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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 33 (2023), No. 1, 119 - 132

A GENERAL FIXED POINT THEOREM FOR MAPPINGS IN ORBITALLY COMPLETE S - METRIC SPACES

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Abstract. The purpose of this paper is to extend Theorem 5 [18] to a S - metric space, without orbitally continuity, generalizing Theorem 4 [2], Theorems 1-4, 6-8 [11], Corollary 2.19 and 2.21 [23], Theorems 23, 24 [20], Theorems 3.2-3.4 [21] and to obtaining a Ćirić-Jotić-type result in S - metric spaces.

1. Introduction

In 1974, Ćirić [1] initiated the study of non-unique fixed points in metric spaces and introduced the notion of orbitally complete metric spaces and orbitally continuous functions.

Definition 1.1 ([1]). Let f be a self mapping on a metric space (X, d). If for $x_0 \in X$, every Cauchy sequence $\{y_n\}$ of the orbit $O_{x_0}(f) = \{x_0, fx_0, f^2x_0, ...\}$ is convergent, then the metric space is said to be f - orbitally complete in $x_0 \in X$. If X is f - orbitally complete at each $x \in X$, then X is said f - orbitally complete.

Remark 1.2. *Every complete metric space is f - orbitally complete for every f.*

Keywords and phrases: fixed point, S - metric space, Ćirić-Jotić-type, orbitally complete, implicit relation.
(2010) Mathematics Subject Classification: 54H25, 47H10.

An orbitally complete metric space may not be a complete metric space (Example [2]).

Definition 1.3 ([1])**.** Let f be a self mapping on a metric space (X, d). The mapping f is said to be f - orbitally continuous at a point $x_0 \in X$ if the sequence $\{f(y_n)\}$ converges to $f(z)$ for any sequence $\{y_n\}$ in $O_{x_0}(f)$ which converges to a point z. The mapping f is said to be orbitally continuous if it has this property at each point $x \in X$.

Remark 1.4. *Any continuous self mappings of a metric space is orbitally continuous. An orbitally continuous mapping may not be continuous (see Examples* [2]*).*

The following theorem is proved in [2].

Theorem 1.5 (Ćirić and Jotić [2])**.** *Let T be an orbitally continuous mapping of a T - orbitally complete metric space (X, d) such that*

$$\min\{d(Tx, Ty), d(y, Ty)\} - \min\{d(x, Tx), d(y, Ty)\} \leq \\ q \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

for all $x \neq y$ in X and $q \in [0, 1]$. Then T has a fixed point.

Other fixed point theorems for orbitally continuous functions in orbitally complete metric spaces are published in [3], [10], [13], [14], [17], [25] and in other papers.

The notion of D - metric space is introduced in [4], [5]. Mustafa and Sims [8], [8] proved that most of the claims concerning the fundamental topological structures of D - metric spaces are incorrect and introduced the appropriate notion of generalized metric spaces, named G - metric spaces. In fact, Mustafa, Sims and other authors studied many fixed point results for self mappings under certain contractive conditions in other paper.

Recently, in [21] the authors introduced, as a generalization of G - metric spaces, a new type of generalized metric spaces, named S - metric spaces. In [6], the authors proved that the notion of S - metric space is not a generalization of G - metric space or vice versa. Hence, the notions of G - metric space and S - metric space are independent.

Other recently results of fixed points in S - metric spaces are obtained in [11], [12], [20], [21], [23], [24] and in other papers.

Quite recently, the notion of orbitally complete metric spaces is introduced in [12].

Definition 1.6 ([12]). Let (X, S) be a S - metric space and T be a self mapping of X. Then (X, S) is said to be T_S - orbitally complete if any Cauchy sequence which is contained in the sequence $\{x, Tx, T^2x, ..., T^nx, ...\}$ for some $x \in X$ converges in X.

The following theorem is proved in [12].

Theorem 1.7 ([12]). *Let (X, S) be T_S - orbitally complete and let T be a self mapping of X satisfying inequality*

$$S(Tx, Tx, Ty) \leq h \max \left\{ \begin{array}{c} S(x, x, y), S(x, x, Tx), S(y, y, Ty), \\ S(y, y, Tx), S(x, x, Ty) \end{array} \right\}$$

for all $x \neq y$ in X and $h \in \left[0, \frac{1}{3}\right)$. Then T has a unique common fixed point.

Several classical fixed point theorems and common fixed point theorems in metric spaces have been unified in [15], [16] considering a general condition by an implicit function. Recently, the method is used in the study of fixed points in metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, convex spaces, Hilbert spaces, compact metric spaces, fuzzy metric spaces, probabilistic metric spaces, intuitionistic metric spaces, partial metric spaces, dislocated metric spaces, for single valued mappings, hybrid pairs of mappings and set - valued mappings.

The study of fixed points in orbitally complete metric spaces for mappings satisfying an implicit relation is initiated in [17], [18].

The study of fixed points for mappings satisfying an implicit relation in G - metric spaces is initiated in [19].

The study of fixed points for mappings satisfying an implicit relation in S - metric spaces is initiated in [23], [24].

In this paper we prove a general fixed point theorem for mappings in orbitally S - metric spaces, satisfying an implicit relation.

2. Preliminaries

Definition 2.1 ([22], [23]). Let X be a nonempty set. A S - metric on X is a function $S : X^3 \longmapsto \mathbb{R}_+$ such that for all $x, y, z, a \in X$:
(S_1) : $S(x, y, z) = 0$ if and only if $x = y = z$;
(S_2) : $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$.

The pair (X, S) is called a S - metric space.

Example 2.2. *Let $X = \mathbb{R}$ and $S(x, y, z) = |x - z| + |y - z|$. Then, (X, S) is a S - metric space.*

Lemma 2.3 ([22]). *If S is a S - metric on a nonempty set X, then $S(x,x,y) = S(y,y,x)$ for all $x, y \in X$.*

Definition 2.4. Let (X,S) be a S - metric space. For $r > 0$ and $x \in X$, we define the open ball

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\}.$$

The topology induced by the S - metric is the topology induced by the base of all open balls in X.

Definition 2.5 ([22], [23]). a) A sequence $\{x_n\}$ in (X,S) converges to x, denoted $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.
b) A sequence $\{x_n\}$ in (X,S) is a Cauchy sequence if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.
c) (X,S) is complete if every Cauchy sequence is convergent.

Example 2.6. *(X,S) from Example 2.2 is a complete S - metric space.*

Lemma 2.7 ([22]). *Let (X,S) be a S - metric space. If $x_n \to x$ and $y_n \to y$, then $S(x_n, x_n, y_n) \to S(x,x,y)$.*

Lemma 2.8 ([22]). *Let (X,S) be a S - metric space and $x_n \to x$. Then $\lim_{n\to\infty} x_n$ is unique.*

Lemma 2.9 ([7]). *Let (X,S) be a S - metric space and A a nonempty set of X. If A is closed for all convergent sequence x_n to x, then $x \in A$.*

3. S - IMPLICIT RELATIONS

In [15], [16] is introduced $\mathcal{F}$ as the set of all continuous functions $F : \mathbb{R}_+^6 \longmapsto \mathbb{R}$ satisfying the following properties:
(F_1) : F is nonincreasing in variable t_6,
(F_2) : There exists $h \in [0,1)$ such that for all $u, v \geq 0$, $F(u,v,v,u,0,u+v) \leq 0$ implies $u \leq hv$;
(F_3) : $F(t,t,0,0,t,t) > 0$, $\forall t > 0$.

The following theorem is proved in [18].

Theorem 3.1. *Let (X,d) be a metric space and $T : X \longmapsto X$ satisfying inequality*

$$F\begin{pmatrix} d(Tx,Ty), d(x,y), d(x,Tx), \\ d(y,Ty), d(y,Tx), d(x,Ty) \end{pmatrix} < 0$$

for all $x \neq y$ in X and F satisfying properties $(F_1), (F_2)$.

If (X,d) is orbitally complete and T is orbitally continuous, then T has a fixed point. If, in addition, F satisfy property (F_3), then the fixed point is unique.

Let $\mathcal{F}_S$ as the set of all lower semi - continuous functions $F:\mathbb{R}_+^6 \longmapsto \mathbb{R}$ satisfying the following properties:
(F_1) : F is nonincreasing in variable t_6,
(F_2) : There exists $h \in [0,1)$ such that for all $u,v \geq 0$, $F(u,v,v,u,0,2u+v) \leq 0$ implies $u \leq hv$;
(F_3) : $F(t,t,0,0,t,t) > 0$, for all $t > 0$.
In the following examples, the functions F satisfy property (F_1).

Example 3.2. $F(t_1,...,t_6) = \min\{t_1,t_4\} - a\min\{t_5,t_6\} - q\max\{t_2,t_3,t_4\}$, *where* $a \geq 0$ *and* $q \in (0,1)$.

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - q\max\{u,v\} \leq 0$. If $u > v$, then $u(1-q) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = q < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+q)] > 0,\ \forall t > 0$.

Example 3.3. $F(t_1,...,t_6) = \min\{t_1^2,t_4^2,t_1t_4\} - a\min\{t_3^2,t_4t_5,t_1t_6\} - qt_2^2$, *where* $a \geq 0$ *and* $q \in (0,1)$.

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u^2 - qv^2 \leq 0$. If $u > v$, then $u^2(1-q) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{q} < 1$.
(F_3) : $F(t,t,0,0,t,t) = -qt^2 < 0,\ \forall t > 0$.

Example 3.4. $F(t_1,...,t_6) = t_1 - k\max\{t_2,t_3,t_4,t_5,t_6\}$, *where* $k \in \left[0,\frac{1}{3}\right)$.

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - k(2u+v) \leq 0$. If $u > v$, then $u(1-3k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < \frac{1}{3}$.
(F_3) : $F(t,t,0,0,t,t) = t(1-k) > 0,\ \forall t > 0$.

Example 3.5. $F(t_1,...,t_6) = t_1 - k\max\left\{t_2,t_3,t_4,\frac{t_5+t_6}{3}\right\}$, *where* $k \in [0,1)$.

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - k\max\{u,v,2u+v\} \leq 0$. If $u > v$, then $u(1-k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = k < 1$.
(F_3) : $F(t,t,0,0,t,t) = t(1-k) > 0,\ \forall t > 0$.

Example 3.6. $F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a,b,c,d,e \geq 0$. $a+b+c+3e < 1$ *and* $a+d+e < 1$.

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - bv - cu - e(2u+v) \leq 0$. If $u > v$, then $u[1-(a+b+c+3e)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+b+c+3e < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+d+e)] > 0, \ \forall t > 0$.

Example 3.7. *$F(t_1,...,t_6) = t_1 - at_2 - b\max\{t_3,t_4,t_5,t_6\}$, where $a,b \geq 0$ and $a+3b < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - b\max\{u,v,2u+v\} \leq 0$. If $u > v$, then $u[1-(a+3b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+3b < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+3b)] > 0, \ \forall t > 0$.

Example 3.8. *$F(t_1,...,t_6) = t_1 - at_2 - bt_3 - ct_4 - d\max\{t_5,t_6\}$, where $a,b,c,d \geq 0$ and $a+b+c+3d < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - bv - cu - d(2u+v) \leq 0$. If $u > v$, then $u[1-(a+b+c+3d)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+b+c+3d < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+d)] > 0, \ \forall t > 0$.

Example 3.9. *$F(t_1,...,t_6) = t_1 - at_2 - bt_5 - ct_6 - d\max\{t_3,t_4\}$, where $a,b,c,d \geq 0$, $a+3c+d < 1$ and $a+b+c < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - av - c(2u+v) - d\max\{u,v\} \leq 0$. If $u > v$, then $u[1-(a+3c+d)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a+3c+d < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(a+b+c)] > 0, \ \forall t > 0$.

Example 3.10. *$F(t_1,...,t_6) = t_1 - a(t_5+t_6) - bt_2 - c\max\{t_3,t_4\}$, where $a,b,c \geq 0$ and $3a+b+c < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = u - a(2u+v) - bv - c\max\{u,v\} \leq 0$. If $u > v$, then $u[1-(3a+b+c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 3a+b+c < 1$.
(F_3) : $F(t,t,0,0,t,t) = t[1-(2a+b)] > 0, \ \forall t > 0$.

Example 3.11. *$F(t_1,...,t_6) = t_1 - a(t_3+t_4) - bt_2 - c\max\{t_5,t_6\}$, where $a,b,c \geq 0$ and $2a+b+3c < 1$.*

(F_2) : Let $u,v \geq 0$ and $F(u,v,v,u,0,2u+v) = v - a(u+v) - bv - c(2u+v) \leq 0$. If $u > v$, then $u[1-(2a+b+3c)] \leq 0$,

a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 2a + b + 3c < 1$.
(F_3): $F(t, t, 0, 0, t, t) = t[1 - (b + c)] > 0, \ \forall t > 0$.

Example 3.12. *$F(t_1, ..., t_6) = t_1 - a \max\{t_3 + t_4, t_5 + t_6\} - bt_2$, where $a, b \geq 0$ and $3a + b < 1$.*

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - a(2u + v) - bv \leq 0$. If $u > v$, then $u[1 - (3a + b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = 3a + b < 1$.
(F_3): $F(t, t, 0, 0, t, t) = t[1 - (2a + b)] > 0, \ \forall t > 0$.

Example 3.13. *$F(t_1, ..., t_6) = t_1 - at_2 - b \max\{t_3 + t_4, t_5 + t_6\}$, where $a, b \geq 0$ and $a + 3b < 1$.*

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - b(2u + v) \leq 0$. If $u > v$, then $u[1 - (a + 3b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = a + 3b < 1$.
(F_3): $F(t, t, 0, 0, t, t) = t[1 - (a + 2b)] > 0, \ \forall t > 0$.

Example 3.14. *$F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, where $a, b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$.*

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - u(av + bv + cu) \leq 0$. If $u > v$, then $u^2[1 - (a + b + c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt{a + b + c} < 1$.
(F_3): $F(t, t, 0, 0, t, t) = t^2[1 - (a + d)] > 0, \ \forall t > 0$.

Example 3.15. *$F(t_1, ..., t_6) = t_1^3 - at_1t_2t_3 - bt_2t_3t_4 - ct_3t_4t_5 - dt_4t_5t_6$, where $a, b, c, d \geq 0$ and $a + b < 1$.*

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^3 - (a + b)uv^2 \leq 0$. If $u > v$, then $u^3[1 - (a + b)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq hv$, where $0 \leq h = \sqrt[3]{a + b} < 1$.
(F_3): $F(t, t, 0, 0, t, t) = t^3 > 0, \ \forall t > 0$.

The purpose of this paper is to extend Theorem 5 [18] to a S - metric space, without orbitally continuity, generalizing Theorem 4 [12], Theorem 1-4, 6-8 [11], Corollaries 2.19 and 2.21 [23], Theorem 3.2-3.4 [21] and to obtain a Ćirić - Jotić type result in S - metric spaces.

4. Main results

Theorem 4.1. *Let (X, S) be a S - metric spaces and $T : X \longmapsto X$ be a mapping with $S(T^2x, T^2x, Tx) \leq hS(Tx, Tx, Tx)$, with $h \in [0, 1)$*

and some $x \in X$. Then, the sequence $\{T^n x\}$, $n = 1, 2, ...$ is a Cauchy sequence.

Proof. Since $S(T^2x, T^2x, Tx) \leq hS(Tx, Tx, Tx)$, by induction we obtain $S(T^n x, T^n x, T^{n-1}x) \leq h^{n-1}S(Tx, Tx, x)$. By (S_2) and Lemma 2.3 we obtain

$$\begin{aligned} S(T^n x, T^n x, T^m x) &\leq 2S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^m x, T^m x, T^{n+1}x\right) \\ &= 2S\left(T^n x, T^n x, T^{n+1}x\right) + S\left(T^{n+1}x, T^{n+1}x, T^m x\right) \\ &\leq ... \leq 2\left(h^n + h^{n+1} + ... + h^{m-1}\right) S(Tx, Tx, x) \\ &\leq \frac{2h^n}{1-h} S(Tx, Tx, x). \end{aligned}$$

Letting n tend to infinity we obtain $\lim_{n\to\infty} S(T^n x, T^n x, T^m x) = 0$. Hence, $\{T^n x\}$ is a Cauchy sequence. □

Theorem 4.2. *Let (X, S) be a S - metric space and $T : X \longmapsto X$ a mapping satisfying the inequality*

$$F\begin{pmatrix} S(Tx, Tx, Ty), S(x, x, y), S(x, x, Tx), \\ S(y, y, Ty), S(y, y, Tx), S(x, x, Ty) \end{pmatrix} \leq 0 \tag{4.1}$$

for all $x \neq y \in \overline{O_{x_0}(T)}$, for some $x_0 \in X$ and F satisfying properties (F_1) and (F_2). If (X, S) is T - orbitally complete, then T has a unique fixed point. Moreover, if F satisfy also property (F_3), then the fixed point is unique.

Proof. By (4.1) for $y = Tx$ we obtain

$$F\begin{pmatrix} S(Tx, Tx, T^2x), S(x, x, Tx), S(x, x, Tx), \\ S(Tx, Tx, T^2x), S(Tx, Tx, Tx), S(x, x, T^2x) \end{pmatrix} \leq 0.$$

By Lemma 2.3 we obtain

$$F\begin{pmatrix} S(T^2x, T^2x, Tx), S(Tx, Tx, x), S(Tx, Tx, x), \\ S(T^2x, T^2x, Tx), 0, S(T^2x, T^2x, x) \end{pmatrix} \leq 0. \tag{4.2}$$

By (S_2)

$$\begin{aligned} S(T^2x, T^2x, Tx) &\leq 2S(T^2x, T^2x, Tx) + S(x, x, Tx) \\ &= 2S(T^2x, T^2x, Tx) + S(Tx, Tx, x). \end{aligned}$$

By (F_1) and (4.2) we get

$$F\begin{pmatrix} S(T^2x, T^2x, Tx), S(Tx, Tx, x), S(Tx, Tx, x), \\ S(T^2x, T^2x, Tx), 0, 2S(T^2x, T^2x, Tx) + S(Tx, Tx, x) \end{pmatrix} \leq 0.$$

By (F_2),

$$S(T^2x, T^2x, Tx) \leq hS\left(Tx, Tx, x\right).$$

By Theorem 4.1, $\{T^n x\}$ is a Cauchy sequence. Since (X, S) is orbitally complete, then $\{T^n x\}$ is convergent to a point z. By Lemma 2.9, $z \in \overline{O_x\left(T\right)}$.

We prove that z is a fixed point of T.

By (4.1) we obtain

$$F\left(\begin{array}{c} S\left(T^n x, T^n x, Tz\right), S\left(T^{n-1}x, T^{n-1}x, z\right), \\ S\left(T^{n-1}x, T^{n-1}x, T^n x\right), S\left(z, z, Tz\right), \\ S\left(z, z, T^n x\right), S\left(T^{n-1}x, T^{n-1}x, Tz\right) \end{array}\right) \leq 0. \tag{4.3}$$

Letting n tend to infinity in (4.3), by Lemma 2.7 we obtain

$$F\left(S\left(z, z, Tz\right), 0, 0, S\left(z, z, Tz\right), 0, S\left(z, z, Tz\right)\right) \leq 0.$$

By (F_1) we have

$$F\left(S\left(z, z, Tz\right), 0, 0, S\left(z, z, Tz\right), 0, 2S\left(z, z, Tz\right)\right) \leq 0.$$

By (F_2) we have $S\left(z, z, Tz\right) = 0$ and by (S_1), $z = Tz$. Hence, z is a fixed point of T in $\overline{O_x\left(T\right)}$.

We prove that there does not exists an other fixed point $u \neq z$ of T in $\overline{O_x\left(T\right)}$. By (4.1) for $x = z$ and $y = u$ we obtain

$$F\left(\begin{array}{c} S\left(Tz, Tz, Tu\right), S\left(z, z, u\right), S\left(z, z, Tz\right), \\ S\left(u, u, Tu\right), S\left(u, u, Tz\right), S\left(z, z, Tu\right) \end{array}\right) \leq 0$$

and by (S_1) we get

$$F\left(S\left(z, z, u\right), S\left(z, z, u\right), 0, 0, S\left(u, u, z\right), S\left(z, z, u\right)\right) \leq 0.$$

By Lemma 2.3,

$$S\left(u, u, z\right) = S\left(z, z, u\right).$$

Hence,

$$F\left(S\left(z, z, u\right), S\left(z, z, u\right), 0, 0, S\left(z, z, u\right), S\left(z, z, u\right)\right) \leq 0,$$

a contradiction of (F_3). Hence z is the unique fixed point of T in $\overline{O_x\left(T\right)}$. □

Remark 4.3. *This theorem extends Theorem 3.1 to a S - metric space.*

Corollary 4.4. *Let (X, S) be a S - metric space, some $x_0 \in X$ and $T : X \mapsto X$ satisfying the inequality*

$$S\left(Tx, Tx, Ty\right) \leq k \max\left\{\begin{array}{c} S\left(x, x, y\right), S\left(x, x, Tx\right), S\left(y, y, Ty\right), \\ S\left(y, y, Tx\right), S\left(x, x, Ty\right) \end{array}\right\},$$

where $k \in \left[0, \frac{1}{3}\right)$, *for all* $x \neq y$ *in* $\overline{O_{x_0}(T)}$. *If* (X, S) *is* T *- orbitally complete, then* T *has a unique fixed point.*

Proof. The proof follows by Theorem 4.2 and Example 3.4. □

Remark 4.5. *Corollary 4.4 is a generalization of Corollary 2.21* [23].

Corollary 4.6. *Let* (X, S) *be a* S *- metric space, some* $x_0 \in X$ *and* $T : X \mapsto X$ *satisfying the inequality*

$$\min\{S(Tx, Tx, Ty), S(y, y, Ty)\} - a\min\{S(y, y, Tx), S(x, x, Ty)\} \leq q\max\{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\},$$

for all $x \neq y$ *in* $\overline{O_{x_0}(T)}$, *where* $a \geq 0$ *and* $0 \leq q < 1$. *If* (X, S) *is* T *- orbitally complete, then* T *has a unique fixed point.*

Proof. The proof follows by Theorem 4.2 and Example 3.2. □

Remark 4.7. *Corollary 4.6 extend Theorem 1.5 to* S *- metric spaces.*

Corollary 4.8. *Let* (X, S) *be a* S *- metric space, some* $x_0 \in X$ *and* $T : X \mapsto X$ *satisfying the inequality*

$$S(Tx, Tx, Ty) \leq k\max\left\{\begin{array}{c} S(x, x, y), S(x, x, Tx), S(y, y, Ty), \\ \frac{S(y,y,Tx)+S(x,x,Ty)}{3} \end{array}\right\},$$

where $k \in [0, 1)$, *for all* $x \neq y$ *in* $\overline{O_{x_0}(T)}$. *If* (X, S) *is* T *- orbitally complete, then* T *has a unique fixed point.*

Proof. The proof follows by Theorem 4.2 and Example 3.5. □

Example 4.9. *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |x - z| + |y - z|$. *By Example 2.2,* (X, S) *is a* S *- metric space. Let* $Tx = \dfrac{1}{2}x$ *and* $x_0 = 1$. *Then*

$$\overline{O_1(T)} = \left\{\left(\frac{1}{2}\right)^k : k \in \mathbb{N}\right\}$$

and

$$\overline{O(T)} = O_1(T) \cup \{0\}.$$

Let $x \neq y \in \overline{O_1(T)}$. *Then*

$$S(Tx, Tx, Ty) = 2|Tx - Ty| = 2|x - y|$$

and

$$S(x, x, y) = 2|x - y|.$$

Then

$$S(Tx, Tx, Ty) \leq kS(x, x, y), \text{ where } k \in \left[\frac{1}{2}, 1\right).$$

Hence

$$S(Tx,Tx,Ty) \le k \max \left\{ \begin{array}{c} S(x,x,y), S(x,x,Tx), S(y,y,Ty), \\ \frac{S(y,y,Tx)+S(x,x,Ty)}{3} \end{array} \right\},$$

where $k \in \left[\frac{1}{2}, 1\right)$, *for all* $x \neq y$ *in* $\overline{O_1(T)}$.

By Theorem 4.8, T *has a unique fixed point* $x = 0$.

Remark 4.10. *1) a) By Example 3.7 and Theorem 4.2 we obtain a generalization of Theorem 1* [11].
b) By Example 3.8 and Theorem 4.2 we obtain a generalization of Theorem 2 [11].
c) By Example 3.9 and Theorem 4.2 we obtain a generalization of Theorem 3 [11].
d) By Example 3.10 and Theorem 4.2 we obtain a generalization of Theorem 5 [11].
e) By Example 3.11 and Theorem 4.2 we obtain a generalization of Theorem 6 [11].
f) By Example 3.12 and Theorem 4.2 we obtain a generalization of Theorem 7 [11].
g) By Example 3.13 and Theorem 4.2 we obtain a generalization of Theorem 8 [11].
2) By Example 3.6 and Theorem 4.2 we obtain a generalization of Corollaries 2.19, 2.21 [23].
3) By Example 3.6 and Theorem 4.2 we obtain theorems 2.3, 2.4 [20] *and theorems 3.2-3.4* [21].

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