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UPPER AND LOWER ALMOST LPT mI-CONTINUOUS MULTIFUNCTIONS

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Abstract. The notions of upper/lower almost nearly continuous (resp. of upper/lower almost c-continuous, of upper/lower l-continuous) multifunctions have been introduced and investigated in [8] (resp. [16], [14]). In [25], the present authors obtained a unified form of generalizations of upper/lower almost nearly continuous multifunctions. In this paper, by using the m-structure $mIO(X)$ defined in an ideal topological space (X, τ, I), we define upper/lower almost LPT mI-continuous multifunctions and obtain their properties, where LPT denotes one of nearly compact, compact, Lindelöf, connected.

1. Introduction

The notion of N-closed sets in a topological space was introduced in [5]. Ekici [8] introduced the notions of upper/lower almost nearly continuous multifunctions as a generalization of upper/lower nearly continuous multifunctions [7] and upper/lower almost continuous multifunctions [26].

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Rychlewicz [32] has introduced the notion of upper/lower almost nearly quasi-continuous multifunctions as a generalization of upper/lower almost nearly continuous multifunctions and upper/lower almost quasi continuous multifunctions [31]. In [29], the present authors introduced and studied the notion of upper/lower m-continuous multifunctions. Furthermore, in [25], they introduced and studied the notion of upper/lower almost nearly m-continuous multifunctions. The notion generalize upper/lower m-continuous multifunctions and upper/lower almost nearly continuous multifunctions.

In this paper, we introduce a unified form of many generalizations of upper and lower almost nearly continuous multifunctions. First, by $mIO(X)$ we denote an m-structure which is constracted by a topology τ and an ideal I in an ideal topological space (X, τ, I). Second, by LPT-property we denote one of N-closed, compact, connected, Lindelöf sets in a topological space. Then we define and investigate the notion of upper/lower almost LPT mI-continuous multifunctions $F : (X, \tau, I) \to (Y, \sigma)$.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{A}$).

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [22] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [18] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [20] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) *b-open* [2] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.

The family of all semi-open (resp. preopen, α-open, β-open, b-open) sets in X is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$).

Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a *δ-cluster point* of A if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set V containing x. The set of all δ-cluster points of A is called the *δ-closure* of A [34] and is denoted by $\mathrm{Cl}_\delta(A)$. If $A = \mathrm{Cl}_\delta(A)$, then A is said to be *δ-closed*. The complement of a δ-closed set is said to be *δ-open*. The union of all δ-open sets of A is called the *δ-interior* of A and is denoted by $\mathrm{Int}_\delta(A)$.

Definition 2.2. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [27], [28] on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*.

Definition 2.3. Let (X, m) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [19] as follows:
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m\}$.

Remark 2.1. Let (X, τ) be a topological space and A be a subset of X. If $m = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$, $\beta(X)$), then we have
(a) $\mathrm{mCl}(A) = \mathrm{Cl(A)}$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, $\beta(A)$),
(b) $\mathrm{mInt}(A) = \mathrm{Int(A)}$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, $\beta\mathrm{Int}(A)$).

Lemma 2.1. (Maki et al. [19]). *Let (X, m) be an m-space. For subsets A and B of X, the following properties hold:*
(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$*,*
(2) If $(X \setminus A) \in m$*, then* $\mathrm{mCl}(A) = A$ *and if* $A \in m$*, then* $\mathrm{mInt}(A) = A$*,*
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$*,*
(4) If $A \subset B$*, then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$*,*
(5) $\mathrm{mInt}(A) \subset A \subset \mathrm{mCl}(A)$*,*
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$*.*

Lemma 2.2. (Popa and Noiri [28]). *Let (X, m) be an m-space and A a subset of X. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing x.*

Definition 2.4. An m-structure m on a nonempty set X is said to have *property $\mathcal{B}$* [19] if the union of any family of subsets belonging to m belongs to m.

Remark 2.2. Let (X, τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\mathrm{BO}(X)$ and $\beta(X)$ are m-structures and have property $\mathcal{B}$.

Lemma 2.3. (Popa and Noiri [30]). *For an m-structure m on a nonempty set X, the following properties are equivalent:*
(1) m has property $\mathcal{B}$;

(2) If $\mathrm{mInt}(A) = A$*, then* $A \in m$*;*
(3) If $\mathrm{mCl}(A) = A$*, then* A *is* m*-closed.*

Throughout the present paper, (X, τ) and (Y, σ) (briefly X and Y) always denote topological spaces and $F : X \to Y$ presents a multivalued function. For a multifunction $F : X \to Y$, we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and}$$
$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Definition 2.5. A subset A of a topological space (X, τ) is said to be *N-closed relative to X* (briefly *N-closed*) [5] if every cover of A by regular open sets of X has a finite subcover.

Definition 2.6. A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(1) *upper almost nearly continuous* [8] at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost nearly continuous* at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an open set U of X containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly continuous* on X if it has this property at each point of X.

3. Almost LPT m-continuous multifunctions

For a multifunction $F : (X, m) \to (Y, \sigma)$, the present authors [25] defined upper/lower almost nearly m-continuous multifunctions as follows:

Definition 3.1. Let (X, m) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m) \to (Y, \sigma)$ is said to be

(1) *upper almost nearly m-continuous* at a point $x \in X$ if for each open set V containing $F(x)$ and having N-closed complement, there exists an m-open set U containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost nearly m-continuous* at a point $x \in X$ if for each open set V meeting $F(x)$ and having N-closed complement, there exists an m-open set U containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost nearly m-continuous* on X if it has this property at every point of X.

In the following, we denote by *LTP property* one of N-closed, connected, compact, Lindelöf property.

Definition 3.2. Let (X, m) be an m-space and (Y, σ) a topological space. A multifunction $F : (X, m) \to (Y, \sigma)$ is said to be

(1) *upper almost LTP m-continuous* at a point $x \in X$ if for each open set V containing $F(x)$ and having LTP complement, there exists an m-open set U containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost LTP m-continuous* at a point $x \in X$ if for each open set V meeting $F(x)$ and having LPT complement, there exists an m-open set U containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost LTP m-continuous* on X if it has this property at every point of X.

Remark 3.1. Let $F : (X, m) \to (Y, \sigma)$ be upper/lower almost LTP m-continuous. If $m = \tau$ is a topology and LTP is N-closed (resp. compact , Lindelöf), then F is upper/lower almost nearly [8] (resp. upper/lower almost c-continuous [16], upper/lower almost l-continuous [14]).

Theorem 3.1. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y containing F(x) and having LTP complement;

(3) $x \in \mathrm{mInt}(F^+(\mathrm{sCl}(V)))$ for each open set V of Y containing F(x) and having LTP complement;

(4) $x \in \mathrm{mInt}(F^+(V))$ for each regular open set V of Y containing F(x) and having LTP complement;

(5) for each regular open set V of Y containing F(x) and having LTP complement, there exists $U \in m$ containing x such that $F(U) \subset V$.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y containing $F(x)$ and having LTP complement. There exists $U \in m$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$. Thus we have $x \in U \subset F^+(\mathrm{Int}(\mathrm{Cl}(V)))$ and hence $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): Since every open set is pre-open, by Lemma 3.2 of [23] $\mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{sCl}(V)$ for every open set V of Y.

(3) $\Rightarrow$ (4): Let V be any regular open set of Y containing $F(x)$ and having LTP complement. Then by Lemma 3.2 of [23], $V = \mathrm{Int}(\mathrm{Cl}(V)) = \mathrm{sCl}(V)$.

(4) $\Rightarrow$ (5): V be any regular open set of Y containing $F(x)$ and

having LTP complement. By (4), $x \in \mathrm{mInt}(F^+(V))$ and hence there exists $U \in m$ such that $x \in U \subset F^+(V)$; hence $F(U) \subset V$.

(5) $\Rightarrow$ (1): Let V be any open set of Y containing $F(x)$ and having LTP complement. Then $\mathrm{Int}(\mathrm{Cl}(V))$ is a regular open set of Y containing $F(x)$ and having LTP complement and hence, by (5), there exists $U \in m$ containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$.

Theorem 3.2. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP m-continuous at $x \in X$;

(2) $x \in \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y meeting F(x) and having LTP complement;

(3) $x \in \mathrm{mInt}(F^-(\mathrm{sCl}(V)))$ for each open set V of Y meeting F(x) and having LTP complement;

(4) $x \in \mathrm{mInt}(F^-(V))$ for each regular open set V of Y meeting F(x) and having LTP complement;

(5) for each regular open set V of Y meeting F(x) and having LTP complement, there exists $U \in m$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Proof. The proof is similar to that of Theorem 3.1.

Theorem 3.3. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous;

(2) $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y having LTP complement;

(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$ for every closed set K of Y having LTP property;

(4) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$ for every subset B of Y such that $\mathrm{Cl}(B)$ has LTP property;

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y such that $Y \setminus \mathrm{Int}(B)$ is LTP;

(6) $F^+(V) = \mathrm{mInt}(F^+(V))$ for every regular open set V of Y having LTP complement;

(7) $F^-(K) = \mathrm{mCl}(F^-(K))$ for every regular closed set K of Y having LTP property.

Proof. (1) $\Rightarrow$ (2): Let V be any open set of Y having LTP complement and let $x \in F^+(V)$. Then we have $F(x) \subset V$. By Theorem 3.1, we have $x \in \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$. This shows that $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(2) $\Rightarrow$ (3): Let K be any closed set K of Y having LTP property.

Then, $Y \setminus K$ is an open set of Y having LTP complement. By (2), we have $X \setminus F^-(K) = F^+(Y \setminus K) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(Y \setminus K)))) = \mathrm{mInt}(X \setminus F^-(\mathrm{Cl}(\mathrm{Int}(K)))) = X - \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K))))$. Therefore, we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$.

(3) $\Rightarrow$ (4): Let B be any subset of Y whose closure has LTP property. Then $\mathrm{Cl}(B)$ is a closed and LPT subset of Y and by (3) we obtain $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$.

(4) $\Rightarrow$ (5): Let B be a subset of Y such that $Y \setminus \mathrm{Int}(B)$ is LTP. Then, $Y \setminus \mathrm{Int}(B)$ is closed and LTP. Then, since $Y \setminus \mathrm{Int}(B)$ is closed and LTP, we have $F^+(\mathrm{Int}(B)) = X \setminus F^-(Y \setminus \mathrm{Int}(B)) = X \setminus F^-(\mathrm{Cl}(Y \setminus B)) \subset X \setminus \mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(Y \setminus B))))) = X \setminus \mathrm{mCl}(F^-(Y \setminus (\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))) = \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$. Therefore, we obtain $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$.

(5) $\Rightarrow$ (6): Let V be any regular open set of Y having LTP complement. Then $Y \setminus \mathrm{Int}(V)$ is LTP and by (5) we have $F^+(V) \subset \mathrm{Int}(F^+(V))$. Therefore, we have $F^+(V) = \mathrm{mInt}(F^+(V))$.

(6) $\Rightarrow$ (7): Let K be any regular closed set K of Y having LTP property. Then $Y \setminus K$ is a regular open set having LTP complement. By (6) $X \setminus F^-(K) = F^+(Y \setminus K) = \mathrm{mInt}(F^+(Y \setminus K)) = \mathrm{mInt}(X \setminus F^-(K)) = X \setminus \mathrm{mCl}(F^-(K))$. Therefore, we obtain $F^-(K) = \mathrm{mCl}(F^-(K))$.

(7) $\Rightarrow$ (1): Let $x \in X$ and V be any regular open set of Y cotaining $F(x)$ and having LTP complement. Then $Y \setminus V$ is regular closed and LPT. By (7), we have $X \setminus F^+(V) = F^-(Y \setminus V) = \mathrm{mCl}(F^-(Y \setminus V)) = X \setminus \mathrm{mInt}(F^+(V))$. Therefore, we have $x \in F^+(V) = \mathrm{mInt}(F^+(V))$. Then, there exist $U \in m$ containing x such that $F(U) \subset V$. It follows from Theorem 3.1 that F is upper almost LTP m-continuous at $x \in X$. Therefore, F is upper almost LTP m-continuous.

Theorem 3.4. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP m-continuous;

(2) $F^-(V) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for each open set V of Y having LTP complement;*

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^+(K)$ *for every closed set K of Y having LTP property;*

(4) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y whose closure has LTP property;*

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ *for every subset B of Y such that* $Y \setminus \mathrm{Int}(B)$ *is LTP;*

(6) $F^-(V) = \mathrm{mInt}(F^-(V))$ *for every regular open set V of Y having LTP complement;*

(7) $F^+(K) = \mathrm{mCl}(F^+(K))$ for every regular closed set K of Y having LTP property.

Proof. The proof is similar to that of Theorem 3.3.

Corollary 3.1. *Let (X, m) be an m-space and m have property $\mathcal{B}$. For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous;

(2) $F^+(V)$ is m-open for each regular open set V of Y having LTP complement;

(3) $F^-(K)$ is m-closed for every regular closed set K of Y having LTP property.

Proof. This is an immediate consequence of Theorem 3.3 and Lemma 2.3.

Corollary 3.2. *Let (X, m) be an m-space and m have property $\mathcal{B}$. For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP m-continuous;

(2) $F^-(V)$ is m-open for each regular open set V of Y having LTP complement;

(3) $F^+(K)$ is m-closed for every regular closed set K of Y having LTP property.

Proof. This is an immediate consequence of Theorem 3.4 and Lemma 2.3.

Remark 3.2. Let (X, τ) and (Y, σ) be topological spaces, $m = \tau$ (resp. $\mathrm{SO}(X)$) and LTP be N-closed. Then we have the following properties:

(1) If $F : (X, m) \to (Y, \sigma)$ upper almost LTP m-continuous, then by Theorem 3.3 and Corollary 3.1 we obtain Theorem 3 of [8] (resp. Theorem 1 of [32]).

(2) If $F : (X, m) \to (Y, \sigma)$ lower almost LTP m-continuous, then by Theorem 3.4 and Corollary 3.2 we obtain Theorem 6 of [8] (resp. Theorem 2 of [32]).

Corollary 3.3. *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction. If $F^{-1}(K) = \mathrm{mCl}(F^{-1}(K))$ (resp. $F^+(K) = \mathrm{mCl}(F^+(K))$) for every set K of Y having LTP property, then F is upper almost LTP m-continuous (resp. lower almost LTP m-continuous).*

Theorem 3.5. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous;

(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*

(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*

(4) $F^+(V)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in \mathrm{PO}(Y)$ *having LTP complement.*

Proof. (1) $\Rightarrow$ (2): Let V be any β-open set of Y such that $\mathrm{Cl}(V)$ is LTP. It is obvious that $\mathrm{Cl}(V)$ is regular closed. Since F is upper almost LTP m-continuous, by Theorem 3.3, $F^-(\mathrm{Cl}(V)) = \mathrm{mCl}(F^-(\mathrm{Cl}(V)))$. Therefore, $\mathrm{mCl}(F^-(V)) \subset \mathrm{mCl}(F^-(\mathrm{Cl}(V))) = F^-(\mathrm{Cl}(V))$.

(2) $\Rightarrow$ (3): Since every semi-open set is β-open, the proof is obvious.

(3) $\Rightarrow$ (4): Let V be any preopen set of Y having LTP complement. Then $\mathrm{Int}(\mathrm{Cl}(V))$ is a regular open set having LTP complement. Then $X \setminus \mathrm{Int}(\mathrm{Cl}(V))$ is regular closed and LTP. Therefore, $X \setminus \mathrm{Int}(\mathrm{Cl}(V))$ is a semi-open set having LTP-closure. By (3), we have $X \setminus \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))) = \mathrm{mCl}(F^-(Y \setminus \mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(Y \setminus \mathrm{Int}(\mathrm{Cl}(V)))) = X \setminus F^+(\mathrm{Int}(\mathrm{Cl}(V))) \subset X \setminus F^+(V)$. Therefore, $F^+(V) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$.

(4) $\Rightarrow$ (1): Let V be any regular open set having LTP complement. Then V is a preopen set having LTP complement and hence $F^+(V)) \subset \mathrm{mInt}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) = \mathrm{mInt}(F^+(V))$. Therefore, $F^+(V)) = \mathrm{mInt}(F^+(V))$. By Theorem 3.3, F is upper almost LTP m-continuous.

Theorem 3.6. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP m-continuous;

(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*

(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*

(4) $F^-(V)) \subset \mathrm{mInt}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in \mathrm{PO}(Y)$ *having LTP complement.*

Proof. The proof is similar to that of Theorem 3.5.

Corollary 3.4. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous;
(2) $\mathrm{mCl}(F^-(V)) \subset F^-(\alpha\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*
(3) $\mathrm{mCl}(F^-(V)) \subset F^-(\mathrm{pCl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property.*

Proof. It is shown in [23] that (1) $\alpha\mathrm{Cl}(V) = \mathrm{Cl}(V)$ for every $V \in \beta(Y)$ and (2) $p\mathrm{Cl}(V) = \mathrm{Cl}(V)$ for every $V \in SO(Y)$. The proof follows from the results and Theorem 3.5.

Corollary 3.5. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*
(1) F is lower almost LTP m-continuous;
(2) $\mathrm{mCl}(F^+(V)) \subset F^+(\alpha\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property;*
(3) $\mathrm{mCl}(F^+(V)) \subset F^+(\mathrm{pCl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *such that* $\mathrm{Cl}(V)$ *has LTP property.*

Proof. The proof follows from Theorem 3.6 similarly with Corollary 3.4.

Definition 3.3. A subset A of a topological space (X, τ) is said to be
(1) *α-paracompact* [35] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,
(2) *α-regular* [15] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

For a multifunction $F : X \to (Y, \sigma)$, a multifunction $\mathrm{Cl}F : X \to (Y, \sigma)$ is defined in [3] as follows: $(\mathrm{Cl}F)(x) = \mathrm{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\beta\mathrm{Cl}F$, and $\mathrm{bCl}F$.

Theorem 3.7. *Let $F : (X, m) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper almost LTP m-continuous if and only if $G : (X, m) \to (Y, \sigma)$ is upper almost LTP m-continuous, where G denotes* $\mathrm{Cl}F, \alpha\mathrm{Cl}F, \mathrm{sCl}F, \mathrm{pCl}F,$ $\mathrm{bCl}F or \beta\mathrm{Cl}F$.

Proof. The proof is similar to that of Theorem 3.7 of [25].

Theorem 3.8. *A multifunction $F : (X, m) \to (Y, \sigma)$ is lower almost LTP m-continuous if and only if $G : (X, m) \to (Y, \sigma)$ is lower almost LTP m-continuous, where G denotes* $\mathrm{Cl}F, \alpha\mathrm{Cl}F, \mathrm{sCl}F, \mathrm{pCl}F, \mathrm{bCl}F or \beta\mathrm{Cl}F$.

Proof. The proof is similar to that of Theorem 3.8 of [25].

Theorem 3.9. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP m-continuous;

(2) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property;*

(3) $\mathrm{mCl}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property.*

Proof. The proof is similar to that of Theorem 3.9 of [25].

Theorem 3.10. *For a multifunction $F : (X, m) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP m-continuous;

(2) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property;*

(3) $\mathrm{mCl}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property.*

Proof. The proof is similar to that of Theorem 3.10 of [25].

Definition 3.4. A topological space (Y, σ) is said to be *LTP-normal* if for each disjoint closed sets K and H of Y, there exist open sets U and V having LTP complement such that $K \subset U, H \subset V$ and $U \cap V = \emptyset$.

If LTP is N-closed, then LTP-normal is said to be *N-normal* [8].

Definition 3.5. An m-space (X, m) is said to be m-T_2 [27] if for each distinct points $x, y \in X$, there exist $U, V \in m$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 3.11. *If $F : (X, m) \to (Y, \sigma)$ is an upper almost LTP m-continuous multifunction satisfying the following conditions:*

(1) F(x) is closed in Y for each $x \in X$,

(2) $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,

(3) (Y, σ) is an LTP-normal space, and

(4) m has property $\mathcal{B}$,

then (X, m) is m-T_2.

Proof. The proof is similar to that of Theorem 5.1 of [25].

As a corollary of Theorem 3.11, we obtain Theorem 15 of [8] as follows:

Corollary 3.6. *Let $F : X \to Y$ be an upper almost nearly continuous multifunction and point closed from a toplogical space X to an N-normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a Hausdorff space.*

Theorem 3.12. *Let (X, m) be an m-space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F from (X, m) into a LTP-normal space (Y, σ) satisfying the following conditions:*
(1) $F(x_1)$ and $F(x_2)$ are closed in Y,
(2) F is upper almost LTP m-continuous at x_1 and x_2, and
(3) $F(x_1) \cap F(x_2) = \emptyset$,
then (X, m) is m-T_2.

Proof. The proof is similar to that of Theorem 5.3 of [25].

Definition 3.6. A topological space (X, σ) is said to be *LTP-connected* if X cannot be written as the union of two disjoint nonempty open sets having LTP complements.

If LTP is N-closed, then LTP-connected is said to be N-*connected* [8].

Definition 3.7. An m-space (X, m) is said to be *m-connected* [24] if X cannot be written as the union of two disjoint nonempty m-open sets.

Theorem 3.13. *Let (X, m) be an m-space, where m has property $\mathcal{B}$. If $F : (X, m) \to (Y, \sigma)$ is an upper almost LTP m-continuous or lower almost LTP m-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and (X, m) is m-connected, then (Y, σ) is LTP-connected.*

Proof. The proof is similar to that of Theorem 5.4 of [25].

As a corollary of Theorem 3.13, we obtain Theorem 14 of [8] as follows:

Corollary 3.7. *Let F be a multifunction from a connected toplogical space X onto a topological space Y such that F is point connected. If F is upper almost nearly continuous multifunction, then Y is an N-connected space.*

4. Ideal topological spaces

Let (X, τ) be a topological space. The notion of ideals has been introduced in [17] and [33] and further investigated in [13].

Definition 4.1. A nonempty collection I of subsets of a set X is called an *ideal* on X [17], [33] if it satisfies the following two conditions:
(1) $A \in I$ and $B \subset A$ implies $B \in I$,
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space (X, τ) with an ideal I on X is called an *ideal topological space* and is denoted by (X, τ, I). Let (X, τ, I) be an ideal topological space. For any subset A of X, $A^\star(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the *local function* of A with respect to τ and I [13]. Hereafter $A^\star(I, \tau)$ is simply denoted by $A^\star$. It is well known that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure operator on X and the topology generated by $\mathrm{Cl}^\star$ is denoted by $\tau^\star$.

Lemma 4.1. *Let (X, τ, I) be an ideal topological space and A, B be subsets of X. Then the following properties hold:*
(1) $A \subset B$ *implies* $\mathrm{Cl}^\star(A) \subset \mathrm{Cl}^\star(B)$,
(2) $\mathrm{Cl}^\star(X) = X$ *and* $\mathrm{Cl}^\star(\emptyset) = \emptyset$,
(3) $\mathrm{Cl}^\star(A) \cup \mathrm{Cl}^\star(B) \subset \mathrm{Cl}^\star(A \cup B)$.

Definition 4.2. Let (X, τ, I) be an ideal topological space. A subset A of X is said to be
(1) *α-I-open* [11] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A)))$,
(2) *semi-I-open* [11] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(3) *pre-I-open* [6] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A))$,
(4) *b-I-open* [4] if $A \subset \mathrm{Int}(\mathrm{Cl}^\star(A)) \cup \mathrm{Cl}^\star(\mathrm{Int}(A))$,
(5) *β-I-open* [12] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A)))$,
(6) *weakly semi-I-open* [9] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(7) *weakly b-I-open* [21] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^\star(A))) \cup \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}(A)))$,
(8) *strongly β-I-open* [10] if $A \subset \mathrm{Cl}^\star(\mathrm{Int}(\mathrm{Cl}^\star(A)))$.

Among the sets in Definition 4.2, we have the following relations:

DIAGRAM

$$\begin{array}{ccccccc}
\text{open} & \Rightarrow & \alpha\text{-}I\text{-open} & \Rightarrow & \text{semi-}I\text{-open} & \Rightarrow & \text{weakly semi-}I\text{-open} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{pre-}I\text{-open} & \Rightarrow & b\text{-}I\text{-open} & \Rightarrow & \text{weakly } b\text{-}I\text{-open} \\
 & & & & \Downarrow & & \Uparrow \\
 & & & & \text{strongly } \beta\text{-}I\text{-open} & \Rightarrow & \beta\text{-}I\text{-open}
\end{array}$$

The family of all α-I-open (resp. semi-I-open, pre-I-open, b-I-open, β-I-open, weakly semi-I-open, weakly b-I-open, strongly β-I-open) sets in an ideal topological space (X, τ, I) is denoted by $\alpha\mathrm{IO}(X)$

(resp. SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X)).

Remark 4.1. If $I = \{\emptyset\}$, then $A^\star = \mathrm{Cl}(A)$ and $\mathrm{Cl}^\star(A) = A^\star \cup A = \mathrm{Cl}(A)$. Therefore,

(1) $\tau^\star = \tau$, $\alpha\mathrm{IO}(X) = \alpha(X)$, $\mathrm{SIO}(X) = \mathrm{SO}(X)$, $\mathrm{PIO}(X) = \mathrm{PO}(X)$, $\mathrm{BIO}(X) = \mathrm{BO}(X)$ and $\beta\mathrm{IO}(X) = \beta(X)$.

(2) WSIO(X), WBIO(X), SβIO(X) and βIO(X) are coincide with $\beta(X)$.

Definition 4.3. By mIO(X), we denote each one of the families $\tau^\star$, αIO(X), SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X).

Lemma 4.2. *Let (X, τ, I) be an ideal topological space. Then* mIO*(X) is an m-structure and has property $\mathcal{B}$.*

Proof. The proof follows from Lemma 4.1(1)(2). As an example, we shall show that αIO(X) has property $\mathcal{B}$. Let A_α be an α-I-open set for each $\alpha \in \Lambda$. Then $A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(A_\alpha))) \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$ for each $\alpha \in \Lambda$ and hence $\cup_{\alpha\in\Lambda} A_\alpha \subset \mathrm{Int}(\mathrm{Cl}^\star(\mathrm{Int}(\cup_{\alpha\in\Lambda} A_\alpha)))$. Therefore, $\cup_{\alpha\in\Lambda} A_\alpha$ is α-I-open.

Remark 4.2. It is shown in Theorem 3.4 of [11] (resp. Theorem 2.10 of [6], Theorem 2.1 of [9], Theorem 2.7 of [21], Proposition 3 of [10]) that SIO(X) (resp. PIO(X), WSIO(X), WBIO(X), SβIO(X)) has property $\mathcal{B}$.

Definition 4.4. Let (X, τ, I) be an ideal topological space. For a subset A of X, the *mIO(X)-closure* $\mathrm{mCl_I}(A)$ and the *mIO(X)-interior* $\mathrm{mInt_I}(A)$ are defined as follows:

(1) $\mathrm{mCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{mIO}(X)\}$,

(2) $\mathrm{mInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{mIO}(X)\}$.

Let (X, τ, I) be an ideal topological space and mIO(X) the m-structure on X. If mIO(X) = αIO(X) (resp. SIO(X), PIO(X), BIO(X), βIO(X), WSIO(X), WBIO(X), SβIO(X)), then we have

(1) $\mathrm{mCl_I}(A) = \alpha\mathrm{Cl_I}(A)$ (resp. $\mathrm{sCl_I}(A)$, $\mathrm{pCl_I}(A)$, $\mathrm{bCl_I}(A)$, $\beta\mathrm{Cl_I}(A)$, $\mathrm{wsCl_I}(A)$, $\mathrm{wbCl_I}(A)$, $\mathrm{s}\beta\mathrm{Cl_I}(A)$),

(2) $\mathrm{mInt_I}(A) = \alpha\mathrm{Int_I}(A)$ (resp. $\mathrm{sInt_I}(A)$, $\mathrm{pInt_I}(A)$, $\mathrm{bInt_I}(A)$, $\beta\mathrm{Int_I}(A)$, $\mathrm{wsInt_I}(A)$, $\mathrm{wbInt_I}(A)$, $\mathrm{s}\beta\mathrm{Int_I}(A)$).

5. Almost LTP mI-continuous multifunctions

In this section, by using the results in Section 3, we obtain many properties of upper/lower almost LTP-continuous multifunction $F : (X, \tau, I) \to (Y, \sigma)$.

Definition 5.1. Let (X, τ, I) be an ideal topological space and (Y, σ) a topological space. A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) *upper almost LTP mI-continuous* at a point $x \in X$ if for each open set V containing $F(x)$ and having LTP complement, there exists an mI-open set U containing x such that $F(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$,

(2) *lower almost LTP mI-continuous* at a point $x \in X$ if for each open set V meeting $F(x)$ and having LTP complement, there exists an mI-open set U containing x such that $F(u) \cap \mathrm{Int}(\mathrm{Cl}(V)) \neq \emptyset$ for each $u \in U$,

(3) *upper/lower almost LTP mI-continuous* on X if it has this property at every point of X.

Lemma 5.1. *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is upper/lower almost LTP mI-continuous if and only if a multifunction $F : (X, mIO(X)) \to (Y, \sigma)$ is upper/lower almost LTP m-continuous.*

Proof. This is obvious from the definition.

Remark 5.1. Let $F : (X, \tau, I) \to (Y, \sigma)$ be upper/lower almost LTP mI-continuous and $LTP = N$-closed (resp. compact, Lindelöf, connected). Moreover, let $I = \{\emptyset\}$, then $mIO(X) = \tau^\star = \tau$. Therefore, we obtain the following definitions: $F : (X, \tau) \to (Y, \sigma)$ is upper/lower almost nearly continuous [8] (resp. upper/lower almost c-continuous [16], upper/lower almost l-continuous [14], upper/lower almost connected continuous).

Theorem 5.1. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous at $x \in X$;

(2) $x \in \mathrm{mInt_I}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y containing F(x) and having LTP complement;

(3) $x \in \mathrm{mInt_I}(F^+(\mathrm{sCl}(V)))$ for each open set V of Y containing F(x) and having LTP complement;

(4) $x \in \mathrm{mInt_I}(F^+(V))$ for each regular open set V of Y containing F(x) and having LTP complement;

(5) for each regular open set V of Y containing F(x) and having

LTP complement, there exists $U \in mIO(X)$ containing x such that $F(U) \subset V$.

Proof. The proof is obvious by Lemma 5.1 and Theorem 3.1.

Theorem 5.2. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous at $x \in X$;

(2) $x \in \mathrm{mInt_I}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y meeting F(x) and having LTP complement;

(3) $x \in \mathrm{mInt_I}(F^-(\mathrm{sCl}(V)))$ for each open set V of Y meeting F(x) and having LTP complement;

(4) $x \in \mathrm{mInt_I}(F^-(V))$ for each regular open set V of Y meeting F(x) and having LTP complement;

(5) for each regular open set V of Y meeting F(x) and having LTP complement, there exists $U \in mIO(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Proof. The proof is obvious by Lemma 5.1 and Theorem 3.2.

Theorem 5.3. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous;

(2) $F^+(V) \subset \mathrm{mInt_I}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y having LTP complement;

(3) $\mathrm{mCl_I}(F^-(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^-(K)$ for every closed set K of Y having LTP complement;

(4) $\mathrm{mCl_I}(F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^-(\mathrm{Cl}(B))$ for every subset B whose closure has LTP property;

(5) $F^+(\mathrm{Int}(B)) \subset \mathrm{mInt_I}(F^+(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ for every subset B of Y such that $Y \setminus \mathrm{Int}(B)$ is LTP;

(6) $F^+(V) = \mathrm{mInt_I}(F^+(V))$ for every regular open set V of Y having LTP complement;

(7) $F^-(K) = \mathrm{mCl_I}(F^-(K))$ for every regular closed set K of Y having LTP property.

Proof. The proof is obvious by Lemma 5.1 and Theorem 3.3.

Theorem 5.4. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous;

(2) $F^-(V) \subset \mathrm{mInt_I}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ for each open set V of Y having LTP complement;

(3) $\mathrm{mCl_I}(F^+(\mathrm{Cl}(\mathrm{Int}(K)))) \subset F^+(K)$ for every closed set K of Y

having LTP property;

(4) $\mathrm{mCl_I}(F^+(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^+(\mathrm{Cl}(B))$ *for every subset B of Y whose closure has LTP property;*

(5) $F^-(\mathrm{Int}(B)) \subset \mathrm{mInt_I}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(B)))))$ *for every subset B of Y such that* $Y \setminus \mathrm{Int}(B)$ *is LTP;*

(6) $F^-(V) = \mathrm{mInt_I}(F^-(V))$ *for every regular open set V of Y having LTP complement;*

(7) $F^+(K) = \mathrm{mCl_I}(F^+(K))$ *for every regular closed set K of Y having LTP property.*

Proof. The proof is obvious by Lemma 5.1 and Theorem 3.4.

Corollary 5.1. *For a multifunction* $F : (X, \tau, I) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous;

(2) $F^+(V)$ *is mI-open for each regular open set V of Y having LTP complement;*

(3) $F^-(K)$ *is mI-closed for every regular closed set K of Y having LTP property.*

Proof. This is an immediate consequence of Corollary 3.1 and Lemma 4.2.

Corollary 5.2. *For a multifunction* $F : (X, \tau, I) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous;

(2) $F^-(V)$ *is mI-open for each regular open set V of Y having LTP complement;*

(3) $F^+(K)$ *is mI-closed for every regular closed set K of Y having LTP property.*

Proof. This is an immediate consequence of Corollary 3.2 and Lemma 4.2.

Theorem 5.5. *For a multifunction* $F : (X, \tau, I) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \beta(Y)$ *such that* Cl*(V) has LTP property;*

(3) $\mathrm{mCl_I}(F^-(V)) \subset F^-(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{SO}(Y)$ *such that* Cl*(V) has LTP property;*

(4) $F^+(V)) \subset \mathrm{mInt_I}(F^+(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every* $V \in \mathrm{PO}(Y)$ *having LTP complement.*

Proof. The proof is obvious by Theorem 3.5 and Lemma 5.1.

Theorem 5.6. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property;*

(3) $\mathrm{mCl_I}(F^+(V)) \subset F^+(\mathrm{Cl}(V))$ *for every $V \in \mathrm{SO}(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property;*

(4) $F^-(V)) \subset \mathrm{mInt_I}(F^-(\mathrm{Int}(\mathrm{Cl}(V))))$ *for every $V \in \mathrm{PO}(Y)$ having LTP complement.*

Proof. The proof is obvious by Theorem 3.6 and Lemma 5.1.

Corollary 5.3. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^-(V)) \subset F^-(\alpha\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property;*

(3) $\mathrm{mCl_I}(F^-(V)) \subset F^-(\mathrm{pCl}(V))$ *for every $V \in \mathrm{SO}(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property.*

Proof. The proof follows from Corollary 3.4.

Corollary 5.4. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^+(V)) \subset F^+(\alpha\mathrm{Cl}(V))$ *for every $V \in \beta(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property;*

(3) $\mathrm{mCl_I}(F^+(V)) \subset F^+(\mathrm{pCl}(V))$ *for every $V \in \mathrm{SO}(Y)$ such that* $\mathrm{Cl}(V)$ *has LTP property.*

Proof. The proof follows from Corollary 3.5.

Theorem 5.7. *Let $F : (X, \tau, I) \to (Y, \sigma)$ be a multifunction such that $F(x)$ is α-regular and α-paracompact for each $x \in X$. Then F is upper almost LTP mI-continuous if and only if $G : (X, \tau, I) \to (Y, \sigma)$ is upper almost LTP mI-continuous, where G denotes* $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* $\beta\mathrm{Cl}F$.

Proof. The proof is obvious by Theorem 3.7.

Theorem 5.8. *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is lower almost LTP mI-continuous if and only if $G : (X, \tau, I) \to (Y, \sigma)$ is lower almost LTP mI-continuous, where G denotes* $\mathrm{Cl}F$, $\alpha\mathrm{Cl}F$, $\mathrm{sCl}F$, $\mathrm{pCl}F$, $\mathrm{bCl}F$ *or* $\beta\mathrm{Cl}F$.

Proof. The proof is obvious by Theorem 3.8.

Theorem 5.9. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is upper almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^{-}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^{-}(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property;*

(3) $\mathrm{mCl_I}(F^{-}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^{-}(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property.*

Proof. The proof is obvious by Theorem 3.9.

Theorem 5.10. *For a multifunction $F : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent:*

(1) F is lower almost LTP mI-continuous;

(2) $\mathrm{mCl_I}(F^{+}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(B))))) \subset F^{+}(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property;*

(3) $\mathrm{mCl_I}(F^{+}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B))))) \subset F^{+}(\mathrm{Cl}_\delta(B))$ *for every subset B of Y such that the δ-closure has LTP property.*

Proof. The proof is obvious by Theorem 3.10.

Theorem 5.11. *If $F : (X, \tau, I) \to (Y, \sigma)$ is an upper almost LTP mI-continuous multifunction satisfying the following conditions:*

(1) $F(x)$ is closed in Y for each $x \in X$,

(2) $F(x) \cap F(y) = \emptyset$ for each distinct points $x, y \in X$,

(3) (Y, σ) is an LTP-normal space,

then $(X, mIO(X))$ is $mIO(X)$-T_2.

Proof. The proof is obvious by Theorem 3.11 and Lemma 4.2.

If we put $LTP = N$-closed and $I = \{\emptyset\}$, then as a corollary of Theorem 5.11, we obtain Theorem 15 of [8]:

Corollary 5.5. *Let $F : X \to Y$ be an upper almost nearly continuous multifunction and point closed from a topological space X to a N-normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a Hausdorff space.*

Theorem 5.12. *Let (X, τ, I) be an ideal topological space. If for each pair of distinct points x_1 and x_2 in X, there exists a multifunction F from $(X, mIO(X))$ into a LTP-normal space (Y, σ) satisfying the following conditions:*

(1) $F(x_1)$ and $F(x_2)$ are closed in Y,

(2) F is upper almost LTP mI-continuous at x_1 and x_2, and

(3) $F(x_1) \cap F(x_2) = \emptyset$,

then $(X, mIO(X))$ is $mIO(X)$-T_2.

Proof. The proof is obvious by Theorem 3.12 and Lemma 4.2.

Theorem 5.13. *If $F : (X, \tau, I) \to (Y, \sigma)$ is an upper almost LTP mI-continuous or lower almost LTP mI-continuous surjective multifunction such that $F(x)$ is connected for each $x \in X$ and $(X, mIO(X))$ is $mIO(X)$-connected, then (Y, σ) is LTP-connected.*

Proof. The proof is similar to that of Theorem 3.13 and Lemma 4.2.

If we put $LTP = N$-closed and $I = \{\emptyset\}$, then as a corollary of Theorem 5.13, we obtain Theorem 14 of [8]:

Corollary 5.6. *Let F be a multifunction from a connected topological space X onto a topological space Y such that F is point connected. If F is upper almost nearly continuous multifunction, then Y is a N-connected space.*

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