(Y \setminus f(A)) \subset X \setminus A$. Using our assumption (2) with $B = Y \setminus f(A)$ and $U = X \setminus A$, we find $V \in \mathcal{U}(m_Y)$ such that $Y \setminus f(A) \subset V$ and $f^{-1}(V) \subset X \setminus A$. But $f^{-1}(V) \subset X \setminus A$ implies $V \subset Y \setminus f(A)$. Therefore, V is uniquely determined in this case, as $V = Y \setminus f(A)$. We proved that $V = Y \setminus f(A) \in \mathcal{U}(m_Y)$, hence $f(A)$ is almost $m-$closed.

(2) $\Rightarrow$ (3): This implication is obvious.

(3) $\Rightarrow$ (2): Let $B \subset Y$ and $U \in \mathcal{U}(m_X)$ with $f^{-1}(B) \subset U$. For each $y \in B$ there exists $V_y \in \mathcal{U}(m_Y)$ such that $y \in V_y$ and $f^{-1}(V_y) \subset U$. Denote $V = \bigcup_{y \in B} V_y$. Then $B \subset V$ and $f^{-1}(V) = \bigcup_{y \in B} f^{-1}(V_y) \subset U$. Since $V_y \in \mathcal{U}(m_Y)$ for each $y \in B$, we see that $V \in \mathcal{U}(m_Y)$. ∎

Theorem 15. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is almost $M-$closed;

(2) $\{y \in Y : f^{-1}(y) \subset U\} \in \mathcal{U}(m_Y)$ for every $U \in \mathcal{U}(m_X)$;

(3) $\{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ is almost m_Y-closed whenever $F \subset X$ is almost m_X-closed.

Proof. (2) $\Rightarrow$ (1): Let A be almost m_X-closed. Then $X \setminus A \in \mathcal{U}(m_X)$. By (2), $\{y \in Y : f^{-1}(y) \subset X \setminus A\} \in \mathcal{U}(m_Y)$. By Lemma

13, $\{y \in Y : f^{-1}(y) \subset X \setminus A\} = Y \setminus f(A)$. Then $Y \setminus f(A) \in \mathcal{U}(m_Y)$. i.e. $f(A)$ is almost m_Y-closed.

(1) $\Rightarrow$ (2): Let $U \in \mathcal{U}(m_X)$. Denote $V = \{y \in Y : f^{-1}(y) \subset U\}$. By Theorem 14 ((1) $\Rightarrow$ (3)), for every $y \in V$ there exists $V_y \in \mathcal{U}(m_Y)$ such that $y \in V_y$ and $f^{-1}(V_y) \subset U$. Since $\{y\} \subset V_y \subset V$ for every $y \in V$, we see that $V = \bigcup_{y \in V} V_y$. Since $V_y \in \mathcal{U}(m_Y)$ for every $y \in V$, it follows that $V \in \mathcal{U}(m_Y)$, as required.

(2) $\Leftrightarrow$ (3): This is obvious by complementarity, since $Y \setminus \{y \in Y : f^{-1}(y) \cap F \neq \emptyset\} = \{y \in Y : f^{-1}(y) \subset X \setminus F\}$, while $X \setminus F \in \mathcal{U}(m_X)$ if and only if $F \subset X$ is almost m_X-closed and $Y \setminus B \in \mathcal{U}(m_Y)$ if and only if B is almost m_Y-closed, for every $B \subset Y$. ∎

Remark 16. *If m_X is closed to arbitrary unions, by Theorem 14 and Theorem 15 we obtain Theorem 3.1 and Theorem 3.3 of* [27], *respectively.*

We mention the following characterizations of weakly $M-$closed functions. The proof follows the lines from the proofs of Theorem 14 and Theorem 15, in which we replace the property of almost $m-$open (almost $m-$closed) by the stronger property of $m-$open (respectively, $m-$closed) and we replace $\mathcal{U}(m_X)$ by m_X.

Theorem 17. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is weakly $M-$closed;

(2) For each $B \subset Y$ and every $U \in m_X$ with $f^{-1}(B) \subset U$, there exists $V \in \mathcal{U}(m_Y)$ such that $B \subset V$ and $f^{-1}(V) \subset U$;

(3) For each $y \in Y$ and every $U \in m_X$ with $f^{-1}(y) \subset U$, there exists $V \in \mathcal{U}(m_Y)$ such that $y \in V$ and $f^{-1}(V) \subset U$.

(4) $\{y \in Y : f^{-1}(y) \subset U\} \in \mathcal{U}(m_Y)$ for every $U \in m_X$;

(5) $\{y \in Y : f^{-1}(y) \cap A \neq \emptyset\}$ is almost m_Y-closed whenever $A \subset X$ is m_X-closed.

The following characterization of almost $M-$closed functions in terms of $m-$closures is very useful, being similar in spirit to some characterization of almost $M-$open functions, see Lemma 24.

Theorem 18. *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*

(1) f is almost $M-$closed;

(2) $m_Y - Cl(f(A)) \subset f(m_X - Cl(A))$ for every $A \subset X$.

Proof. (1) $\Rightarrow$ (2): Let $A \subset X$. Since $m_X - Cl$ is idempotent, the set $m_X - Cl(A)$ is almost m_X-closed. By our assumption (1), $f\left(m_X - Cl(A)\right)$ coincides with its m_Y-closure. Clearly, $f(A) \subset f\left(m_X - Cl(A)\right)$. We conclude that

$$m_Y - Cl\left(f(A)\right) \subset m_Y - Cl\left(f\left(m_X - Cl(A)\right)\right) = f\left(m_X - Cl(A)\right).$$

(2) $\Rightarrow$ (1): Let $C \subset X$ be almost m_X-closed. Then $m_X - Cl\left(C\right) = C$, hence $f\left(m_X - Cl\left(C\right)\right) = f\left(C\right)$. By our assumption (2), $m_Y - Cl(f(C)) \subset f\left(m_X - Cl(C)\right)$. Then $m_Y - Cl(f(C)) \subset f\left(C\right)$, therefore $f\left(C\right)$ is almost m_Y-closed, q.e.d. ∎

Remark 19. *If m_X and m_Y are both closed to arbitrary unions, by Theorem 18 we obtain Theorem 3.2 of* [27].

4. Characterizations of the $M-$continuity of an inverse function

We recall the fundamental definition of $M-$continuity [32], respectively the definitions of $M-$openness and almost $M-$openness [14].

Definition 20 (Popa and Noiri, [32]). *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M-$continuous at $x \in X$ if for every $V \in m_Y$ containining $f\left(x\right)$ there exists $U \in m_X$ containing x such that $f\left(U\right) \subset V$. The function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M-$continuous on a nonempty subset A of X if it is $M-$continuous at every $x \in A$.*

Definition 21 ([14]). *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M-$open at $x \in X$ if for each $U \in m_X$ containing x we have $f(U) \in m_Y$. The function $f : (X, m_X) \to (Y, m_Y)$ is said to be $M-$open on a nonempty subset A of X if it is $M-$open at every $x \in A$.*

Definition 22 ([14]). *A function $f : (X, m_X) \to (Y, m_Y)$ is said to be almost $M-$open at $x \in X$ if for each $U \in m_X$ containing x there exists $V \in m_Y$ such that $f\left(x\right) \in V \subset f(U)$. The function $f : (X, m_X) \to (Y, m_Y)$ is said to be almost $M-$open on a nonempty subset A of X if it is almost $M-$open at every $x \in A$.*

We need the known characterizations of $M-$continuity, respectively of almost $M-$openness.

Lemma 23 (Popa and Noiri, [32]). *For a function $f : (X, m_X) \to (Y, m_Y)$, the following properties are equivalent:*
(1) f is $M-$continuous;
(2) $f^{-1}(V) = m_X - Int\left(f^{-1}(V)\right)$ for every $V \in m_Y$;

(3) $f(m_X - Cl(A)) \subset m_Y - Cl(f(A))$ *for every* $A \subset X$*;*
(4) $m_X - Cl(f^{-1}(B)) \subset f^{-1}(m_Y - Cl(B))$ *for every* $B \subset Y$*;*
(5) $f^{-1}(m_Y - Int(B)) \subset m_X - Int(f^{-1}(B))$ *for every* $B \subset Y$*.*

Lemma 24. [14] *The following properties are equivalent for* $f : (X, m_X) \to (Y, m_Y)$*:*
(i) f *is almost* $M-$*open;*
(ii) $f(m_X - Int(A)) \subset m_Y - Int(f(A))$ *for every* $A \subset X$*;*
(iii) If $U \in m_X$*, then* $f(U) \in \mathcal{U}(m_X)$*;*
(iv) $m_X - Int(f^{-1}(B)) \subset f^{-1}(m_Y - Int(B))$ *for every* $B \subset Y$*;*
(v) $f^{-1}(m_Y - Cl(B)) \subset m_X - Cl - (f^{-1}(B))$ *for every* $B \subset Y$*.*
(vi) f *maps almost* $m-$*open sets to* $m-$*almost* $m-$*open sets.*

Remark 25. *A function* $f : (X, m_X) \to (Y, m_Y)$ *is* $M-$*continuous and almost* $M-$*open if and only if* $m_X - Int(f^{-1}(B)) = f^{-1}(m_Y - Int(B))$ *for all* $B \subset X$*, respectively it is* $M-$*continuous and almost* $M-$*closed if and only if* $f(m_X - Cl(A)) = m_Y - Cl(f(A))$ *for every* $A \subset X$*.*

Comparing Theorem 18 with Lemma 23 and with Lemma 24 we arrive to the following generalization of a well-known result from General Topology.

Theorem 26. *For a bijective function* $f : (X, m_X) \to (Y, m_Y)$ *the following properties are equivalent:*
(1) The inverse $f^{-1} : (Y, m_Y) \to (X, m_X)$ *is* $M-$*continuous;*
(2) $f : (X, m_X) \to (Y, m_Y)$ *is almost* $M-$*open;*
(3) $f : (X, m_X) \to (Y, m_Y)$ *is almost* $M-$*closed.*

Proof. We choose to prove the equivalence of each pair of properties among (1), (2) and (3).

(2) $\Leftrightarrow$ (3): We use for $f : (X, m_X) \to (Y, m_Y)$ Lemma 24 $(i) \Leftrightarrow (vi)$ and the definition of almost $M-$closed functions, as well as the set identity $f(X \setminus A) = Y \setminus f(A)$ for every $A \subset X$, valid for bijective f. Alternatively, we may use the fact that the inclusion (v) in Lemma 24 with $B = f(A)$ and $A \subset X$ is equivalent to property (2) in Theorem 18, since f is bijective.

(1) $\Leftrightarrow$ (2): This was proved in [14, Theorem 4.1], using the following characterizations: $g : (Y, m_Y) \to (X, m_X)$ is $M-$continuous if and only if $g^{-1}(U) \in \mathcal{U}(m_Y)$ whenever $U \in m_X$, while $f : (X, m_X) \to (Y, m_Y)$ is almost $M-$open if and only if $f : (X, m_X) \to (Y, \mathcal{U}(m_Y))$ is $M-$open. If f is bijective, we apply the above characterizations with $g = f^{-1}$.

However, we can give here a different more direct proof. We may use Lemma 23 ((1) $\Leftrightarrow$ (3)), where we replace f by f^{-1}, and Lemma 24 ((i) $\Leftrightarrow$ (v)). Alternatively, we may use Lemma 23 ((1) $\Leftrightarrow$ (5)), where we replace f by f^{-1}, and Lemma 24 ((i) $\Leftrightarrow$ (ii)).

(1) $\Leftrightarrow$ (3): We use Lemma 23 (1) $\Leftrightarrow$ (5) for f^{-1} instead of f. The inverse $f^{-1}:(Y,m_Y)\to(X,m_X)$ is $M-$continuous if and only if $m_Y-Cl(f(A))\subset f\left(m_X-Cl(A)\right)$ for every $A\subset X$.

According to Theorem 18, the latter assertion is equivalent to the almost $M-$closedness of $f:(X,m_X)\to(Y,m_Y)$. ∎

Corollary 27. *For a bijective function $f:(X,m_X)\to(Y,m_Y)$ the following properties are equivalent:*

(1) f and its inverse $f^{-1}:(Y,m_Y)\to(X,m_X)$ are $M-$continuous;
(2) $f\left(m_X-Int(A)\right)=m_Y-Int(f(A))$ for every $A\subset X$;
(3) $f\left(m_X-Cl(A)\right)=m_Y-Cl(f(A))$ for every $A\subset X$;
(4) $f^{-1}\left(m_Y-Int(B)\right)=m_X-Int(f^{-1}(B))$ for every $B\subset Y$;
(5) $f^{-1}\left(m_Y-Cl(B)\right)=m_X-Cl(f^{-1}(B))$ for every $B\subset Y$.

5. Invariance of $m-\theta-$openness

Definition 28. [24] *Let (X,m_X) be a space with minimal structure and $S\subset X$. A point $x\in X$ is called:*

(a) an $m-\theta-$adherent point of S if $m_X-Cl(U)\cap S\neq\emptyset$ for every $U\in m_X$ containing x;

(b) an $m-\theta-$interior point of S if $m_X-Cl(V)\subset S$ for some $V\in m_X$ containing x.

Definition 29. [24] *The set $m_X-Cl_\theta(S)$ containing all the $m-\theta-$adherent points of S is called the $m-\theta-$closure of S. The set $m_X-Int_\theta(S)$ containing all the $m-\theta-$interior points of S is called the $m-\theta-$interior of S.*

A set $A\subset X$ is said to be $m-\theta-$closed in (X,m_X) if $m_X-Cl_\theta(A)=A$. The complements of $m-\theta-$closed sets are called $m-\theta-$open sets.

The following properties have been proved in [24]. The GCOs m_X-Cl_θ and m_X-Int_θ are dual to each other, grounded and isotone. A set $A\subset X$ is $m-\theta-$open in (X,m_X) if and only if $m_X-Int_\theta(A)=A$. For every $A\subset X$,

$$m_X-Int_\theta(A)\subset m_X-Int(A)\subset A\subset m_X-Cl(A)\subset m_X-Cl_\theta(A).$$

In general, m_X-Cl_θ and m_X-Int_θ are not idempotent, see [20].

Lemma 30. [18] *Let (X,m_X) be a space with minimal structure, A and B subsets of X, and $x\in X$. The following properties hold:*

(i) $m_X - Int_\theta(A) \in \mathcal{U}(m_X)$, *in particular every* $\theta - m-$*open set is almost* $m-$*open;*

(ii) $m_X - Cl_\theta(B)$ *is almost* $m-$*closed, in particular every* $\theta - m-$*closed set is almost* $m-$*closed;*

(iii) If $x \in m_X - Cl_\theta(A)$, *then for every* $m - \theta-$*open set* $D \subset X$ *containing* x *we have* $D \cap A \neq \emptyset$.

Corollary 31. [24, Lemma 3.6 (6)] *If* m_X *has property* $(\mathcal{B})$, *then* $m_X - Int_\theta(A)$ *is* $m-$*open and* $m_X - Cl_\theta(A)$ *is* $m-$*closed, for every* $A \subset X$.

Moreover, by [24, Lemma 3.6 (5)], $m_X - Cl(A) = m_X - Cl_\theta(A)$ whenever $A \in m_X$. A more general property holds.

Lemma 32. [18] *Let* (X, m_X) *be a space with minimal structure. If* $A \in \mathcal{U}(m_X)$, *then* $m_X - Cl_\theta(A) = m_X - Cl(A)$.

We prove the invariance of $m - \theta-$openness under maps which are almost $M-$open and almost $M-$closed.

Theorem 33. *Every function* $f : (X, m_X) \to (Y, m_Y)$ *which is almost* $M-$*open and almost* $M-$*closed preserves* $m - \theta-$*open sets: if* $A \subset X$ *is* $m - \theta-$*open, then* $f(A) \subset Y$ *is* $m - \theta-$*open.*

Proof. Let $A \subset X$ be $m - \theta-$open. In order to prove that $f(A)$ is $m-\theta-$open, we will show that for every $y \in f(A)$ there exists $V \in m_Y$ such that $y \in V \subset m_Y - Cl(V) \subset f(A)$.

Fix $y \in f(A)$. Choose $x \in A$ such that $y = f(x)$. Since $A \subset X$ is $m-\theta-$open, there exists $U \in m_X$ such that $x \in U \subset m_X - Cl(U) \subset A$. Consequently, $y = f(x) \in f(U) \subset f(m_X - Cl(U)) \subset f(A)$.

As $U \in m_X$ and f is almost $M-$open, it follows that $f(U) \in \mathcal{U}(m_Y)$, therefore there exists a family $\{V_i : i \in I\} \subset m_Y$ such that $f(U) = \bigcup_{i \in I} V_i$. Then $y \in f(U)$ implies the existence of some $j \in I$ such that $y \in V_j$.

On the other hand, since f is almost $M-$closed, by Theorem 18 we obtain $m_Y - Cl(f(U)) \subset f(m_X - Cl(U))$. Obviously, $f(U) \subset m_Y - Cl(f(U))$, hence $f(U) \subset m_Y - Cl(f(U)) \subset f(m_X - Cl(U))$. For every $i \in I$ we have $V_i \subset f(U)$, therefore $m_Y - Cl(V_i) \subset m_Y - Cl(f(U))$.

For $j = i$ we obtain

$$y \in V_j \subset m_Y - Cl(V_j) \subset m_Y - Cl(f(U)) \subset f(m_X - Cl(U)) \subset f(A).$$

In particular, $y \in V_j \subset m_Y - Cl(V_j) \subset f\,(A)$, but $V_j \in m_Y$ and so the proof is completed. ∎

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