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SOME REMARKS ON BIHARMONIC QUADRATIC MAPS BETWEEN SPHERES

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Abstract. In this note, we prove a characterization formula for biharmonic maps in the Euclidean spheres of radius R, whose image lies in a small hypersphere. This formula represents a generalization of a result in [10]. Then we apply it for quadratic maps between spheres.

1. Introduction

Biharmonic maps represent a natural generalization of the well known harmonic maps. As suggested by J. Eells and J.H. Sampson in [5, 6], or J. Eells and L. Lemaire in [4], biharmonic maps $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds are critical points of the bienergy functional

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \qquad E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \ v_g,$$

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where M is compact and $\tau(\varphi) = \text{trace}\nabla d\varphi$ is the tension field associated to the smooth map φ. In 1986, G.Y. Jiang proved in [8, 9] that the biharmonic maps are characterized by the vanishing of their bitension field, that is

$$0 = \tau_2(\varphi) = -\Delta\tau(\varphi) - \text{trace}R^N\left(d\varphi(\cdot), \tau(\varphi)\right)\mathrm{d}\varphi(\cdot).$$

The equation $\tau_2(\varphi) = 0$ is called the biharmonic equation and it is a forth order semilinear elliptic equation.

Since any harmonic map is automatically biharmonic, we study the biharmonic maps which are not harmonic, which are called proper biharmonic.

While the most examples and classification results for proper biharmonic maps have been obtained in the submanifolds theory (see, for example, [7] and [14]), many other examples of proper biharmonic maps were obtained when the maps are not immersions (see, for example, [11], [12], [13] and [15]).

In this paper, we first establish a characterization formula for biharmonic maps that take values in the Euclidean sphere, but whose image lies in a small hypersphere. This formula is a generalization to a previous result obtained by E. Loubeau and C. Oniciuc in [10]. Then, as an application, we consider the particular case of homogeneous polynomial maps of degree 2 between spheres, that are called quadratic forms. Further, for a better understanding of the structure of the proper biharmonic quadratic forms, we construct a special example of such a map.

Conventions. In this paper, the following sign conventions for the rough Laplacian, that acts on the set $C\left(\varphi^{-1}TN\right)$ of all sections of the pull-back bundle $\varphi^{-1}TN$, and for the curvature tensor field are used

$$\Delta\sigma = -\text{trace}\nabla^2\sigma, \quad R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_Y Z - \nabla_{[X,Y]}Z.$$

By $\mathbb{S}^m(r)$ we indicate the m-dimensional Euclidean sphere of radius r and, when $r = 1$, we write $\mathbb{S}^m$ instead of $\mathbb{S}^m(1)$.

2. Preliminaries

In the following, we will denote by i the canonical inclusion of the small hypersphere $\mathbb{S}^{n-1}\left(R/\sqrt{2}\right)$ into $\mathbb{S}^n(R)$.

First, we recall a result from [10] which can be given in the following way.

Theorem 2.1. *[10] Let M be a compact manifold and $\psi : M \to \mathbb{S}^{n-1}\left(R/\sqrt{2}\right)$ a non constant map. If the map $\varphi = \mathrm{i} \circ \psi : M \to \mathbb{S}^n(R)$ is proper biharmonic, then ψ is harmonic and $e(\psi)$ is constant.*

We can give the converse, where we do not need the compactness of M.

Theorem 2.2. *[10] Let $\psi : M \to \mathbb{S}^{n-1}\left(R/\sqrt{2}\right)$ be a non constant map. If ψ is harmonic and $e(\psi)$ is constant, then the map $\varphi = \mathrm{i} \circ \psi : M \to \mathbb{S}^n(R)$ is proper biharmonic.*

We also recall that for a small hypersphere $\mathbb{S}^{n-1}(r) \times \left\{\sqrt{R^2 - r^2}\right\} \equiv \mathbb{S}^{n-1}(r)$ of $\mathbb{S}^n(R)$, the set of all vector fields tangent to $\mathbb{S}^{n-1}(r)$ is given by

$$\begin{aligned} C\left(T\mathbb{S}^{n-1}(r)\right) = \big\{X = \left(X^1, \ldots, X^n, 0\right), \\ x^1X^1 + x^2X^2 + \cdots + x^nX^n = 0\big\} \end{aligned}$$

and a unit section in the normal bundle of $\mathbb{S}^{n-1}(r)$ in $\mathbb{S}^n(R)$ is

$$\eta_p = \frac{1}{R}\left(\frac{\sqrt{R^2 - r^2}}{r}x^1, \ldots, \frac{\sqrt{R^2 - r^2}}{r}x^n, -r\right),$$

where $p = \left(x^1, x^2, \ldots, x^n, \sqrt{R^2 - r^2}\right) \in \mathbb{S}^{n-1}(r)$.

Lemma 2.3. *Let $\mathbb{S}^{n-1}(r)$ be a small hypersphere of $\mathbb{S}^n(R)$ having the radius $0 < r < R$. Then the second fundamental form of $\mathbb{S}^{n-1}(r)$ in $\mathbb{S}^n(R)$ is*

$$B(X, Y) = -\frac{\sqrt{R^2 - r^2}}{Rr}\langle X, Y\rangle\eta.$$

Further, we present how the composition with an homothety modifies several quantities that we will appear in our computations.

Lemma 2.4. *Let $\tilde{\varphi} : M \to \mathbb{S}^{n-1}$ be a smooth map and let $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}(r)$ be an homothety, $h(\overline{x}) = r\overline{x}$. Consider $\psi : M \to \mathbb{S}^{n-1}(r)$ the composition $\psi = h \circ \tilde{\varphi}$. Then,*

$$\begin{aligned} |\mathrm{d}\psi|^2 &= r^2 \left|\mathrm{d}\tilde{\varphi}\right|^2, \\ \tau(\psi) &= r\tau(\tilde{\varphi}), \\ \Delta^{\psi}\tau(\psi) &= r\Delta^{\varphi}\tau(\tilde{\varphi}), \\ \tau_2(\psi) &= r\tau_2(\tilde{\varphi}). \end{aligned} \tag{2.1}$$

3. Biharmonic maps into spheres, whose image lies in a smaller hypersphere

The next result gives a formula for the bitension field in the more general case of maps $\varphi : M \to \mathbb{S}^n(R)$ whose image lie in a hypersphere of an arbitrary radius $\mathbb{S}^{n-1}(r)$.

Theorem 3.1. *Let $\psi : M \to \mathbb{S}^{n-1}(r)$ be a non constant map, and consider $\varphi = \mathrm{i} \circ \psi : M \to \mathbb{S}^n(R)$, where $0 < r < R$. Then, the bitension field of φ is given by*

(3.1)
$$\tau_2(\varphi) = \tau_2(\psi) + 2\left(\frac{1}{R^2} - \frac{1}{r^2}\right)\mathrm{d}\psi\left(\mathrm{grad}\,|\mathrm{d}\psi|^2\right) + 4\left(\frac{1}{R^2} - \frac{1}{r^2}\right)e(\psi)\tau(\psi)$$
$$+\frac{\sqrt{R^2-r^2}}{Rr}\left(2\Delta\left(e(\psi)\right) - 2\mathrm{div}\theta^\sharp + |\tau(\psi)|^2 - 4\left(\frac{2}{R^2} - \frac{1}{r^2}\right)\left(e(\psi)\right)^2\right)\eta,$$

where $\theta(X) := \langle \mathrm{d}\psi(X), \tau(\psi)\rangle$, for any vector field X tangent to M.

Proof. First, using Lemma (2.3) we get

$$\tau(\varphi) = \tau(\psi) - 2\frac{\sqrt{R^2-r^2}}{Rr}e(\psi)\eta. \tag{3.2}$$

We note that φ cannot be harmonic.
For $\sigma \in C\left(\psi^{-1}T\mathbb{S}^n(r)\right)$, we have $\mathrm{di}(\sigma) \in C\left(\varphi^{-1}T\mathbb{S}^{n+1}(R)\right)$ and
(3.3)
$$\nabla^\varphi_{X_p}\mathrm{di}(\sigma) = \mathrm{di}_{\psi(p)}\left(\nabla^\psi_{X_p}\sigma\right) - (\nabla\mathrm{di})_{\psi(p)}\left(\mathrm{d}\psi_p\left(X_p\right), \sigma(p)\right), \quad \forall p \in M.$$

Thus,

$$\nabla^\varphi_X \sigma = \nabla^\psi_X(\sigma) - \frac{\sqrt{R^2-r^2}}{Rr}\langle \mathrm{d}\psi(X), \sigma\rangle\eta. \tag{3.4}$$

In order to compute $\tau_2(\varphi)$ in terms of ψ, we consider an arbitrary point $p \in M$ and a geodesic frame field $\{X_i\}_{i=1}^m$ around p. Around p we have

$$\nabla^\varphi_{X_i}\tau(\psi) = \nabla^\psi_{X_i}\tau(\psi) - \frac{\sqrt{R^2-r^2}}{Rr}\langle \mathrm{d}\psi(X_i), \tau(\psi)\rangle\eta.$$

Thus,

$$\nabla^\psi_{X_i}\tau(\psi) = \nabla^\varphi_{X_i}\tau(\psi) + \frac{\sqrt{R^2-r^2}}{Rr}\langle \mathrm{d}\psi(X_i), \tau(\psi)\rangle\eta. \tag{3.5}$$

For $\theta(X) := \langle \mathrm{d}\psi(X), \tau(\psi)\rangle$, using Equationa (3.2) and (3.5) we get

$$\begin{aligned}\nabla^{\psi}_{X_i}\tau(\psi) =&\nabla^{\varphi}_{X_i}\tau(\psi) + \frac{\sqrt{R^2-r^2}}{Rr}\theta(X_i)\eta\\ =&\nabla^{\varphi}_{X_i}\tau(\varphi) + 2\frac{\sqrt{R^2-r^2}}{Rr}X_i\left(e(\psi)\right)\eta + 2\frac{\sqrt{R^2-r^2}}{Rr}e(\psi)\nabla^{\varphi}_{X_i}\eta\\ &+ \frac{\sqrt{R^2-r^2}}{Rr}\theta(X_i)\eta.\end{aligned}$$

Thus, we obtain

(3.6)
$$\begin{aligned}\nabla^{\psi}_{X_i}\tau(\psi) =&\nabla^{\varphi}_{X_i}\tau(\varphi) + 2\frac{\sqrt{R^2-r^2}}{Rr}X_i\left(e(\psi)\right)\eta + 2\frac{R^2-r^2}{R^2r^2}e(\psi)\mathrm{d}\varphi(X_i)\\ &+ \frac{\sqrt{R^2-r^2}}{Rr}\theta(X_i)\eta.\end{aligned}$$

For $\sigma = \nabla^{\psi}_{X_i}\tau(\psi)$, using Equation (3.4) we get around p

$$\nabla^{\varphi}_{X_i}\nabla^{\psi}_{X_i}\tau(\psi) =\nabla^{\psi}_{X_i}\nabla^{\psi}_{X_i}\tau(\psi) - \frac{\sqrt{R^2-r^2}}{Rr}\langle \mathrm{d}\psi(X_i), \nabla^{\psi}_{X_i}\tau(\psi)\rangle\eta.$$

Thus, we obtain

$$\nabla^{\psi}_{X_i}\nabla^{\psi}_{X_i}\tau(\psi) =\nabla^{\varphi}_{X_i}\nabla^{\psi}_{X_i}\tau(\psi) + \frac{\sqrt{R^2-r^2}}{Rr}\langle \mathrm{d}\psi(X_i), \nabla^{\psi}_{X_i}\tau(\psi)\rangle\eta. \tag{3.7}$$

Replacing Equation (3.6) in Equation (3.7) we get

$$\begin{aligned}\nabla^{\psi}_{X_i}\nabla^{\psi}_{X_i}\tau(\psi) =& \nabla^{\varphi}_{X_i}\nabla^{\varphi}_{X_i}\tau(\varphi) + 2\frac{\sqrt{R^2-r^2}}{Rr}X_iX_ie(\psi)\eta\\ &+ 4\left(\frac{1}{r^2}-\frac{1}{R^2}\right)X_ie(\psi)\mathrm{d}\varphi(X_i) + 2\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\nabla^{\varphi}_{X_i}\mathrm{d}\varphi(X_i)\\ &+ \left(\frac{1}{r^2}-\frac{1}{R^2}\right)\langle\theta^{\sharp}, X_i\rangle\mathrm{d}\varphi(X_i)\\ &+ \frac{\sqrt{R^2-r^2}}{Rr}\Big(\langle\nabla_{X_i}\theta^{\sharp}, X_i\rangle + \langle\theta^{\sharp}, \nabla_{X_i}X_i\rangle + X_i\langle \mathrm{d}\psi(X_i), \tau(\varphi)\rangle\\ &-\langle\nabla^{\varphi}_{X_i}\mathrm{d}\varphi(X_i), \tau(\varphi)\rangle + 2\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\langle \mathrm{d}\psi(X_i), \mathrm{d}\varphi(X_i)\rangle e(\psi)\Big)\eta.\end{aligned}$$

Taking sum in the above equation, at p, we obtain

$$\begin{aligned}-\Delta^{\psi}\tau(\psi) =& -\Delta^{\varphi}\tau(\varphi) + \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \mathrm{d}\varphi\left(4\mathrm{grad}\left(e(\psi)\right) + \theta^{\sharp}\right) \\ & + 2\left(\frac{1}{r^2} - \frac{1}{R^2}\right) e(\psi)\tau(\varphi) + \frac{\sqrt{R^2 - r^2}}{Rr}\left(-2\Delta(e(\psi))\right. \\ & \left. + 2\mathrm{div}\theta^{\sharp} - \left|\tau(\varphi)\right|^2 + 4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)\left(e(\psi)\right)^2\right)\eta.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta^{\varphi}\tau(\varphi) =& \Delta^{\psi}\tau(\psi) + \left(\frac{1}{r^2} - \frac{1}{R^2}\right) \mathrm{d}\varphi\left(4\mathrm{grad}\left(e(\psi)\right) + \theta^{\sharp}\right) \\ & + 2\left(\frac{1}{r^2} - \frac{1}{R^2}\right) e(\psi)\tau(\varphi) + \frac{\sqrt{R^2 - r^2}}{Rr}\left(-2\Delta(e(\psi))\right. \\ & \left. + 2\mathrm{div}\theta^{\sharp} - \left|\tau(\varphi)\right|^2 + 4\left(\frac{1}{r^2} - \frac{1}{R^2}\right)\left(e(\psi)\right)^2\right)\eta.\end{aligned} \tag{3.8}$$

Further,

$$\begin{aligned}& R^{\mathbb{S}^n(R)}\left(\mathrm{d}\varphi(X_i), \tau(\varphi)\right)\mathrm{d}\varphi(X_i) = \\ & \qquad = \frac{1}{R^2}\langle\tau(\varphi), \mathrm{d}\varphi(X_i)\rangle\mathrm{d}\varphi(X_i) - \frac{1}{R^2}\left|\mathrm{d}\varphi(X_i)\right|^2\tau(\varphi) \\ & \qquad = \frac{1}{R^2}\langle\tau(\varphi), \mathrm{d}\varphi(X_i)\rangle\mathrm{d}\varphi(X_i) \\ & \qquad\quad - \frac{1}{R^2}\left|\mathrm{d}\varphi(X_i)\right|^2\left(\tau(\psi) - 2\frac{\sqrt{R^2 - r^2}}{Rr}e(\psi)\eta\right).\end{aligned}$$

Taking sum in the above equation, we get

$$\begin{aligned}& \mathrm{trace}R^{\mathbb{S}^n(R)}\left(\mathrm{d}\varphi, \tau(\varphi)\right)\mathrm{d}\varphi = \\ & \qquad = \frac{1}{R^2}\mathrm{d}\varphi(\theta^{\sharp}) - \frac{2}{R^2}e(\psi)\tau(\psi) + 4\frac{\sqrt{R^2 - r^2}}{R^3 r}\left(e(\psi)\right)^2\eta.\end{aligned} \tag{3.9}$$

Using Equations (3.8) and (3.9), it follows that

$$\begin{aligned}\tau_2(\varphi) =& - \Delta^{\varphi}\tau(\varphi) - \mathrm{trace} R^{\mathbb{S}^n(R)}\left(\mathrm{d}\varphi, \tau(\varphi)\right)\mathrm{d}\varphi \\ =& - \Delta^{\psi}\tau(\psi) - \frac{R^2 - r^2}{R^2 r^2}\mathrm{d}\varphi\left(4\mathrm{grad}\left(e(\psi)\right)\right) - \frac{1}{r^2}\mathrm{d}\varphi\left(\theta^\sharp\right) \\ & - 2\left(\frac{2}{R^2} - \frac{1}{r^2}\right)e(\psi)\tau(\psi) + \frac{\sqrt{R^2 - r^2}}{Rr}\Big(2\Delta\left(e(\psi)\right) \\ & -2\mathrm{div}\theta^\sharp + |\tau(\psi)|^2 - 4\left(\frac{2}{R^2} - \frac{1}{r^2}\right)\left(e(\psi)\right)^2\Big)\eta. \end{aligned} \tag{3.10}$$

Consider

$$\frac{1}{r}\tilde{\Phi} = \mathrm{i} \circ \tilde{\varphi} : M \to \mathbb{R}^n,$$

where $\mathrm{i} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is the canonical inclusion. More precisely, we have the following diagram

$$\begin{tikzcd} M \arrow[r, "\frac{1}{r}\tilde{\Phi}"] \arrow[d, "\psi"'] \arrow[dr, "\tilde{\varphi}"] & \mathbb{R}^n \\ \mathbb{S}^{n-1}(r) & \mathbb{S}^{n-1} \arrow[l, "h"] \arrow[u, "\mathrm{i}"'] \end{tikzcd}$$

We note that $\tilde{\Phi} = \mathrm{i} \circ \psi$, where $\mathrm{i} : \mathbb{S}^{n-1}(r) \to \mathbb{R}^n$ is the canonical inclusion. From [1] we have the following

$$\begin{aligned}\tau_2(\tilde{\varphi}) =& - \Delta^{\tilde{\varphi}}\tau(\tilde{\varphi}) - \mathrm{d}\tilde{\varphi}(\tilde{\theta}^\sharp) + \left|\mathrm{d}\left(\frac{1}{r}\tilde{\Phi}\right)\right|^2 \tau\left(\frac{1}{r}\tilde{\Phi}\right) + \left|\mathrm{d}\left(\frac{1}{r}\tilde{\Phi}\right)\right|^4 \frac{1}{r}\tilde{\Phi} \\ =& - \Delta^{\tilde{\varphi}}\tau(\tilde{\varphi}) - \mathrm{d}\tilde{\varphi}(\tilde{\theta}^\sharp) + \left|\mathrm{d}\left(\frac{1}{r}\tilde{\Phi}\right)\right|^2 \tau(\tilde{\varphi}).\end{aligned}$$

Using Equations (2.1) and the last formula, we obtain

$$\tau_2(\psi) = -\Delta^{\psi}\tau(\psi) - \frac{1}{r^2}\mathrm{d}\psi\left(\theta^\sharp\right) + \frac{1}{r^2}\left|\mathrm{d}\psi\right|^2\tau(\psi).$$

Thus, we have

$$- \Delta^{\psi}\tau(\psi) = \tau_2(\psi) + \frac{1}{r^2}\mathrm{d}\psi\left(\theta^\sharp\right) - \frac{1}{r^2}\left|\mathrm{d}\psi\right|^2\tau(\psi). \tag{3.11}$$

We replace Equation (3.11) in Equation (3.10) and the proof is complete. □

Remark 3.2. For $R = 1$, Equation (3.1) becomes

$$\tau_2(\varphi) = \tau_2(\psi) + 2\left(1 - \frac{1}{r^2}\right) \mathrm{d}\psi\left(\operatorname{grad}|\mathrm{d}\psi|^2\right) + 4\left(1 - \frac{1}{r^2}\right) e(\psi)\tau(\psi) + \frac{\sqrt{1-r^2}}{r}\left(2\Delta\left(e(\psi)\right) - 2\mathrm{div}\theta^\sharp + |\tau(\psi)|^2 - 4\left(2 - \frac{1}{r^2}\right)\left(e(\psi)\right)^2\right)\eta. \tag{3.12}$$

Remark 3.3. If $R = 1$ and ψ has constant energy density, then Equation (3.12) becomes

$$\tau_2(\varphi) = \tau_2(\psi) + 4\left(1 - \frac{1}{r^2}\right) e(\psi)\tau(\psi) + \frac{\sqrt{1-r^2}}{r}\left(-2\mathrm{div}\theta^\sharp + |\tau(\psi)|^2 - 4\left(2 - \frac{1}{r^2}\right)\left(e(\psi)\right)^2\right)\eta. \tag{3.13}$$

Using the above notations, we give the following results.

Proposition 3.4. *Assume that the map φ is biharmonic. If M is compact, then $r \geq 1/\sqrt{2}$.*

Proposition 3.5. *Assume that φ is biharmonic, $e(\varphi)$ is constant and* $\mathrm{div}\theta^\sharp = 0$. *Then $r \geq 1/\sqrt{2}$.*

Proposition 3.6. *If the map ψ is harmonic with constant energy density, then φ is biharmonic if and only if $r = 1/\sqrt{2}$.*

Further, we recall a result regarding the biharmonicity of homogeneous polynomial maps of degree 2 between spheres, i.e. quadratic forms (we follow the notations and terminology in [1]). For more details about quadratic forms, see [3] and [16].

Theorem 3.7. *(see [1]) Let $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ be a quadratic form given by*

$$F(\overline{x}) = \left(X^t A_1 X, \ldots, X^t A_{n+1}\, X\right),$$

such that if $|\overline{x}| = 1$ then $|F(\overline{x})| = 1$. We consider $\varphi : \mathbb{S}^m \to \mathbb{S}^n$ defined by $\varphi(\overline{x}) = F(\overline{x})$ and $\Phi = \mathrm{i} \circ \varphi : \mathbb{S}^m \to \mathbb{R}^{n+1}$. If we denote $S = A_1^2 + A_2^2 + \cdots + A_{n+1}^2$ then, at a point $\overline{x} \in \mathbb{S}^m$, the bitension field of φ has the following expression

$$\begin{aligned}\tau_2(\varphi)_{\overline{x}} = &- 4\left(m + 5 - 4X^t S X\right)\left(\mathrm{tr}A_1, \mathrm{tr}A_2, \ldots, \mathrm{tr}A_{n+1}\right)\\ &+ 4\left((m+3)(m+5) - 6(m+5)X^t S X + 8\left(X^t S X\right)^2\right)\Phi(\overline{x})\\ &+ 32\left(X^t A_1 S X, \ldots, X^t A_{n+1} S X\right).\end{aligned} \tag{3.14}$$

Further, we consider the particular case of quadratic forms when their image lies in a small hypersphere. As an apllication to Theorem (3.1), we compute the bitension field from Equation (3.12) taking into account Theorem (3.7). In order to accomplish that, we use Lemma (2.4) and some computations from [1]. Then, we recover Proposition (3.6) for this particular case.

Consider the diagram below

$$\begin{tikzcd}
\mathbb{R}^{m+1} \arrow[rrrr, "F"] \arrow[drr, "\widetilde{F}"] & & & & \mathbb{R}^{n+1} \\
& & \mathbb{R}^{n} \arrow[urr, "G"] & & \\
& & \mathbb{S}^{n-1}(r) \arrow[u, "\mathrm{i}"'] \arrow[ddrr, "H"] & & \\
& & \mathbb{S}^{n-1} \arrow[u, "h"'] & & \\
\mathbb{S}^{m} \arrow[uuuu, "\mathrm{i}"] \arrow[rrrr, "\varphi"'] \arrow[uuurr, "\widetilde{\Phi}"] \arrow[uurr, "\psi"] \arrow[urr, "\widetilde{\varphi}"'] \arrow[uuuurrrr, "\Phi"] & & & & \mathbb{S}^{n} \arrow[uuuu, "\mathrm{i}"']
\end{tikzcd}$$

where the small hypersphere $\mathbb{S}^{n-1}(r)$ of $\mathbb{S}^n$ is identified with the $(n-1)$-dimensional sphere of $\mathbb{R}^n$, and $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ is a quadratic form given by

$$F(\overline{x}) = \left(X^t A_1 X, \ldots, X^t A_{n+1}\ X \right),$$

such that if $|\overline{x}| = 1$ then $|F(\overline{x})| = 1$. In this case, $\tilde{F} = \mathrm{Pr} \circ F$, where $\mathrm{Pr} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the vector function given by

$$\mathrm{Pr}\left(y^1, \ldots, y^n, y^{n+1}\right) = \left(y^1, \ldots, y^n\right),$$

and the map $\tilde{\Phi} : \mathbb{S}^m \to \mathbb{R}^n$ is given by

$$\tilde{\Phi} = \tilde{F}_{|\mathbb{S}^m} : \mathbb{S}^m \to \mathbb{R}^n.$$

The vector function $G : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is given by

$$G(\overline{y}) = G(y^1, \ldots, y^n) = \left(y^1, \ldots, y^n, \frac{\sqrt{1-r^2}}{r^2} |\overline{y}|^2 \right),$$

the map $H : \mathbb{S}^{n-1}(r) \to \mathbb{S}^n$ is given by

$$H(\overline{y}) = H(y^1, \ldots, y^n) = \left(y^1, \ldots, y^n, \sqrt{1-r^2} \right),$$

and the map $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}(r)$ is given by

$$h(\overline{y}) = r\overline{y}.$$

We will compute all terms from the right hand side member of equation (3.1). For simplicity, we set

$$\begin{aligned} T =& \tau_2(\psi) + 2\left(\frac{1}{R^2} - \frac{1}{r^2} \right) \mathrm{d}\psi \left(\operatorname{grad} |\mathrm{d}\psi|^2 \right) + 4\left(\frac{1}{R^2} - \frac{1}{r^2} \right) e(\psi)\tau(\psi), \\ N =& 2\Delta \left(e(\psi) \right) - 2\operatorname{div}\theta^\sharp + |\tau(\psi)|^2 - 4\left(\frac{2}{R^2} - \frac{1}{r^2} \right) \left(e(\psi) \right)^2. \end{aligned}$$

First, we compute N. In [1], we have already computed some of the terms that form N, but for quadratic maps between unit spheres. Now we need to modify the radius of the target sphere, thus we compose $\tilde{F}$ with the right homothety. Consider the following diagram

$$\begin{tikzcd}
\mathbb{R}^{m+1} \arrow[r, "\frac{1}{r}\tilde{F}"] & \mathbb{R}^n \\
\mathbb{S}^m \arrow[u, "\mathrm{i}"] \arrow[ur, "\frac{1}{r}\tilde{\Phi}"] \arrow[r, "\tilde{\varphi}"'] & \mathbb{S}^{n-1} \arrow[u, "\mathrm{i}"']
\end{tikzcd}$$

We denote

$$\tilde{S} = A_1^2 + \cdots + A_n^2.$$

From [1] we recall that

$$\left| \overset{o}{\mathrm{d}} \left(\frac{1}{r}\tilde{F} \right) \right|^2 = 4X^t \left(\frac{1}{r^2}\tilde{S} \right) X = \frac{4}{r^2} X^t \tilde{S} X.$$

First, using the definition of θ and Lemma (2.4) we have

$$\theta\left(X \right) = \left\langle \mathrm{d}\psi(X), \tau(\psi) \right\rangle = r^2 \left\langle \mathrm{d}\tilde{\varphi}(X), \tau(\tilde{\varphi}) \right\rangle = r^2 \tilde{\theta}(X).$$

We note that θ and $\tilde{\theta}$ are 1-forms on $\mathbb{S}^m$. Thus, $\mathrm{div}\theta^\sharp = r^2\mathrm{div}\tilde{\theta}^\sharp$. Using [1], we have

$$\mathrm{div}\tilde{\theta}^\sharp = \left|\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\right|^2 + 2(m+1)\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - 4(m+1)(m+3).$$

Therefore,

(3.15)

$$\mathrm{div}\theta^\sharp = r^2\left\{\left|\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\right|^2 + \frac{8(m+1)}{r^2}X^t\tilde{S}X - 4(m+1)(m+3)\right\}.$$

Further,

$$\begin{aligned} 2\Delta e(\psi) =& \Delta\left|\mathrm{d}\psi\right|^2 = r^2\Delta\left|\mathrm{d}\tilde{\varphi}\right|^2 \\ =& r^2\left\{\overset{o}{\Delta}\left(\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2\right) + 2(m+1)\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2\right\} \\ =& r^2\left\{-8\mathrm{tr}\left(\frac{1}{r^2}\tilde{S}\right) + \frac{8(m+1)}{r^2}X^t\tilde{S}X\right\}. \end{aligned} \tag{3.16}$$

Since

$$\tau(\tilde{\varphi}) = -\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right) + \left(\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - 2(m+3)\right)\frac{1}{r}\tilde{\Phi},$$

we get

$$\left|\tau(\tilde{\varphi})\right|^2 = \left|\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - \left(\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - 2(m+3)\right)^2.$$

Thus,

$$\begin{aligned} \left|\tau(\psi)\right|^2 =& r^2\left|\tau(\tilde{\varphi})\right|^2 \\ =& r^2\left\{\left|\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - \frac{16}{r^4}\left(X^t\tilde{S}X\right)^2 + \frac{16(m+3)}{r^2}X^t\tilde{S}X\right. \\ & \left. -4(m+3)^2\right\}. \end{aligned} \tag{3.17}$$

Next, we have that

$$\begin{aligned} 2e(\psi) =& \left|\mathrm{d}\psi\right|^2 = r^2\left|\mathrm{d}\tilde{\varphi}\right|^2 \\ =& r^2\left\{\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2 - 4\right\}. \end{aligned} \tag{3.18}$$

Using a general property of quadratic forms (see [1]), we obtain

$$8\operatorname{tr}\left(\frac{1}{r^2}\tilde{S}\right)+\left|\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\right|^2=4(m+1)(m+3). \tag{3.19}$$

We replace Equations (3.15), ..., (3.19) in the equation of N and by direct calculations, we obtain
(3.20)

$$N=-8(m+1)r^2-32r^4+8\left(m+1+8r^2\right)X^t\tilde{S}X-32\left(X^t\tilde{S}X\right)^2.$$

In order to compute T, using again [1], we have

$$\begin{aligned}
&2\mathrm{d}\psi\left(\operatorname{grad}|\mathrm{d}\psi|^2\right)=2r^3\mathrm{d}\tilde{\varphi}\left(\operatorname{grad}|\mathrm{d}\tilde{\varphi}|^2\right)\\
&=r^3\left\{2\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\left(\overset{o}{\operatorname{grad}}\left(\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right)\right)-8\left|\overset{o}{\mathrm{d}}\left(\frac{1}{r}\tilde{F}\right)\right|^2\frac{1}{r}\tilde{\Phi}\right\} \qquad (3.21)\\
&=32\left(X^tA_1\tilde{S}X,\ldots,X^tA_n\tilde{S}X\right)-32X^t\tilde{S}X\tilde{\Phi}.
\end{aligned}$$

Using Theorem (3.7), at $\overline{x}\in\mathbb{S}^m$ we have

$$\begin{aligned}
\tau_2(\psi)=&r\tau_2(\tilde{\varphi})\\
=&2r\left(m+5-4X^t\tilde{S}X\right)\overset{o}{\Delta}\left(\frac{1}{r}\tilde{F}\right)\\
&+4\left((m+3)(m+5)-\frac{6(m+5)}{r^2}X^t\tilde{S}X+\frac{8}{r^4}\left(X^t\tilde{S}X\right)^2\right)\tilde{\Phi}\\
&+\frac{32}{r^2}\left(X^tA_1\tilde{S}X,\ldots,X^tA_n\tilde{S}X\right).
\end{aligned}$$

We replace the above equations in the expression of T and by direct calculations we obtain

$$\begin{aligned}
T=&2\left(m+1+4r^2-4X^t\tilde{S}X\right)\overset{o}{\Delta}\tilde{F}+32\left(X^tA_1\tilde{S}X,\ldots,X^tA_n\tilde{S}X\right)\\
&+\left(4(m+3)(m+1)-\frac{8}{r^2}(m+1)X^t\tilde{S}X\right. \qquad (3.22)\\
&\left.+\frac{32}{r^2}\left(X^t\tilde{S}X\right)^2-16(m+7)X^t\tilde{S}X+16(m+3)r^2\right)\tilde{\Phi}.
\end{aligned}$$

Now we are ready to reobtain the result in Proposition (3.6). Indeed, if we suppose that ψ is harmonic, using a result in [1], it follows that $\overset{o}{\Delta}\tilde{F}=\overline{0}$. Moreover, the energy density of ψ is constant and $\tilde{S}=$

$(m+3)(r^2/2)I_{m+1}$. As a consequence, T vanishes and

$$N = -8(m+1)^2r^4 + 4(m+1)^2r^2.$$

Therefore, φ is proper biharmonic if and only if $r = 1/\sqrt{2}$.

Special example. Based on the examples from [1], we may think that if a proper biharmonic quadratic form lies in a small hypersphere $\mathbb{S}^{n-1}(r) \times \left\{\sqrt{1-r^2}\right\}$ of $\mathbb{S}^n$, then $r = 1/\sqrt{2}$ and the corresponding map ψ is a harmonic map.

In other words, one may ask the following question: if the image of φ lies in a hyperplane $(\Pi) : \left\langle \overline{N}, \overline{y} \right\rangle = \alpha$, then does it follow that the distance $\mathrm{d}\left(O, \Pi\right)$ is $1/\sqrt{2}$? The answer is negative, meaning that: according to a result of [2], if $\overline{N}$ is parallel to $\overset{o}{\Delta} F$, then $\mathrm{d}\left(O, \Pi\right)$ is indeed $1/\sqrt{2}$, but if $\overline{N}$ is not parallel to $\overset{o}{\Delta} F$, then $\mathrm{d}\left(O, \Pi\right)$ is not $1/\sqrt{2}$. In order to support this affirmation, we construct the following example.

Let $F = (F^1, F^2, F^3, F^4) : \mathbb{R}^4 \to \mathbb{R}^4$,

$$\begin{aligned} F\left(\overline{x}\right) = \Bigg(&\frac{1}{\sqrt{2}}\left(\left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right), \sqrt{2}\left(x^1x^3 - x^2x^4\right), \\ &\sqrt{2}\left(x^1x^4 + x^2x^3\right), \frac{1}{\sqrt{2}}\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2\right)\Bigg). \end{aligned}$$

By simple computations, we can see that

$$\overset{o}{\Delta} F = \left(0, 0, 0, -\frac{8}{\sqrt{2}}\right),$$

and

$$\left\langle -\overset{o}{\Delta} F, F\left(\overline{x}\right)\right\rangle = 4\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2\right).$$

If $|\overline{x}| = 1$, then $|F\left(\overline{x}\right)| = 1$. Thus, F takes $\mathbb{S}^3$ into $\mathbb{S}^3$. Therefore, the image of φ lies in the hyperplane (Π) in $\mathbb{R}^4$ given by

$$(\Pi) : \left\langle -\overset{o}{\Delta} F, \overline{y}\right\rangle = 4.$$

Using directly Theorem (3.7), or using [2], one can prove that $\varphi = F_{|\mathbb{S}^3}$ is a biharmonic map. We note that F^4 is constant on $\mathbb{S}^3$ and the first 3 components of φ are harmonic polynomials on $\mathbb{R}^4$ and form a harmonic map $\psi : \mathbb{S}^3 \to \mathbb{S}^2(1/\sqrt{2})$.

We construct a new map $G : \mathbb{R}^4 \to \mathbb{R}^5$, given by

$$G\left(\overline{x}\right) = \left(F\left(\overline{x}\right), 0\right).$$

Since $F\big|_{\mathbb{S}^m}$ is a biharmonic map, it follows that also $G\big|_{\mathbb{S}^m}$ is a biharmonic map. We apply the next transformation

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which is an isometry, on the components of G. Thus, we obtain the map

$$T_1 \circ G = \left(F^1, F^2, F^3, \frac{1}{2}F^4, \frac{1}{2}F^4\right).$$

Next, we apply another transformation,

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is also an isometry, to $T_1 \circ G$, and we obtain

$$H = T_2 \circ T_1 \circ G = \left(F^1, F^2, \frac{1}{\sqrt{2}}\left(F^3 - \frac{1}{2}F^4\right), \frac{1}{\sqrt{2}}\left(F^3 + \frac{1}{2}F^4\right), \frac{1}{2}F^4\right).$$

More precisely,

$$\begin{aligned} H\left(\overline{x}\right) = \Bigg(& \frac{1}{\sqrt{2}}\left(\left(x^1\right)^2 + \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right), \sqrt{2}\left(x^1x^3 - x^2x^4\right), \\ & \frac{1}{\sqrt{2}}\left(\sqrt{2}\left(x^1x^4 + x^2x^3\right) - \frac{1}{2}\left|\overline{x}\right|^2\right), \\ & \frac{1}{\sqrt{2}}\left(\sqrt{2}\left(x^1x^4 + x^2x^3\right) + \frac{1}{2}\left|\overline{x}\right|^2\right), \frac{1}{2}\left|\overline{x}\right|^2\Bigg). \end{aligned}$$

We can see that

$$\overset{o}{\Delta} H = \left(0, 0, \frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}}, -4\right).$$

and

$$\left\langle -\overset{o}{\Delta} H, H\left(\overline{x}\right)\right\rangle = 4\left(\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2\right).$$

We consider the hyperplane (Π_1) in $\mathbb{R}^5$, given by

$$(\Pi_1) : \left\langle -\overset{o}{\Delta} H, \overline{y} \right\rangle = 4.$$

The image of restriction $\varphi_1 = H_{|\mathbb{S}^3} : \mathbb{S}^3 \to \mathbb{S}^4$ lies in the hyperplane (Π_1), but also in the hyperplane

$$(\Pi_2) : \langle \overline{e}_5, \overline{y} \rangle = \frac{1}{2}.$$

The map φ_1 is proper biharmonic and, indeed, the distance from the origin to the hyperplane (Π_1) is equal to $1/\sqrt{2}$. But, the last component of φ_1 is constant $1/2$, the first 4 components are not constant and the corresponding map $\psi : \mathbb{S}^3 \to \mathbb{S}^3\left(\sqrt{3}/2\right)$ is not harmonic; the distance from the origin to (Π_2) is $1/2$.

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