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SIR DYNAMICAL MODEL WITH DEMOGRAPHY AND LAGRANGE-HAMILTON GEOMETRIES

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Abstract. The aim of this paper is to develop, via the least squares variational method, the Lagrange-Hamilton geometries (in the sense of nonlinear connections, d-torsions and Lagrangian Yang-Mills electromagnetic-like energy) produced by the SIR dynamical system with demography in epidemiology. From a geometrical point of view, the Jacobi instability of this SIR dynamical system with demography is established. At the same time, some possible epidemiological and demographic interpretations are also derived.

1. SIR dynamical system with demography

There is a vast recent literature focused on epidemic mathematical models, part of it dealing with the spread of a disease from a temporal and spatial perspective. The resulting advantages consist in the possibility of tracking and forecasting the trend of infectious diseases, on the basis of which public health policies can be built, contributing to the prevention or reduction of future propagation.

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The mathematical epidemic models have been constructed either as *continuous-time* models by differential equations (considered as more tractable mathematically than the discrete models while having a direct physical interpretation, thus being widely used), or as *discrete-time* ones by difference equations (proved to be particularly helpful in building the overall picture related to the present situation and the future evolution of a disease). See, for instance, De la Sen and Ibeas [5] or Parsamanesh et al. [15].

In mathematical modeling of epidemiology, the demographic component is neglected in the case of fast diseases, whose evolution is taking place for a significantly shorter period of time than the lifespan of the individuals (such as in the case of childhood diseases or influenza) (see De la Sen et al. [6]), instead for slowly progressive diseases (such as HIV, hepatitis C, tuberculosis), given that the disease extends over a long period, sometimes for the entire lifespan of the individuals, the effects of demography have to be considered (see Litră [8]). The demographic events of birth and death influence the number and the links of nodes in networks, interfering with the mechanisms of infection and recovery and creating a dynamic network with effect in epidemic spreading (see Jing et al. [7]). Thus, the use of the SIR model with demography allows a fundamental perspective on the dynamics of infectious diseases and how they can be controlled, insights that are almost unachievable only from tracking the disease itself. Therefore the SIR model with demography constitutes the theoretical framework for designing interventions in the field of public health (see Weiss [17]).

Note that the SIR dynamical system with demography is used in epidemiology as a three compartmental mathematical model which is useful to predict how a disease spreads or to estimate the duration of an epidemic. For more details about compartmental mathematical models in epidemiology, consult the works: Adda and Bichara [1], Ozioko et al. [14] or Wikipedia [18], and references therein.

The SIR (**S**usceptible, **I**nfectious and **R**ecovered) dynamical system with demography is expressed by [18]

$$(1)\qquad \begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - \dfrac{\beta IS}{N} \\ \dfrac{dI}{dt} = \dfrac{\beta IS}{N} - \gamma I - \mu I \\ \dfrac{dR}{dt} = \gamma I - \mu R, \end{cases}$$

where $N = S(t) + I(t) + R(t)$ represents a population divided into classes of susceptible, infectious and removed individuals, whose numbers at the moment t are denoted by $S(t)$, $I(t)$ and $R(t)$, and μ, Λ, β and γ are some parameters representing the *per capita death rate*, the *total birth rate*, the *effective per capita contact rate of infective individuals* and the *per capita rate of recovery* respectively.

2. From a given dynamical system to Lagrange-Hamilton geometries

Let M be a n-dimensional smooth manifold, whose coordinates are $(x^i)_{i=\overline{1,n}}$. Let TM (respectively T^*M) be the tangent (respectively cotangent) bundle, whose coordinates are $(x^i, y^i)_{i=\overline{1,n}}$ (respectively $(x^i, p_i)_{i=\overline{1,n}}$).

Let us consider a vector field $X = (X^i(x))_{i=\overline{1,n}}$ on M, which produces the dynamical system

$$\frac{dx^i}{dt} = X^i(x(t)), \quad i = \overline{1,n}. \tag{2}$$

Because the solutions of class C^2 of the dynamical system (2) are the global minimum points for the *least squares Lagrangian* $L : TM \to \mathbb{R}$, given by[1]

(3)

$$L(x,y) = \delta_{ij}\left(y^i - X^i(x)\right)\left(y^j - X^j(x)\right) \Leftrightarrow$$
$$L(x,y) = \left(y^1 - X^1(x)\right)^2 + \left(y^2 - X^2(x)\right)^2 + ... + \left(y^n - X^n(x)\right)^2,$$

it follows that, via its Euler-Lagrange equations ($k = \overline{1,n}$)

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^k}\right) = 0, \quad y^k = \frac{dx^k}{dt},$$

$$\Leftrightarrow \frac{d^2x^k}{dt^2} + 2G^k(x,y) = 0, \quad G^k = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k}\right), \tag{4}$$

we can construct an entire natural collection of nonzero Lagrangian geometrical objects (such as nonlinear connection, d-torsions and Yang-Mills electromagnetic-like energy) that characterize the initial dynamical system (2). For more details about Lagrange geometry on tangent bundles and the Lagrangian least squares variational method for dynamical systems, consult the works: Miron and Anastasiei [9], Udrişte and Neagu [16], [13], and Balan and Neagu [2].

[1] The Einstein convention of summation is adopted all over this paper.

Also, if we construct the *least squares Hamiltonian* $H : T^*M \to \mathbb{R}$, associated with the Lagrangian (3), which is defined by
(5)
$$H(x,p) = \frac{\delta^{ij}}{4} p_i p_j + X^k(x) p_k \Leftrightarrow$$
$$H(x,p) = \frac{1}{4}\left(p_1^2 + p_2^2 + ...p_n^2\right) + X^1(x)p_1 + X^2(x)p_2 + ... + X^n(x)p_n,$$

where $p_r = \partial L/\partial y^r$ and $H = p_r y^r - L$, we can build again a natural and distinct collection of nonzero Hamiltonian geometrical objects (such as nonlinear connection and d-torsions), which also characterize the system (2). For all details about Hamilton geometry on cotangent bundles and the Hamiltonian least squares variational method for dynamical systems, see the monographs: Miron et al. [10] and Neagu and Oană [12].

It is important to note that the above Lagrange-Hamilton geometries produced by the Lagrangian (3) and Hamiltonian (5) are exposed in detail in the monographs [2] and [12]. These are achieved via the nonzero geometrical objects, where $J(X) = \left(\partial X^i/\partial x^j\right)_{i,j=\overline{1,n}}$ is the Jacobian matrix of X:

(1) $\mathcal{N} = \left(N_j^i\right)_{i,j=\overline{1,n}} = -\frac{1}{2}\left[J(X) - {}^T J(X)\right]$ is the *Lagrangian nonlinear connection* on the tangent bundle TM, where $N_j^i = \partial G^i/\partial y^j$;

(2) $R_k = \left(R_{jk}^i\right)_{i,j=\overline{1,n}} = \dfrac{\partial \mathcal{N}}{\partial x^k}$, $\forall\ k = \overline{1,n}$, are the *Lagrangian d-torsions*, where
$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}, \quad \frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N_k^r \frac{\partial}{\partial y^r};$$

(3) $\mathcal{EYM}(x) = \frac{1}{2} \cdot Trace\left[F \cdot {}^T F\right]$, where $F = -\mathcal{N}$, is the *Lagrangian Yang-Mills electromagnetic-like energy*;

(4) $\mathbf{N} = \left(N_{ij}\right)_{i,j=\overline{1,n}} = J(X) + {}^T J(X)$ is the *Hamiltonian nonlinear connection* on the cotangent bundle T^*M, where
$$N_{ij} = \frac{\partial^2 H}{\partial x^j \partial p_i} + \frac{\partial^2 H}{\partial x^i \partial p_j};$$

(5) $\mathbf{R}_k = (R_{kij})_{i,j=\overline{1,n}} = \dfrac{\partial}{\partial x^k}\left[J(X) - {}^TJ(X)\right] = -2R_k,\ \forall\ k = \overline{1,n}$, are the *Hamiltonian d-torsions*, where

$$R_{kij} = \frac{\delta N_{ki}}{\delta x^j} - \frac{\delta N_{kj}}{\delta x^i}, \quad \frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N_{rj}\frac{\partial}{\partial p_r}.$$

3. Lagrange-Hamilton geometries for SIR dynamical system with demography

It is obvious that, in the context of the SIR dynamical system (1), we have the particular 3-dimensional manifold $M = \mathbb{R}^3$, whose coordinates are

$$\left(x^1 = S,\ x^2 = I,\ x^3 = R\right),$$

and we consider the vector field $X = (X^i(S,I,R))_{i=\overline{1,3}}$, which is given by

$$X^1(S,I,R) = \Lambda - \mu S - \frac{\beta IS}{N}, \quad X^2(S,I,R) = \frac{\beta IS}{N} - \gamma I - \mu I,$$

$$X^3(S,I,R) = \gamma I - \mu R.$$

Applying now the geometrical ideas from the preceding section, we deduce that the Jacobian matrix $J = J(X)$ of the vector field $X(S,I,R)$ is expressed by

$$J = \begin{pmatrix} -\mu - \dfrac{\beta IN - \beta IS}{N^2} & -\dfrac{\beta SN - \beta IS}{N^2} & \dfrac{\beta IS}{N^2} \\ \dfrac{\beta IN - \beta IS}{N^2} & \dfrac{\beta SN - \beta IS}{N^2} - \gamma - \mu & -\dfrac{\beta IS}{N^2} \\ 0 & \gamma & -\mu \end{pmatrix},$$

and, consequently, we find the Lagrange-Hamilton geometrical objects that characterize the SIR dynamical system (1):

(1) the Lagrangian nonlinear connection skew-symmetric matrix:

$$\mathcal{N} = -\frac{1}{2}\left[J - {}^TJ\right] =$$

$$= -\frac{1}{2}\begin{pmatrix} 0 & \dfrac{-\beta SN + 2\beta IS - \beta IN}{N^2} & \dfrac{\beta IS}{N^2} \\ \dfrac{\beta SN - 2\beta IS + \beta IN}{N^2} & 0 & -\dfrac{\beta IS}{N^2} - \gamma \\ -\dfrac{\beta IS}{N^2} & \dfrac{\beta IS}{N^2} + \gamma & 0 \end{pmatrix};$$

(2) the Lagrangian d-torsion skew-symmetric matrices:

$$R_1 = \frac{\partial \mathcal{N}}{\partial S} = \begin{pmatrix} 0 & -a & -b \\ a & 0 & b \\ b & -b & 0 \end{pmatrix},$$

where

$$a = \frac{1}{2}\left(\frac{\beta I - \beta N + \beta S}{N^2} + \frac{2\beta IN - 4\beta IS}{N^3}\right), \quad b = \frac{\beta IN - 2\beta IS}{2N^3};$$

$$R_2 = \frac{\partial \mathcal{N}}{\partial I} = \begin{pmatrix} 0 & -c & -d \\ c & 0 & d \\ d & -d & 0 \end{pmatrix},$$

where

$$c = \frac{1}{2}\left(\frac{-\beta N + \beta IN + \beta S}{N^2} + \frac{2\beta SN - 4\beta IS}{N^3}\right), \quad d = \frac{\beta SN - 2\beta IS}{2N^3};$$

$$R_3 = \frac{\partial \mathcal{N}}{\partial R} = \begin{pmatrix} 0 & -e & -f \\ e & 0 & f \\ f & -f & 0 \end{pmatrix},$$

where

$$e = \frac{1}{2}\left(\frac{\beta I + \beta S}{N^2} - \frac{4\beta IS}{N^3}\right), \quad f = -\frac{\beta IS}{N^3};$$

(3) the Lagrangian Yang-Mills electromagnetic-like energy:

$$\mathcal{EYM}(S, I, R) = \frac{1}{4N^4}\left[(\beta SN - 2\beta IS + \beta IN)^2 + \beta^2 I^2 S^2 + \right.$$

$$\left. + \left(\beta IS + \gamma N^2\right)^2\right];$$

(4) the Hamiltonian nonlinear connection symmetric matrix:

$$\mathbf{N} = J + {}^TJ =$$

$$= \begin{pmatrix} -2\mu - 2\dfrac{\beta IN - \beta IS}{N^2} & \dfrac{\beta IN - \beta SN}{N^2} & \dfrac{\beta IS}{N^2} \\ \dfrac{\beta IN - \beta SN}{N^2} & 2\dfrac{\beta SN - \beta IS}{N^2} - 2\gamma - 2\mu & -\dfrac{\beta IS}{N^2} + \gamma \\ \dfrac{\beta IS}{N^2} & -\dfrac{\beta IS}{N^2} + \gamma & -2\mu \end{pmatrix};$$

(5) the Hamiltonian d-torsion matrices are $\mathbf{R}_k = -2R_k$, $\forall\, k = \overline{1,3}$.

The matrix of deviation curvature from Kosambi-Cartan-Chern (KCC) geometrical theory is given by the formula (see Böhmer et al. [4])

$$P=\left(P_j^i\right)_{i,j=\overline{1,3}}=R_k y^k=\left(\begin{array}{ccc}0 & -A & -B\\ A & 0 & B\\ B & -B & 0\end{array}\right),$$

where $A=ay^1+cy^2+ey^3$, $B=by^1+dy^2+fy^3$. Its eigenvalues are

$$\lambda_1=0,\ \lambda_{2,3}=\pm i\sqrt{A^2+2B^2}\Rightarrow \operatorname{Re}\lambda_1=0,\ \operatorname{Re}\left(\lambda_{2,3}\right)=0.$$

In conclusion, the behavior of neighboring solutions of the Euler-Lagrange equations (4) is Jacobi unstable. This means that the trajectories of the vector field X of the SIR dynamical model (1) are neither bunching together and are neither dispersing. For more details about KCC theory and Jacobi stability, consult the paper [4].

4. Conclusions

Along with AI-based methods and also the statistical modeling methods, mathematical modeling was frequently used in epidemiology, contributing to the understanding of some aspects and leading to certain findings that some authors admit they could not have reached from the data alone, such as determining a herd immunity threshold which makes vaccination against a disease to be effective without making everyone immune (see Weiss [17]). Mathematical modeling based on SIR model and artificial intelligence have both shown to be reliable tools in the fight against pandemics (see Mohamadou et al. [11]). Moreover, the reduction of the transmission rate in the SIR model might find applications for timing the implementation of epidemic control measures, helping for an epidemic to fade out before it becomes endemic (i.e., established in a population). See, for example, Ballard et al. [3].

It is important to note that the interaction between the demography and epidemic dynamics is complex. Therefore it is important to accurately estimate the parameters of demography in order to determine the disease transmission dynamics. The mathematical epidemic models, like the SIR-model, allow a better understanding of the relationship between real-world dynamics and the varying incidence of an endemic disease, and can be used to assess the spread of epidemics and the impact of government intervention strategies. In such a context, from our new geometric-physical approach, the surfaces of constant level of the Lagrangian Yang-Mills electromagnetic-like energy produced by the SIR dynamical system (1) could have important connotations for the epidemiological phenomena taken in study. For such a reason, it is an open problem to find the epidemiological information contained in the shape of the surfaces of constant level ($C \geq 0$)

$$\Sigma_C : \frac{1}{4N^4}\left[(\beta SN - 2\beta IS + \beta IN)^2 + \beta^2 I^2 S^2 + \left(\beta IS + \gamma N^2\right)^2\right] = C.$$

In this direction, we believe that the computer drawn graphics of these surfaces are important for the study of the epidemiological phenomena involved in the SIR dynamical system (1). For example, note that if the effective per capita contact rate of infective individuals β is zero, then the Lagrangian Yang-Mills electromagnetic-like energy produced by the SIR dynamical system is constant and equal to $\mathcal{EYM} = \gamma^2/4$, where γ is the per capita rate of recovery. Is there an epidemiological meaning of this fact?

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