"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 33 (2023), No. 1, 75 - 86

A MEAN ERGODIC THEOREM IN A SUBALGEBRA OF GENERALIZED WEIGHTED GRAND LEBESGUE SPACES

İLKER ERYILMAZ

Abstract. The paper introduces a new type of grand Lebesgue space, for which special cases are weighted grand Lebesgue spaces introduced by Fiorenza, Gupta and Jain (2008) and a generalization of grand Lebesgue spaces introduced by Greco, Iwaniec and Sbordone (1997). The main result is a mean ergodic theorem, in the Von Neumann sense, for some operator acting on the closure of the set of compactly supported in the newly introduced grand Lebesgue space.

1. Introduction

Iwaniec and Sbordone in [12], extrapolated the concept of Lebesgue spaces and presented a new space of measurable, almost everywhere equal integrable function classes, which they called grand Lebesgue spaces. Let Y be locally compact Hausdorff space and (Y, Σ, μ) be a finite measure space. According to [12], grand Lebesgue spaces are the space of equivalence classes that are obtained according to equality almost everywhere of all measurable functions defined on (Y, Σ, μ) and denoted by $L^{r)}$ for $1 < r < \infty$. A measurable function u on Y belongs to $L^{r)}$ if the functional

Keywords and phrases: Mean ergodic theorem, Iwaniec-Sbordone space, weighted Lebesgue spaces, averages of iterates.
(2020) Mathematics Subject Classification: 47A35, 28D05,46E30, 47B33.

$$\|u\|_{r)}=\sup_{0<\varepsilon<r-1}\varepsilon^{\frac{1}{r-\varepsilon}}\left(\int_Y |u(x)|^{r-\varepsilon}d\mu(x)\right)^{\frac{1}{r-\varepsilon}}$$

is finite. The above functional is a Banach function norm on $L^{r)}$ and the Banach function space $L^{r)}$ is rearrangement invariant. Also $L^r\subset L^{r)}\subset L^{r-\varepsilon}$ if $0<\varepsilon\le r-1$. New results on grand Lebesgue spaces can be observed in current studies [3, 6, 8, 11, 17, 18] and [19]. Presented in terms of the Jacobian integrability problem, these works have proven useful in assorted applications of P.D.E's and related applied mathematics' problems, where they are utilized in the study of extrapolation theory, maximum functions, etc. The harmonic analysis of grand Lebesgue spaces, and the related associate spaces, small Lebesgue spaces, has been advanced intensively in recent years and goes on to invite the researchers to study on itself because of their various applications.

Let $1<r<\infty$ and w be a weight function. This means that w is real valued, measurable, integrable and locally bounded function on Y which also has absolute minimum value 1 on Y. Weighted grand Lebesgue spaces denoted by $L_w^{r)}$ are the space of equivalence classes of measurable functions defined on (Y,Σ,μ) such that

$$\|u\|_{r),w}=\sup_{0<\varepsilon<r-1}\left(\varepsilon\int_Y |u(x)|^{r-\varepsilon}w(x)\,d\mu(x)\right)^{\frac{1}{r-\varepsilon}}$$

is finite for any $u\in L_w^{r)}$. These spaces were defined in [7] and the boundedness property of the Hardy-Littlewood maximal operator on $L_w^{r)}$ are examined in the same paper. "In [15], it is proved that the Hilbert transform in a certain framework is bounded in the weighted grand Lebesgue space if and only if the weight belongs to the corresponding Muckenhoupt class. For more on weighted grand Lebesgue spaces and its fundamental properties, we can refer to [13, 14, 15].

In the generalization of grand Lebesgue spaces given by Greco, Iwaniec and Sbordone in [10], the definition of that notion says: For $1<p<\infty$ and $0\le\theta<\infty$ the generalized grand Lebesgue space

$L^{r),\theta}(Y)$ consists of the functions $u \in \bigcap\limits_{0<\varepsilon<r-1} L^{r-\varepsilon}$ such that

$$\|u\|_{r),\theta} = \sup_{0<\varepsilon<r-1} \varepsilon^{\frac{\theta}{r-\varepsilon}} \left(\int_Y |u(x)|^{r-\varepsilon} d\mu \right)^{\frac{1}{r-\varepsilon}} = \sup_{0<\varepsilon<r-1} \varepsilon^{\frac{\theta}{r-\varepsilon}} \|u\|_{r-\varepsilon}$$

is finite. According to [1, Proposition 2.1] and [10], $L^{r),\theta}(Y)$ is a rearrangement-invariant Banach function space with this norm. $L^{r),\theta}(Y)$ reduces to classical Lebesgue spaces $L^r(Y)$ when $\theta = 0$ and brings down to grand Lebesgue spaces $L^{r)}(Y)$ when $\theta = 1$ [3, 10]. Also we have $L^r(Y) \subset L^{r),\theta}(Y) \subset L^{r-\varepsilon}(Y)$ for $0 < \varepsilon \le r-1$ and $L^r(Y) \subset L^{r),\theta_1}(Y) \subset L^{r),\theta_2}(Y)$ for $0 \le \theta_1 < \theta_2$ [1, 10]. It is important to recall that the subspace of test functions $C_0^\infty(Y)$ is not dense in $L^{r),\theta}(Y)$. If we denote the closure of $C_0^\infty(Y)$ in $L^{r),\theta}(Y)$ by $E^{r),\theta}(Y)$, then

$$E^{r),\theta}(Y) = \left\{ u \in L^{r),\theta}(Y) : \lim_{\varepsilon \to 0^+} \varepsilon^{\frac{\theta}{r-\varepsilon}} \|u\|_{r-\varepsilon} = 0 \right\}$$

and $L^{r),\theta_1}(Y) \subset E^{r),\theta_2}(Y)$ for $0 \le \theta_1 < \theta_2$ [10].

2. Mean Ergodic Theorem

For the rest of the paper, let $X = (X, \Sigma, \mu)$ be a finite measure space, $\mathfrak{M}$ denote the collection of all extended scalar-valued (real or complex) $\mu-$measurable functions on X, $\mathfrak{M}_0$ stand for the class of functions in $\mathfrak{M}$ that are finite $\mu-$a.e and χ_A be the characteristic function of a set A. For any two non-negative functions or functionals, say M and N, this symbol $M \prec N$ means that $M \le cN$, for some positive constant c independent from the variables in M and N. If both $M \prec N$ and $N \prec M$ are satisfied, then we will write $M \approx N$ and say that M and N are equivalent.

Definition 1. *Let X be locally compact Hausdorff space, (X, Σ, μ) be a finite measure space, w be a weight function and $1 < p \le \infty$. The weighted grand Lebesgue space denoted by $L_w^{p),\theta}(X)$ is the space of measurable functions defined on (X, Σ, μ) such that*

$$\|u\|_{p),\theta}^{w} = \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X |u(x)|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}}$$

is finite for any $u \in L_w^{p),\theta}(X)$.

It is easy to see from the previously mentioned that $L_w^{p),\theta}(X) \subset \bigcap\limits_{1\leq q<p} L_w^{q),\theta}(X)$ and $L_w^{p),\theta}(X)$ is a Banach function space with this norm for $p>1$ and $\theta\geq 0$ and not a rearrangement-invariant space. Also $L_w^{p),\theta}(X)$ reduces to classical weighted Lebesgue spaces $L_w^p(X)$ when $\theta=0$ and to weighted grand Lebesgue spaces $L_w^{p)}(X)$ when $\theta=1$ [3, 10]. Besides $L_w^p(X)\subset L_w^{p),\theta}(X)\subset L_w^{p-\varepsilon}(X)$ for $0<\varepsilon\leq p-1$ and $L_w^p(X)\subset L_w^{p),\theta_1}(X)\subset L_w^{p),\theta_2}(X)$ for $0\leq\theta_1<\theta_2$ [1, 10].

The weighted Marcinkiewicz class, denoted by *weak*$-L_w^p(X)$ or $L_w^{p,\infty}(X)$, consists of all $\mu-$measurable functions $u: X\to\mathbb{C}$ such that

$$\sup_{\lambda>0}\lambda^p D_{u,w}(\lambda)<\infty$$

where

$$D_{u,w}(\lambda)=w\left(\{x\in X:\ |u(x)|>\lambda\}\right)=\int\limits_{\{x\in X:\ |u(x)|>\lambda\}} w(x)\,d\mu(x)\,,\ \lambda\geq 0$$

is the distribution function of u. Then $L_w^{p,\infty}(X)\subset L_w^{p),\theta}(X)$. Since $w(x)\geq 1$ for all $x\in X$, it is easy to see that $L_w^{p),\theta}(X)\subset L^{p),\theta}(X)$ and $\|u\|_{p),\theta}\prec\|u\|_{p),\theta}^w$ for any $u\in L_w^{p),\theta}(X)$. Moreover the relation $w_1\prec w_2$ implies that $L_{w_2}^{p),\theta}(X)\subset L_{w_1}^{p),\theta}(X)$. Since the test functions $C_0^\infty(X)$ is not dense in $L_w^{p),\theta}(X)$, we can define the closure of $C_0^\infty(X)$ as a subclass of $L_w^{p),\theta}(X)$ consisting of all functions such that $\lim\limits_{\varepsilon\to 0^+}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|u\|_{p-\varepsilon}^w=0$. If we denote this subclass by $E_w^{p),\theta}(X)$, i.e.

$$E_w^{p),\theta}(X)=\overline{[C_0^\infty(X)]}_{p),\theta}^w=\left\{u\in L_w^{p),\theta}(X):\ \lim_{\varepsilon\to 0^+}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|u\|_{p-\varepsilon}^w=0\right\},$$

then it is easy to prove that $E_w^{p),\theta}(X)$ is a closed subspace of $L_w^{p),\theta}(X)$ and so a Banach space when $p>1$. By using [9, Lemma 3.4], we can conclude that the set of simple functions $\mathbb{S}$ and the set of smooth functions with compact support $C_0^\infty(X)$ are dense subsets of $E_w^{p),\theta}(X)$.

The research on Ergodic theory began in the 1930s, initiated by Birkhoff in [2] and Neumann in [20], and originated from applied physics and statistical mechanics. The fundamental problem in Ergodic theory is to study and find the necessary conditions for when the sequences of Cesàro averages $\sum\limits_{j=1}^{n}T^n(\cdot)$ are convergent where T was a mapping defined on a suitable space. The theorem of mean ergodicity was extended for bounded linear operators on Banach spaces by Yosida and implemented to Markoff processes by Yosida and Kakutani in [21] and [22], respectively. Moreover, Dudley studied and published a paper on Lorentz-invariant Markoff processes in relativistic phase space in [4]. Thenceforward, ergodic theory and its applications have certainly evolved in various mathematical and statistical problems and has been studied by many researchers. For a systematic preparation and development of ergodic theorems, we can refer to the classic book [16], which contains rich literature in this area. In [5, Section VIII.5], the averages of iterates of a linear operator T is examined and discussed and then tried to throw some light upon the problems which are occured in probability and statistical mechanics. The conditions of an operator T in an arbitrary complex Banach space Y were given which are necessary and sufficient for the convergence in Y of the averages

$$A(n) = \frac{1}{n} \sum_{j=0}^{n-1} T^j$$

of the iterates of T. These general conditions have been interpreted for operators in Lebesgue spaces which occur in statistical mechanics and probability.

The main aim of this paper is to prove the mean ergodic theorem which can be written for averages of iterates of T on $E_w^{p),\theta}(X)$ where $1 < p < \infty$, w is a weight function and $\theta \geq 0$.

Lemma 2. *Let (X, Σ, μ) be a finite positive measure space, $\aleph \neq 0$ be a complex Banach space and φ be a map of X into itself which satisfies the following conditions:*

$$\begin{aligned} (i)\ \varphi^{-1}(E) &\in \Sigma \text{ for all } E \in \Sigma \\ (ii)\ \text{If } \mu(E) &= 0 \text{ then } \mu\left(\varphi^{-1}(E)\right) = 0. \end{aligned} \tag{1}$$

Then for every function u from X to $\aleph$ the following operator T defined as

$$T(u)(\cdot) = u(\varphi(\cdot)) \tag{2}$$

maps measurable functions into measurable functions and μ-equivalent functions into μ-equivalent functions. Furthermore T is a continuous linear map of the space of all $\aleph$-valued $\mu-$measurable functions into itself.

Proof. See [5, VIII.5.6, Lemma 6] ∎

Lemma 3. *Let (X, Σ, μ) be a finite positive measure space, $Y \neq 0$ be a complex Banach space. Assume that φ is a map of X into itself which satisfies (1). Then for any $p > 1$, the linear operator T defined in the complex linear space Y^X of all functions on X into Y by*

$$Tu\,(x) = u\,(\varphi\,(x))\,, \quad x \in X, \quad u \in Y^X \tag{3}$$

maps $E_w^{p),\theta}(X)$ into itself if and only if

$$\sup_{E \in \Sigma} \frac{w\,(\varphi^{-1}\,(E))}{w\,(E)} =: M < \infty. \tag{4}$$

Furthermore, when this condition is satisfied T is a continuous linear map on $E_w^{p),\theta}(X)$ and $\|T\| = M$.

Proof. If $w\,(E) = 0$, then $\frac{w\left(\varphi^{-1}(E)\right)}{w(E)}$ will be taken zero. Now assume that $w\,(E) > 0$. If u is a measurable function and φ is defined as in (1), then is easy to see that Tu is measurable by (3). Firstly suppose that T maps $E_w^{p),\theta}(X)$ into itself. It will be shown that T is closed and hence continuous. Since T is defined on $E_w^{p),\theta}(X)$, then it maps μ-equivalent functions into μ-equivalent functions and also measurable functions into measurable functions. Now let $\alpha \neq 0$ be a fixed vector in Y and let E be a μ-null set in Σ. Then $\mu\,(E) = 0$, $\chi_E = 0$ (a.e.) and

$$T\,(\alpha \chi_E) = \alpha \chi_{\varphi^{-1}(E)}$$

by the definition of T. Also, linearity of T implies that $\mu\,(\varphi^{-1}\,(E)) = 0$. This means that φ is a measure-preserving map of X into itself and satisfies (1). Since $\overline{\mathbb{S}} = E_w^{p),\theta}(X)$, for any $u \in E_w^{p),\theta}(X)$ a sequence of simple functions $(u_n) \subset \mathbb{S}$ can be formed such that $\|u_n - u\|_{p),\theta}^{\,w} \to 0$. This convergence implies convergence in measure and so the graph of T is closed. Therefore T is bounded and continuous on $E_w^{p),\theta}(X)$ by Closed graph theorem. On the other hand, for any $0 \neq \alpha \in Y$ and E

$\in \Sigma$, we have

$$\begin{aligned}
\|T(\alpha\chi_E)\|_{p),\theta}^{w} &= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X |T(\alpha\chi_E)(x)|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X \left|\alpha\chi_{\varphi^{-1}(E)}(x)\right|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= |\alpha| \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_{\varphi^{-1}(E)} w(x) d\mu \right)^{\frac{1}{p-\varepsilon}} \qquad (5) \\
&= |\alpha| \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} w\left(\varphi^{-1}(E)\right)^{\frac{1}{p-\varepsilon}} \\
&= |\alpha| (p-1)^{\theta} w\left(\varphi^{-1}(E)\right).
\end{aligned}$$

Therefore, one can get that

$$\begin{aligned}
|\alpha| (p-1)^{\theta} w\left(\varphi^{-1}(E)\right) &= \|T(\alpha\chi_E)\|_{p),\theta}^{w} \leq |a| \|T\| \|\chi_E\|_{p),\theta}^{w} \\
&= |a| \|T\| (p-1)^{\theta} w(E)
\end{aligned}$$

which means $M \leq \|T\|$.

Conversely, let s be a μ-integrable simple function having values $\beta_1, \beta_2, \ldots, \beta_k$ on the disjoint sets $E_1, E_2, \ldots, E_k$ of positive measure. Then Ts has the values $\beta_1, \beta_2, \ldots, \beta_k$ on the sets $\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_k)$. Since the family $\{E_1, E_2, \ldots, E_k\}$ is a decomposition of X, property (1) of φ implies that the family $\{\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_k)\}$ is also a decomposition of X. Therefore, if we use the notation $\sum_{i=1}^{k} \beta_i \chi_{E_i}$ for s, then

$$Ts(x) = s(\varphi(x)) = \sum_{i=1}^{k} \beta_i \chi_{E_i}(\varphi(x)) = \sum_{i=1}^{k} \beta_i \chi_{\varphi^{-1}(E_i)}(x)$$

and

$$\begin{aligned}
\|Ts\|_{p),\theta}^{w} &= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X |Ts(x)|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X |s(\varphi(x))|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X \left| \sum_{i=1}^{k} \beta_i \chi_{\varphi^{-1}(E_i)} \right|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&\leq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_X \sum_{i=1}^{k} \left| \beta_i \chi_{\varphi^{-1}(E_i)} \right|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}}
\end{aligned}$$

can be written. Since the elements of the family $\{\varphi^{-1}(E_1), \varphi^{-1}(E_2), \ldots, \varphi^{-1}(E_k)\}$ are disjoint,

$$\begin{aligned}
\|Ts\|_{p),\theta}^{w} &\leq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\int_{\cup_{i=1}^{k} \varphi^{-1}(E_i)} \sum_{i=1}^{k} \left| \beta_i \chi_{\varphi^{-1}(E_i)} \right|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \left(\sum_{i=1}^{k} \int_{\varphi^{-1}(E_i)} |\beta_i|^{p-\varepsilon} w(x)\, d\mu \right)^{\frac{1}{p-\varepsilon}} \\
&= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} |\beta_i| \left(\sum_{i=1}^{k} w\left(\varphi^{-1}(E_i)\right) \right)^{\frac{1}{p-\varepsilon}} \\
&\leq \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} M^{\frac{1}{p-\varepsilon}} \left(\sum_{i=1}^{k} |\beta_i|^{p-\varepsilon} w(E_i) \right)^{\frac{1}{p-\varepsilon}} = M \|s\|_{p),\theta}^{w}
\end{aligned}$$

is found.

Since the μ-integrable simple functions $\mathbb{S}$ are dense in $E_w^{p),\theta}(X)$ and T is a continuous operator acting on a dense subset of $E_w^{p),\theta}(X)$, we can say that T possesses a unique bounded, continuous extension $\widetilde{T}$ defined on all of $E_w^{p),\theta}(X)$ with norm $\left\|\widetilde{T}\right\| \leq M$. Furthermore, by the definition of M,

$$
\begin{aligned}
M &= \sup_{E\in\Sigma}\frac{w\left(\varphi^{-1}\left(E\right)\right)}{w\left(E\right)} = \sup_{E\in\Sigma}\frac{(p-1)^{\theta}w\left(\varphi^{-1}\left(E\right)\right)}{w\left(E\right)(p-1)^{\theta}} \\
&= \sup_{E\in\Sigma}\frac{\left\|\chi_{\varphi^{-1}(E)}\right\|_{p),\theta}^{w}}{\left\|\chi_{E}\right\|_{p),\theta}^{w}} = \sup_{E\in\Sigma}\frac{\left\|T\chi_{E}\right\|_{p),\theta}^{w}}{\left\|\chi_{E}\right\|_{p),\theta}^{w}} \leq \left\|T\right\|
\end{aligned}
$$

can be found. Therefore $\|T\| = M$. ∎

Proposition 4. *Assume that (X,Σ,μ) be a finite positive measure space and let φ be a mapping of X into itself with $\varphi^{-1}(\Sigma)\subset\Sigma$. Also, suppose that there is a constant M for which*

$$
\frac{1}{n}\sum_{j=0}^{n-1} w\left(\varphi^{-j}\left(E\right)\right) \leq M w\left(E\right) \tag{6}
$$

for all $n\in\mathbb{N}$ and $E\in\Sigma$. Then, for every $p\in(1,\infty)$, the operator T defined in (3) maps $E_{w}^{p),\theta}(X)$ into itself. Also the averages $A(n)$ as operators acting on $E_{w}^{p),\theta}(X)$ are uniformly bounded.

Proof. If we write $n = 2$ in (6), then we get $w\left(\varphi^{-1}\left(E\right)\right) \leq (2M-1)\, w\left(E\right)$ for any $E\in\Sigma$. Therefore T is bounded by (4). This inequality also shows that $w\left(\varphi^{-1}\left(E\right)\right) = 0$ whenever $w\left(E\right) = 0$. Now let $E\in\Sigma$ be any set, $n\in\mathbb{N}$ and $j = 1,2,\ldots,n-1$. Since

$$
\begin{aligned}
T^{j}\left(\chi_{E}\right)(\cdot) &= T^{j-1}\left(T\left(\chi_{E}\right)\right)(\cdot) = T^{j-1}\left(\chi_{E}\left(\varphi\right)\right)(\cdot) = T^{j-1}\left(\chi_{\varphi^{-1}(E)}\right)(\cdot) \\
&= T^{j-2}\left(T\left(\chi_{\varphi^{-1}(E)}\right)\right)(\cdot) = T^{j-2}\left(\chi_{\varphi^{-2}(E)}\right)(\cdot) = \cdots \qquad (7)
\end{aligned}
$$

we can conclude that $T^{j}\left(\chi_{E}\right)(\cdot) = \left(\chi_{\varphi^{-j}(E)}\right)(\cdot)$. For any simple function $s(\cdot) = \sum_{i=1}^{k}\beta_{i}\chi_{E_{i}}(\cdot)$, we have

$$
\begin{aligned}
A(n)(s)(\cdot) &= \frac{1}{n}\sum_{j=0}^{n-1}T^{j}(s)(\cdot) = \frac{1}{n}\sum_{j=0}^{n-1}T^{j}\left(\sum_{i=1}^{k}\beta_{i}\chi_{E_{i}}\right)(\cdot) \\
&= \sum_{i=1}^{k}\beta_{i}\left(\frac{1}{n}\sum_{j=0}^{n-1}T^{j}\chi_{E_{i}}\right)(\cdot) = \sum_{i=1}^{k}\beta_{i}\left(\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_{i})}\right)(\cdot)
\end{aligned}
$$

by (7). Thus

$$\begin{aligned}
\|A(n)(s)\|_{p),\theta}^{w} &= \left\|\sum_{i=1}^{k}\beta_i\left(\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right)\right\|_{p),\theta}^{w} \le \sum_{i=1}^{k}|\beta_i|\left\|\frac{1}{n}\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right\|_{p),\theta}^{w} \\
&= \sum_{i=1}^{k}|\beta_i|\left(\frac{1}{n}\left\|\sum_{j=0}^{n-1}\chi_{\varphi^{-j}(E_i)}\right\|_{p),\theta}^{w}\right) \\
&\le \sum_{i=1}^{k}|\beta_i|\left(\frac{1}{n}\sum_{j=0}^{n-1}\left\|\chi_{\varphi^{-j}(E_i)}\right\|_{p),\theta}^{w}\right) \\
&= \sum_{i=1}^{k}|\beta_i|\left(\frac{1}{n}\sum_{j=0}^{n-1}w\left(\varphi^{-j}(E_i)\right)\right) \\
&\le \sum_{i=1}^{k}|\beta_i|\,Mw(E_i) = M\|s\|_{p),\theta}^{w}
\end{aligned}$$

for all $s \in \mathbb{S}$. By using the density of $\mathbb{S}$ in $E_w^{p),\theta}(X)$ and Tietze extension theorem, $\|A(n)\| \le M$ for any $n \in \mathbb{N}$. The averages of iterates, namely $A(n)$, are uniformly bounded as operators acting on $E_w^{p),\theta}(X)$. ∎

Theorem 5. *(Mean Ergodic Theorem) Assume that (X,Σ,μ) is a finite positive measure space and φ is a mapping of X into itself which satisfies $\varphi^{-1}(\Sigma) \subset \Sigma$. If the inequality*

$$\frac{1}{n}\sum_{j=0}^{n-1}w\left(\varphi^{-j}(E)\right) \le Mw(E) \tag{8}$$

is satisfied for all $n \in \mathbb{N}$ and $E \in \Sigma$, then for every $p \in (1,\infty)$, the operator T defined by the equation (2) is a continuous linear map on $E_w^{p),\theta}(X)$ and the sequence of averages $A(n)$, as operators acting on $E_w^{p),\theta}(X)$, is strongly convergent. Here M is independent of E, n and $\varphi^0(E) = E$.

Proof. With (8), it can be written that $\frac{w\left(\varphi^{-1}(E)\right)}{w(E)} \le (2M-1)$ for any $E \in \Sigma$. Therefore the linear operator T defined by the equation (2) is a bounded and continuous map on $E_w^{p),\theta}(X)$ by Lemma 3. If we denote the space of all linear and continuous operators on $E_w^{p),\theta}(X)$ by $\mathfrak{B}\left(E_w^{p),\theta}(X)\right)$, then it can be easily seen that $A(n)$, the averages are in this complete space. Since the averages $A(n)$ are uniformly bounded while operating on $E_w^{p),\theta}(X)$, we can write that the sequence $\{A(n)f\} \subset E_w^{p),\theta}(X)$ converges for all $f \in E_w^{p),\theta}(X)$ by Riesz–Thorin convexity theorem. By the way, when the averages $A(n)$ are operating on $E_w^{p),\theta}(X)$, we obtained that $A(n)f \in E_w^{p),\theta}(X)$ for all $n \in \mathbb{N}$ and for each $f \in E_w^{p),\theta}(X)$. It is known that the characteristic functions of elements of Σ form a fundamental set for $E_w^{p),\theta}(X)$. Then, for any $E \in \Sigma$ and $x \in X$, since we have $|\chi_E| \leq 1$ and

$$\left|\frac{T^n(\chi_E)(x)}{n}\right| = \left|\frac{\chi_E(\varphi^n)(x)}{n}\right| \leq \frac{1}{n} \to 0,$$

we can say that the sequence $\{A(n)\}$ converges in strongly operator topology by [5, VIII.5.1]. ∎

Remark 6. *It should be noted that the weighted measure preserving transformation (i.e. one for which* $w(\varphi^{-1}(E)) = w(E)$ *for every* $E \in \Sigma$*) satisfies the assumptions of Theorem 5. It has already been known that precisely this type of transformation occurs when studying conservative mechanical systems.*

References

[1] G. Anatriello, **Iterated grand and small Lebesgue spaces**, Collect. Math., 65 (2014), 273–284.

[2] G.D. Birkhoff, **Proof of the ergodic theorem**, Proceedings of the National Academy of Sciences, 17 (1931), no.12, 656-660.

[3] R.E. Castillo, H. Rafeiro, **An introductory course in Lebesgue spaces**, CMS Books in Mathematics, Springer, 2016.

[4] R.M. Dudley, **Lorentz-invariant Markov processes in relativistic phase space**, Arkiv för Matematik, 6 (1966), no. 3, 241-268.

[5] N. Dunford, J.T.Schwartz, **Linear operators, part 1: General Theory**, John Wiley & Sons, no. 10, 1988.

[6] A. Fiorenza, **Duality and reflexivity in grand Lebesgue spaces**, Collect. Math., 51 (2000), no. 2, 131–148.

[7] A. Fiorenza, B. Gupta, P. Jain, **The maximal theorem for weighted grand Lebesgue spaces**, Studia Math., 188 (2008), no. 2, 123–133.

[8] A. Fiorenza, G.E. Karadzhov, **Grand and small Lebesgue spaces and their analogs**, Z. Anal. Anwendungen, 23 (2008), no. 4, 657–681.

[9] L. Greco, **A remark on the equality** $detDf = DetDf$, Differ. Integral Equ., 6 (1193), no. 5, 1089–1100.

[10] L. Greco, T. Iwaniec, C. Sbordone, **Inverting the p-harmonic operator**, Manuscripta Mathematica, 92, (1997), no. 1, 249-258.

[11] A.T. Gürkanlı, **Multipliers of grand and small Lebesgue spaces**, https://arxiv.org/abs/1903.06743v1.

[12] T. Iwaniec, C. Sbordone, **On the integrability of the Jacobian under minimal hypotheses**, Arch. Rational Mech. Anal., 119 (1992), no. 2, 129–143.

[13] V. Kokilashvili, **Boundedness criterion for the Cauchy singular integral operator in weighted grand Lebesgue spaces and application to the Riemann problem**, Proc. A. Razmadze Math. Inst., 151 (2009), 129–133.

[14] V. Kokilashvili, **Boundedness criteria for singular integrals in weighted grand Lebesgue spaces**, Problems in Mathematical Analysis 49, J. Math. Sci. (N.Y.), 170 (2010), no. 1, 20–33.

[15] V. Kokilashvili, A. Meskhi, **A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces**, Georgian Math. J., 16 (2009), no. 3, 547–551.

[16] U. Krengel, **Ergodic theorems**, Walter de Gruyter, no. 6, 2011.

[17] A. Meskhi, **Weighted criteria for the Hardy transform under the Bp condition in grand Lebesgue spaces and some applications**, J. Math. Sci., Springer, 178 (2011), no. 6, 622–636.

[18] A. Meskhi, **Criteria for the boundedness of potential operators in grand Lebesgue spaces**, Proc. A. Razmadze Math. Inst., 169 (2015), 119–132.

[19] S.G. Samko, S.M. Umarkhadzhiev, **On Iwaniec-Sbordone spaces on sets which may have infinite measure**, Azerb. J. Math., 1 (2011), no. 1, 67–84.

[20] J. Von Neumann, **Proof of the quasi-ergodic hypothesis**, Proceedings of the National Academy of Sciences, 18 (1932), no. 1, 70-82.

[21] K. Yosida, **Mean ergodic theorem in Banach spaces**, Proceedings of the Imperial Academy, 14 (1938), no.8, 292-294.

[22] K. Yosida, S. Kakutani,**Operator-Theoretical Treatment of Markoff's Process and Mean Ergodic Theorem**, Annals of Mathematics, 42 (1941), no.1, 188–228.

Ondokuz Mayıs University,
Faculty of Sciences, Department of Mathematics,
Atakum-Samsun, TURKEY
e-mail:rylmz@omu.edu.tr