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# ON QUASI-REGULAR AND NILPOTENT ELEMENTS IN SUPERTOPOLOGICAL NEAR-RINGS

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**Abstract.** We study supertopological near-rings in which either the set of quasi-regular element is $d$-open (called Q-supertopological rings) or the set of nilpotent elements is $d$-open. We have also characterized Q-supertopological rings among near-rings in terms of right ideals. Several properties regarding the Jacobson radical of such a supertopological ring are proved. Assuming that the set of nilpotent elements is $d$-open and the supertopological ring is right $d$-bounded, it is shown that the Jacobson radical is $d$-open.

## 1. Introduction

Topological rings have been studied by many mathematicians along with S.Warner in [8] and I.Kaplansky in [4], [5]. Topological modules were extensively studied by Arnautov in [1]. Many interesting results related to near-rings have been developed in the past of which a lot have been proved in [2] and [3]. We have used D-supercontinuity and supertopological rings defined in [7] to study supertopological near-rings. In section one, we have provided all the major prerequisites for near-rings and definitions that will be used in later sections.
In section two, on applying various conditions on the set of quasi-regular elements of a supertopological near-ring, we have studied the structure of $d$-absorbing ideals, maximal ideals and $d$-bounded ideals. In last section we have determined the structure of Jacobson ideal in relation with nil subsets. In this paper we denote a supertopological near-ring by $A$.

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**Definition 1.** [6] *A function $f : X \to Y$ from topological space $X$ to topological space $Y$ is said to be D-supercontinuous if for each $x \in X$ and each open set $U \subset Y$ containing $f(x)$ there exists an open $F_\sigma$-set $V \subset X$ containing $x$ such that $f(V) \subset U$.*

**Definition 2.** [7] *A topological space $R$ that is also a ring with operations '+' and '·' where $R \times R$ carries product topology, is a supertopological ring if the mappings*

$$r_1 : R \times R \to R \text{ such that } (x, y) \to x + y$$

*,*

$$r_2 : R \to R \text{ such that } x \to -x$$

*and*

$$r_3 : R \to R \text{ such that } (x, y) \to x \cdot y$$

*are D-supercontinuous.*

For the convenience of the reader, we recall some definitions from the theory of (supertopological) near-rings which are already popular in mathematics literature, for example in [2] and [3]. A *near-ring* is a triplet $(R, +, \cdot)$ where $R$ is non-empty set, and $+$ and $\cdot$ are associative binary operation called addition and multiplication respectively. $(R, +)$ is a group with identity 0 which is not necessarily abelian and the following are satisfied"

(1) if each of $x, y$ and $z$ is in $R$, then $x \cdot (y + z) = x \cdot y + x \cdot z$, and
(2) if $x \in R$, then $0 \cdot x = 0$.

$S$ is an *$R$-subgroup* of near-ring $R$ means that $S$ is a subgroup of $\{R, +\}$ such that $SR \subseteq R$ and $I$ is a *right ideal* of $\{R, +, \cdot\}$ means that $I$ is a normal $R$-subgroup of $\{R, +\}$ and if each of $r$ and $s$ is in $R$ and $c \in I$ then $(r + c)s - rs \in I$.

**Definition 3.** *A supertopological near-ring is a quadruple $\{A, +, \cdot, \tau\}$ such that $\{A, +, \cdot\}$ is a near-ring where $\tau$ is a Hausdorff topology on $A$ and both $+$ and $\cdot$ are D-supercontinuous i.e. for each $a \in A$, the four functions, defined for each $x \in A$ as*

- $f(x) = x + a$,
- $g(x) = a + x$,
- $h(x) = xa$, *and*
- $k(x) = ax$

  *are D-supercontinuous on $A$.*

In [4] Kaplansky defined that $+$ and $\cdot$ be continuous on product space, and hence we assume joint supercontinuity in our article,

and not coordinate-wise supercontinuity although in some cases only coordinate-wise supercontinuity is necessary.
Henceforth, we shall assume that $\{A, +, \cdot, \tau\}$ is a supertopological near-ring (which we shall denote by $A$ and semigroup $\{A, \cdot\}$ has an identity element denoted by 1 which is not equal to 0. Thus if $a \in A$, then $-a = a(-1)$ and the function $l$ defined for each $x \in A$ as $l(x) = a - x$, is also D-supercontinuous. Furthermore, if $O$ is open in $A$ and $a \in A$, then $-O$ and both translations $O + a$ and $a + O$ are also open in $A$.

**Definition 4.** [6] *A set $U$ in a topological space $X$ is said to be d-open if for each $x \in U$, there exists an open $F_\sigma$-set $F$ such that $x \in F \subset U$. Complement of a d-open set is called d-closed.*

We also have that if $O$ is $d$-open in $A$ and $a \in A$, then $-O$ and both translations $O + a$ and $a + O$ are also $d$-open in $A$.

**Definition 5.** [7] *For a set $M \subset X$, the intersection of all the d-closed sets in $X$ containing $M$ is called the d-closure of $M$ which is denoted by $[M]_d$.*

**Definition 6.** *A subset $S$ of topological space $X$ is said to be a d-dense space if $[S]_d = X$.*

More theory on supertopological rings can be found in [7].

Throughout this paper a *d-neighborhood* of a point $x$ will mean an open $F_\sigma$-set containing $x$.

## 2. Q-supertopological near-rings

We recall that $A$ denotes a supertopological near-ring.

**Definition 7.** *By a maximal A-subgroup of A is meant a proper A-subgroup which is not contained in any other proper A-subgroup.*

Using Zorn's lemma, we can assume that $A$ contains a maximal $A$-subgroup, and the intersection of all maximal $A$ subgroups of $A$ is called *radical subgroup* of $A$ and will be denoted by S.

**Definition 8.** *An element $a$ of $A$ is said to be quasi-regular if there exists an element $x$ in $A$ such that $(1 - a)x = 1$.*

We will always denote the set of quasi-regular elements by $Q$.

**Lemma 9.** *If $x \in Q$ and $s$ is in $S$, then $s + x$ is in $Q$.*

This lemma implies that $S \subseteq Q$.

**Theorem 10.** *If $I$ is a quasi-regular right ideal, then $Q + I = Q$.*

*Proof.* For $x \in Q$ and $i \in I$, $(x+i)-x \in I$. As $x+i = ((x+i)-x)+x$, $(x+i) \in Q$ by lemma 9. As $0 \in I$, we are done. □

**Corollary 11.** *If $I$ is a quasi-regular right ideal, then both $I+Q$ and $Q+I$ is $Q$.*

We say that a supertopological near-ring $A$ is a $Q$-supertopological near-ring provided $Q$ i.e. the set of quasi-regular elements, is $d$-open.

**Definition 12.** *A right ideal $I$ of $A$ is said to be d-absorbing if there is a d-open set $U \subset Q$ such that, if $x \in Q$ then there is a $t \in I$ such that $x - t \in U$.*

**Theorem 13.** *The near-ring $R$ is a $Q$-supertopological near-ring if and only if $Q$ contains a d-absorbing right ideal of $R$.*

*Proof.* If $Q$ is $d$-open, let $I = \{0\}$ and let $U = Q$. Then clearly $I$ is $d$-absorbing right ideal. For converse, suppose $I$ is a $d$-absorbing right ideal in $Q$ and $x \in Q$. Let $U$ be any $d$-open set contained in $Q$ and $i \in I$ such that $x - i \in U$. As $I$ is a right ideal, we have that $(x + i) - x \in I$ and since $I$ is quasi-regular, by corollary 11 we have $(x + i) - x + U \subset Q$ which contains $x$. As this subset is also $d$-open, we are done. □

**Lemma 14.** *If $B$ is an $A$-subgroup (right ideal) of $A$,then $cl(B)$ is also an $A$-subgroup (right ideal) of $A$.*

**Theorem 15.** *If $B$ is either a maximal $A$-subgroup or a maximal right ideal of $A$ and $Q$ is d-open , then $B$ is d-closed.*

*Proof.* If $Q$ is $d$-open then $1-Q$ is $d$-open and disjoint from $B$. Therefore $1 \notin cl(B)$, hence we are done by lemma 14. □

The *radical* $J(A)$ of a supertopological near-ring $A$ is the intersection of all right ideals of $A$ that are maximal as $A$-subgroups. If $A$ contains no maximal ideals that are maximal as $A$-subgroups, then $J(A) = A$ and $A$ is called a *radical supertopological near-ring.* Hence we will always consider $J(A)$ as proper subset of $A$. Also, if $Q$ is $d$-open in $A$, then $J(R)$ is $d$-closed and so is radical-subgroup $S$ of $A$.

**Theorem 16.** *If $Q$ is d-open then the direct sum $T$ of all the quasi-regular right ideals of $A$ is a d-closed right ideal of $A$.*

*Proof.* It is known that $T$ is a right ideal that is contained in radical-subgroup $S$. Since $Q$ is $d$-open , $S$ is $d$-closed and thus $cl(T) \subseteq S$ and by lemma 9, $cl(T)$ is quasi-regular. Therefore, $cl(T) \subset T$, and $T$ is $d$-closed. $\square$

**Theorem 17.** *If $T$ is direct sum of the quasi-regular right ideals of $A$, $Q$ is d-open and $T$ is d-dense in $J(A)$, then $J(A) = S$.*

*Proof.* By theorem 16 and definition of $J(A)$,

$$S \subseteq J(A) = cl(T) \subseteq S.$$

$\square$

**Theorem 18.** *If $A$ contains no proper d-closed $A$-subgroups and $Q$ is d-open, then the set of non-zero elements of $A$ is a multiplicative group.*

*Proof.* Let $a$ be an element of $A$. Assume that $aA \subsetneq A$ and as $A$ contains 0, by Zorn's Lemma $A$-subgroup $aA$ is contained in a maximal $A$-subgroup $B$. By theorem 15, $B$ is $d$-closed which is a contradiction. Hence $aA = A$ and $a$ has right inverse. Similarly $a$ has a left inverse and the theorem follows. $\square$

**Theorem 19.** *If $Q$ is d-closed then the direct sum $T$ of all quasi-regular right ideals of $A$ is d-closed.*

*Proof.* Since $T \subseteq Q$ and $Q$ is $d$-closed, $cl(T)$ is also a subset of $Q$. By lemma 14, $T$ is $d$-closed as $cl(T) \subseteq T$. $\square$

**Theorem 20.** *If $A$ contains a unique maximal $A$-subgroup and $Q$ is d-closed, then $B$ is d-closed and each $S$ and $J(A) = B$.*

*Proof.* Observe that $A = B \subseteq cl(A) \subseteq Q$ and the rest follows by lemma 14. $\square$

**Definition 21.** [7] *A subset $D$ of a supertopological ring $A$ is right d-bounded if for any neighborhood $U$ of 0, there exists a d-neighborhood $V$ such that $V.D \subset U$, where $V.S$ is the set of all products of elements in $V$ and $S$.*

**Theorem 22.** *If $A$ is right d-bounded and $Q$ is open $F_\sigma$, then $J(A)$ is open $F_\sigma$.*

*Proof.* Observe that if $x \in V$, then $xA$ is a quasi-regular $A$-subgroup and $xA \subseteq J(A)$ implying $V \subseteq J(A)$. If $y \in J(A)$ and $z \in V$, then $y + (-z) + V$ is open $F_\sigma$ containing $y$ and lies in $J(A)$. $\square$

**Corollary 23.** *If $A$ is d-compact, right d-bounded and $Q$ is open $F_\sigma$ and $J(A) = 0$, then $A$ is finite.*

## 3. Nilpotent elements in supertopological near-rings

Let $N$ denote the set of all nilpotent elements of $A$ where $A$ denotes a supertopological near-ring. A non-empty set $M$ of $A$ is called a *nil subset* of $A$ if $M \subseteq N$.

**Theorem 24.** *If $N$ is d-open and $M$ is a maximal right ideal of $A$, then $M$ is d-closed.*

*Proof.* Suppose $[B]_d = A$. Since $N$ is $d$-open, so is $-N+1$ and there exists a $x \in B$ and $y \in N$, such that $-y+1 = x$. Let $n \in \mathbb{N}$ such that $y^n = 0$.Since $y^{n-1} = (y+x)y^{n-1} - yy^{n-1}$, we see that $y^{n-1} \in B$. By induction on $n$ it follows that $y \in B$ and thus $1 \in B$. This contradiction proves that $[B]_d \neq A$, and as $B$ is a maximal ideal, $B = [B]_d$, implies $B$ is $d$-closed. □

**Corollary 25.** *If $N$ is d-open, then $J(A)$ is d-closed.*

**Lemma 26.** *The Jacobson radical of a near-ring $R$ contains all nil-$R$ subgroups of $R$.*

**Theorem 27.** *If $A$ is right d-bounded and $N$ is d-open, then $J(A)$ is d-open.*

*Proof.* Let $V$ be a $d$-open set such that $V.A \subseteq N$. If $x \in V$ then $xA$ is a nil $A$-subgroup of $A$ hence $xA \subseteq J(A)$ by lemma 26. Since $A$ contains identity element, $V \subseteq J(A)$. If $x \in J(A)$ and $z \in V$, then $x - z + V$ is an open subset contained in radical of $A$ which contains x. Hence, the radical is $d$-open. □

**Corollary 28.** *If $A$ is right d-bounded and $N$ is d-open, then $A$ is not connected.*

## References

[1] V.I. Arnautov , S.T. Glavatsky , A.V. Mikhalev, **Introduction to the Theory of Topological Rings and Modules**, (Shtiinca, Kishinev, 1981) (in Russian).
[2] J.C. Beidleman, **A radical for near-ring modules**, Mich. Math.J. **12**, (1965), 377-383.
[3] J.C. Beidleman, **Quasi-regularity in near-rings**, Math. Z. **89**, (1965), 224-229.
[4] I. Kaplansky, **Topological rings**, Amer. J. Math. **69** (1947), 153–183.
[5] I. Kaplansky, **Topological rings**, Bull. Amer. Math. Soc. **54** (1948), 809–826.
[6] J.K. Kohli , D.Singh, **D-supercontinuous functions**, Indian J. Pure Appl. Math. **32** (2) , (2001), 227-235.

[7] B. Vashishth , D. Singh, **On radical and zero divisors in supertopological rings**, Scientific Studies and Research Series Mathematics and Informatics, **30**, (1), (2020).
[8] S. Warner, **Topological rings**, North Holland, (1993).

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