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s^*-REGULARITY IN FUZZY M-SPACES

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Abstract. This paper deals with a new type of open-like set in fuzzy minimal space [2], viz. fuzzy $m - s^*$-open set taking fuzzy m-semiopen sets [3] as a basic tool. Afterwards, we introduce an idempotent operator, viz. fuzzy $m - s^*$-closure operator. With the help of this operator we introduce and study two new types of functions, viz. fuzzy almost $(m, m_1) - s$-continuous function and fuzzy almost $(m, m_1) - s^*$-continuous function. It is shown that every fuzzy almost $(m, m_1) - s^*$-continuous function is fuzzy almost $(m, m_1) - s$-continuous function, but the reverse implication is not necessarily true in general. Furthermore, we introduce fuzzy $m - s^*$-regular spaces, in which the above mentioned reverse implication holds and, in addition, the classes of fuzzy m-open sets and fuzzy $m - s^*$-open sets coincide.

1. Introduction

In [11], L.A. Zadeh introduced fuzzy set as follows : a fuzzy set A is a mapping from a non-empty set X into the closed interval $[0, 1]$, i.e., $A \in I^X$. In 1968, C.L. Chang introduced fuzzy topology [6]. In [8] Popa and Noiri introduced the notion of minimal structure in general topology, generalizing some properties of continuous functions. Afterwards, Alimohammady and Roohi introduced a more general version of fuzzy topology by introducing fuzzy minimal structure, as follows: a family $\mathcal{M}$ of fuzzy sets in a non-empty set X is said to be a fuzzy minimal structure on X if $\alpha 1_X \in \mathcal{M}$ for every $\alpha \in [0, 1]$ [1].

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However a more general version of it (in the sense of Chang) is introduced in [5, 7] as follows: a family $\mathcal{F}$ of fuzzy sets in a non-empty set X is a fuzzy minimal structure on X if $0_X \in \mathcal{F}$ and $1_X \in \mathcal{F}$. In this paper, we use the notion of fuzzy minimal structure in the sense of Chang. In [2], we introduced fuzzy minimal space (fuzzy m-space, for short), as follows. Let X be a non-empty set and $m \subset I^X$. Then (X, m) is called fuzzy m-space if $0_X \in m$ and $1_X \in m$. The members of m are called fuzzy m-open sets and the complement of a fuzzy m-open set is called fuzzy m-closed set [2]. Many mathematicians have investigated different types of functions in the setting of fuzzy minimal spaces. In this context, we have to mention [4, 10].

2. Preliminaries

Throughout this paper, we shall denote by (X, m) or simply by X a fuzzy minimal space (fuzzy m-space, for short). The support [11] of a fuzzy set A, denoted by $suppA$ or A_0 and is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The complement [11] of a fuzzy set A in a X is denoted by $1_X \setminus A$ and is defined by $(1_X \setminus A)(x) = 1 - A(x)$, for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$, for all $x \in X$ [11] while AqB means A is quasi-coincident (q-coincident, for short) [9] with B, i.e., there exists $x \in X$ such that $A(x) + B(x) > 1$. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively. For any two fuzzy sets A and B in a fuzzy m-space (X, m), the union is defined by $(A \bigvee B)(x) = max\{A(x), B(x)\}$, for all $x \in X$ and the intersection is defined by $(A \bigwedge B)(x) = min\{A(x), B(x)\}$, for all $x \in X$. More general, for any collection of fuzzy sets $\{A_i : i \in I\}$, one defines $C = \bigvee \{A_i : i \in I\}$ by $C(x) = \sup \{A_i(x) : i \in I\}$ for all $x \in X$, respectively $D = \bigwedge \{A_i : i \in I\}$ by $D(x) = \inf \{A_i(x) : i \in I\}$ for all $x \in X$.

For a fuzzy set A and a fuzzy point x_α in X, $x_\alpha \in A$ means that $A(x) \geq \alpha$. A fuzzy set A in a fuzzy m-space (X, m) is called a fuzzy m-neighbourhood (fuzzy m-nbd, for short) of a fuzzy point x_α in X if there exists a fuzzy m-open set U in X such that $x_\alpha \in U \leq A$ [2]. If, in addition, A is fuzzy m-open, then A is called a fuzzy m-open nbd of x_α [2]. A fuzzy set A in a fuzzy m-space (X, m) is called a fuzzy m-q-nbd of a fuzzy point x_α in X if there exists a fuzzy m-open set U

in X such that $x_\alpha q U \leq A$ [2]. If, in addition, A is fuzzy m-open, then A is called a fuzzy m-open q-nbd of x_α [2].

3. Fuzzy $m - s^*$-Open and $m - s^*$-Closed Sets: Some Properties

Using fuzzy m-semiopen set as a basic tool, here we introduce fuzzy $m - s^*$-open sets, the class of which is strictly larger than that of fuzzy $m-$open sets as well as that of fuzzy $m-$preopen sets. Afterwards, we introduce fuzzy $m - s^*$-closure operator which is an idempotent operator.

We first recall some definitions from [2, 3] for ready references.

Definition 3.1 [2]. Let X be a non-empty set and $m \subset I^X$ an m-structure on X. For $A \in I^X$, the m-closure of A and m-interior of A are defined as follows :

$$m - clA = \bigwedge\{F : A \leq F, 1_X \setminus F \in m\}$$

$$m - intA = \bigvee\{D : D \leq A, D \in m\}$$

It can be observed that a given fuzzy minimal structure on X, $A \in I^X$ does not imply that $m - intA \in m$ or that $m - clA$ is fuzzy m-closed. But if m satisfies M-condition (i.e., m is closed under arbitrary union), then $m - intA \in m$ and $m - clA$ is fuzzy m-closed.

Proposition 3.2 [2]. Let X be a non-empty set and m, an m-structure on X. Then for any $A \in I^X$, a fuzzy point $x_\alpha \in m - clA$ if and only if for any $U \in m$ with $x_\alpha q U$, UqA.

Lemma 3.3 [2]. *Let X be a non empty set and $m \subset I^X$ be an m-structure on X.*

For $A, B \in I^X$, the following hold :

(i) If $A \leq B$, then $m - intA \leq m - intB$ and $m - clA \leq m - clB$.

(ii) (a) $m - cl\,(0_X) = 0_X$, $m - cl\,(1_X) = 1_X$,

(b) $m - int\,(0_X) = 0_X$, $m - int\,(1_X) = 1_X$.

(iii) $m - int\,(A) \leq A \leq m - cl\,(A)$.

(iv) (a) $m - cl\,(A) = A$ if $1_X \setminus A \in m$, (b) $m - int\,(A) = A$, if $A \in m$.

(v) $m - cl(1_X \setminus A) = 1_X \setminus m - int\,(A)$, (b) $m - int(1_X \setminus A) = 1_X \setminus m - cl\,(A)$.

(vi) (a) $m - cl(m - clA) = m - cl\,(A)$, (b) $m - int(m - intA) = m - int\,(A)$.

(vii) $m - cl(A \bigwedge B) \leq m - cl\,(A) \bigwedge m - cl\,(B)$,

(b) $m - int(A \bigvee B) \geq m - int\,(A) \bigvee m - int\,(B)$.

Definition 3.4 [3]. Let (X, m) be a fuzzy m-space and $A \in I^X$. Then A is called

(i) fuzzy m-regular open if $A = m - int(m - clA)$
(ii) fuzzy m-semiopen if $A \leq m - cl(m - intA)$

The complement of fuzzy m-semiopen set is called fuzzy m-semiclosed.
The union (intersection) of all fuzzy m-semiopen (resp., fuzzy m-semiclosed) sets contained in (resp., containing) a fuzzy set A is called fuzzy m-semiinterior (resp., fuzzy m-semiclosure) of A denoted by $m - sint\,(A)$ (resp., $m - scl\,(A)$).
The collection of all fuzzy m-semiopen (resp., fuzzy m-semiclosed) sets in a fuzzy m-space X is denoted by $FmSO(X)$ (resp., $FmSC(X)$).

Proposition 3.5 [3]. Let (X, m) be a fuzzy m-space and $A \in I^X$. Then a fuzzy point $x_\alpha \in m - sclA$ if and only if for every fuzzy m-semiopen set U in X, $x_\alpha qU$, UqA, i.e., for every fuzzy m-semiopen q-nbd U of x_α, UqA.

Result 3.6 [3]. *Let (X, m) be a fuzzy m-space and $A, B \in I^X$. Then the following statements hold :*
(i) (a) $A \leq B$ *implies* $m - sintA \leq m - sintB$,
(b) $A \leq B$ *implies* $m - sclA \leq m - sclB$.
(ii) (a) $m - scl\,(0_X) = 0_X$, $m - scl1_X = 1_X$,
(b) $m - sint\,(0_X) = 0_X$, $m - sint1_X = 1_X$.
(iii) $m - sint\,(A) \leq A \leq m - sclA$.
(iv) (a) $m - scl\,(A) = A$ *if* $A \in FmSC(X)$,
(b) $m - sint\,(A) = A$, if $A \in FmSO(X)$.
(v) (a) $m - scl(1_X \setminus A) = 1_X \setminus m - sint\,(A)$,
(b) $m - sint(1_X \setminus A) = 1_X \setminus m - scl\,(A)$.
(vi) (a) $m - scl(m - sclA) = m - scl\,(A)$,
(b) $m - sint(m - sintA) = m - sint\,(A)$.
(vii) (a) $m - scl(A \bigwedge B) \leq m - scl\,(A) \bigwedge m - scl\,(B)$,
(b) $m - sint(A \bigvee B) \geq m - sint\,(A) \bigvee m - sint\,(B)$,
(viii) (a) $m - scl(A \bigvee B) \geq m - scl\,(A) \bigvee m - scl\,(B)$,
(b) $m - scl(A \bigwedge B) \leq m - sint\,(A) \bigwedge m - sint\,(B)$.

Definition 3.7. A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy $m - s^*$-open if $A \leq m - int(m - sclA)$. The complement of this set is called fuzzy $m - s^*$-closed set.
The collection of fuzzy $m - s^*$-open (resp., fuzzy $m - s^*$-closed) sets in (X, m) is denoted by $FmS^*O(X)$ (resp., $FmS^*C(X)$).
The union (resp., intersection) of all fuzzy $m - s^*$-open (resp., fuzzy $m - s^*$-closed) sets contained in (containing) a fuzzy set A is called fuzzy $m - s^*$-interior (resp., fuzzy $m - s^*$-closure) of A, denoted by $m - s^*intA$ (resp., $m - s^*clA$).

Definition 3.8. A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy $m - s^*$-nbd of a fuzzy point x_α if there exists a fuzzy $m - s^*$-open set U in X such that $x_\alpha \in U \leq A$. If, in addition, A is fuzzy $m - s^*$-open, then A is called fuzzy $m - s^*$-open nbd of x_α.

Definition 3.9. A fuzzy set A in a fuzzy m-space (X, m) is called fuzzy $m - s^*$-q-nbd of a fuzzy point x_α if there exists a fuzzy $m - s^*$-open set U in X such that $x_\alpha q U \leq A$. If, in addition, A is fuzzy $m - s^*$-open, then A is called fuzzy $m - s^*$-open q-nbd of x_α.

Result 3.10. *Union (resp., intersection) of any two fuzzy $m - s^*$-open (resp., fuzzy $m - s^*$-closed) sets is also so.*

Proof. Let A, B be two fuzzy $m - s^*$-open (resp., fuzzy $m - s^*$-closed) sets in a fuzzy m-space X.

Then $A \leq m - int(m - sclA), B \leq m - int(m - sclB)$
(resp., $m - cl(m - sintA) \leq A, m - cl(m - sintB) \leq B$).

Now $m - int(m - scl(A \bigvee B)) \geq m - int(m - sclA \bigvee m - sclB)$
$\geq m - int(m - sclA) \bigvee m - int(m - sclB) \geq A \bigvee B$

and

$m - cl(m - sint(A \bigwedge B)) \leq m - cl(m - sintA \bigwedge m - sintB)$
$\leq m - cl(m - sintA) \bigwedge m - cl(m - sintB) \leq A \bigwedge B$.

Remark 3.11. Intersection (resp., union) of two fuzzy $m - s^*$-open (resp., fuzzy $m - s^*$-closed) sets may not be so as it seen from the following example.

Example 3.12. Let $X = \{a, b\}$, $m = \{0_X, 1_X, A, B\}$ where $A(a) = 0.5$, $A(b) = 0.4$ and $B(a) = 0.5$, $B(b) = 0.55$. Then (X, m) is a fuzzy m-space. Now $FmSO(X) = \{0_X, 1_X, U, V\}$ where $A \leq U \leq 1_X \setminus B$, $B \leq V \leq 1_X \setminus A$, and $FmSC(X) = \{0_X, 1_X, 1_X \setminus U, 1_X \setminus V\}$ where $B \leq 1_X \setminus U \leq 1_X \setminus A$, $A \leq 1_X \setminus V \leq 1_X \setminus B$. Consider two fuzzy sets C, D in X defined by $C(a) = C(b) = 0.5, D(a) = 0.6, D(b) = 0.43$. Now $m - int(m - sclC) = B \geq C, m - int(m - sclD) = 1_X > D$, which implies that C, D are fuzzy $m - s^*$-open sets in (X, m). Let $E = C \bigwedge D$. Then $E(a) = 0.5, E(b) = 0.43$. Now $m - int(m - sclE) = A < E$, therefore E is not fuzzy $m - s^*$-open in X.
Again, $1_X \setminus C, 1_X \setminus D$ are fuzzy $m - s^*$-closed sets in (X, m). Now $F = (1_X \setminus C) \bigvee (1_X \setminus D)$ is defined by $F(a) = 0.5, F(b) = 0.57$ and $m - cl(m - sintF) = 1_X \setminus A \not\leq F$, therefore F is not fuzzy $m - s^*$-closed in (X, m).

Theorem 3.13. *For any fuzzy set A in a fuzzy m-space (X, m), a fuzzy point $x_\alpha \in m - s^*clA$ if and only if every fuzzy $m - s^*$-open q-nbd U of x_α, UqA.*

Proof. Let $x_\alpha \in m - s^*clA$ for any fuzzy set A in a fuzzy m-space

(X, m). Let $U \in FmS^*O(X)$ with $x_\alpha qU$. Then $U(x) + \alpha > 1$, hence $x_\alpha \notin 1_X \setminus U \in FmS^*C(X)$. By Definition 3.7, $A \not\leq 1_X \setminus U$, therefore there exists $y \in X$ such that $A(y) > 1 - U(y)$, hence $A(y) + U(y) > 1$, therefore UqA.
Conversely, assume that the given condition holds. Let $U \in FmS^*C(X)$ with $A \leq U$ (1).

We have to show that $x_\alpha \in U$, i.e., $U(x) \geq \alpha$. If possible, let $U(x) < \alpha$. Then $1 - U(x) > 1 - \alpha$, which implies $x_\alpha q(1_X \setminus U)$ where $1_X \setminus U \in FmS^*O(X)$. By hypothesis, $(1_X \setminus U)qA$, hence there exists $y \in X$ such that $1 - U(y) + A(y) > 1$, which implies by (1) that $1 - A(y) + A(y > 1$, which is absurd.

Theorem 3.14. *$m - s^*cl(m - s^*clA) = m - s^*clA$ for any fuzzy set A in a fuzzy m-space (X, m).*

Proof. Let $A \in I^X$. Then $A \leq m - s^*clA$, hence
$m - s^*clA \leq m - s^*cl(m - s^*clA)$ (1).
Conversely, let $x_\alpha \in m - s^*cl(m - s^*clA)$. Assume that $x_\alpha \notin m - s^*clA$.

Then there exists $U \in FmS^*O(X)$ such that
$x_\alpha qU, U\not{q}A$ (2).

But as $x_\alpha \in m - s^*cl(m - s^*clA)$, $Uq(m - s^*clA)$, there exists $y \in X$ such that $U(y) + (m - s^*clA)(y) > 1$, which implies $U(y) + t > 1$ where $t = (m - s^*clA)(y)$. Then $y_t \in m - s^*clA$ and y_tqU where $U \in FmS^*O(X)$. Then by definition, UqA, contradicts (2).

So $m - s^*cl(m - s^*clA) \leq m - s^*clA$ (3).

Combining (1) and (3), we get the proof.

Note 3.15. Fuzzy m-semiopen set and fuzzy $m - s^*$-open set are independent notions, as follows from the next two examples.

Example 3.16. $FmSO(X) \not\subseteq FmS^*O(X)$
Consider Example 3.12. Here $E \in FmSO(X)$, but $E \notin FmS^*O(X)$.

Example 3.17. $FmS^*O(X) \not\subseteq FmSO(X)$
Consider Example 3.12. Here $C \in FmS^*O(X)$, but $C \notin FmSO(X)$.

Note 3.18. It is obvious that every fuzzy m-open set is fuzzy $m - s^*$-open. But the converse is not true, in general, as follows from Example 3.12. Here $C \in FmS^*O(X)$, but $C \notin m$.

4. Fuzzy Almost (m, m_1)-s-Continuous Functions: Some Characterizations

In this section we introduce the notion of fuzzy almost (m, m_1)-s-continuous function between two fuzzy m-spaces and then characterize it via several ways.

Definition 4.1. A function $f : (X, m) \to (Y, m_1)$ is said to be fuzzy almost (m, m_1)-s-continuous if for each fuzzy point x_α in X and every fuzzy m_1-nbd V of $f(x_\alpha)$ in Y, $m - scl(f^{-1}(V))$ is a fuzzy m-nbd of x_α in X.

Theorem 4.2. *For a function $f : (X, m) \to (Y, m_1)$ where m_1 satisfies M-condition, the following statements are equivalent :*
(a) f is fuzzy almost (m, m_1)-s-continuous,
(b) $f^{-1}(B) \le m - int(m - scl(f^{-1}(B)))$, for all fuzzy m_1-open set B of Y,
(c) $f(m - clA) \le m_1 - cl(f(A))$, for all $A \in FmSO(X)$.

Proof (a) $\Rightarrow$ (b). Let B be any fuzzy m_1-open set in Y and $x_\alpha \in f^{-1}(B)$. Then $f(x_\alpha) \in B$, therefore B is a fuzzy m_1-nbd of $f(x_\alpha)$. By (a), $m - scl(f^{-1}(B))$ is a fuzzy m-nbd of x_α in X, hence $x_\alpha \in m - int(m - scl(f^{-1}(B)))$. Then $f^{-1}(B) \le m - int(m - scl(f^{-1}(B)))$.
(b) $\Rightarrow$ (a). Let x_α be a fuzzy point in X and B be a fuzzy m_1-nbd of $f(x_\alpha)$ in Y. Then $x_\alpha \le f^{-1}(B) \le m - int(m - scl(f^{-1}(B)))$ (by (b)) $\le m - scl(f^{-1}(B))$, so $m - scl(f^{-1}(B))$ is a fuzzy m-nbd of x_α in X.
(b) $\Rightarrow$ (c). Let $A \in FmSO(X)$. Then $1_Y \setminus m_1 - cl(f(A))$ is a fuzzy m_1-open set in Y (as m_1 satisfies M-condition).
By (b), $f^{-1}(1_Y \setminus m_1 - cl(f(A)))$
$\le m - int(m - scl(f^{-1}(1_Y \setminus m_1 - cl(f(A)))))$
$= m - int(m - scl(1_X \setminus f^{-1}(m_1 - cl(f(A)))))$
$\le m - int(m - scl(1_X \setminus f^{-1}(f(A)))) \le m - int(m - scl(1_X \setminus A)) = 1_X \setminus m - cl(m - sint\,(A))$
$= 1_X \setminus m - cl\,(A)$. Then $1_X \setminus f^{-1}(m_1 - cl(f(A))) \le 1_X \setminus m - cl\,(A)$, therefore $m - cl\,(A) \le f^{-1}(m_1 - cl(f(A)))$, hence $f(m - cl\,(A)) \le m_1 - cl(f(A))$.
(c) $\Rightarrow$ (b). Let B be any fuzzy m_1-open set in Y. Then
$m - sint(f^{-1}(1_Y \setminus B)) \in FmSO(X)$ (as m_1 satisfies M-condition).
By (c),
$f(m - cl(m - sint(f^{-1}(1_Y \setminus B)))) \le m_1 - cl(f(m - sint(f^{-1}(1_Y \setminus B)))) \le m_1 - cl(f(f^{-1}(1_Y \setminus B))) \le m_1 - cl(1_Y \setminus B) = 1_Y \setminus B$ (as m_1 satisfies M-condition), therefore $f^{-1}(B) = 1_X \setminus f^{-1}(1_Y \setminus B) \le 1_X \setminus m - cl(m - sint(f^{-1}(1_Y \setminus B))) = 1_X \setminus m - cl(m - sint(1_X \setminus f^{-1}(B))) = m - int(m - scl(f^{-1}(B)))$.

Remark 4.3. It is clear from Theorem 4.2 that the inverse image of any fuzzy m_1-open set under a fuzzy almost (m, m_1)-s-continuous function is fuzzy $m - s^*$-open.

Theorem 4.4. *For a function $f : (X, m) \to (Y, m_1)$ where m_1 satisfies M-condition, the following statements are equivalent :*
(a) f is fuzzy almost (m, m_1)-s-continuous,

(b) $f^{-1}(B) \leq m - int(m - scl(f^{-1}(B)))$*, for all fuzzy* m_1*-open sets* B *of* Y*,*
(c) for each fuzzy point x_α *in* X *and each fuzzy* m_1*-open nbd* V *of* $f(x_\alpha)$ *in* Y*, there exists* $U \in FmS^*O(X)$ *containing* x_α *such that* $f(U) \leq V$*,*
(d) $f^{-1}(F) \in FmS^*C(X)$*, for all fuzzy* m_1*-closed sets* F *in* Y*,*
(e) for each fuzzy point x_α *in* X*, the inverse image under* f *of every fuzzy* m_1*-nbd of* $f(x_\alpha)$ *in* Y *is a fuzzy* $m - s^*$*-nbd of* x_α *in* X*,*
(f) $f(m - s^*cl\,(A)) \leq m_1 - cl(f(A))$*, for all* $A \in I^X$*,*
(g) $m - s^*cl(f^{-1}(B)) \leq f^{-1}(m_1 - cl\,(B))$*, for all* $B \in I^Y$*,*
(h) $f^{-1}(m_1 - int\,(B)) \leq m - s^*int(f^{-1}(B))$*, for all* $B \in I^Y$*.*

Proof (a) $\Leftrightarrow$ (b). Follows from Theorem 4.2 (a) $\Leftrightarrow$ (b).
(b) $\Rightarrow$ (c). Let x_α be a fuzzy point in X and V be a fuzzy open m_1-nbd of $f(x_\alpha)$ in Y. By (b), $f^{-1}(V) \leq m - int(m - scl(f^{-1}(V)))$ (1) . Now $f(x_\alpha) \in V$ implies $x_\alpha \in f^{-1}(V)$ ($= U$, say).

Then $x_\alpha \in U$ and by (1), $U = f^{-1}(V)) \in FmS^*O(X)$ and $f(U) = f(f^{-1}(V)) \leq V$.
(c) $\Rightarrow$ (b). Let V be a fuzzy m_1-open set in Y and let $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V$, therefore V is a fuzzy m_1-open nbd of $f(x_\alpha)$ in Y. By (c), there exists $U \in FmS^*O(X)$ containing x_α such that $f(U) \leq V$. Then $x_\alpha \in U \leq f^{-1}(V)$. Now $U \leq m - int(m - scl\,(U))$. Then $U \leq m - int(m - scl\,(U)) \leq m - int(m - scl(f^{-1}(V)))$, hence $x_\alpha \in U \leq m - int(m - scl(f^{-1}(V)))$, which implies
$f^{-1}(V) \leq m - int(m - scl(f^{-1}(V)))$.
(b) $\Leftrightarrow$ (d). Obvious.
(b) $\Rightarrow$ (e). Let W be a fuzzy m_1-nbd of $f(x_\alpha)$ in Y. Then there exists a fuzzy m_1-open set V in Y such that $f(x_\alpha) \in V \leq W$, hence V is a fuzzy m_1-open nbd of $f(x_\alpha)$ in Y. Then by (b), $f^{-1}(V) \in FmS^*O(X)$ and $x_\alpha \in f^{-1}(V) \leq f^{-1}(W)$, therefore $f^{-1}(W)$ is a fuzzy $m - s^*$-nbd of x_α.
(e) $\Rightarrow$ (b). Let V be a fuzzy m_1-open set in Y and $x_\alpha \in f^{-1}(V)$. Then $f(x_\alpha) \in V$, hence V is a fuzzy m_1-open nbd of $f(x_\alpha)$ in Y. By (e), $f^{-1}(V)$ is a fuzzy $m - s^*$-nbd of x_α. Then there exists $U \in FmS^*O(X)$ containing x_α such that $U \leq f^{-1}(V)$. Then $x_\alpha \in U \leq m - int(m - scl\,(U)) \leq m - int(m - scl(f^{-1}(V)))$, hence $f^{-1}(V) \leq m - int(m - scl(f^{-1}(V)))$.
(d) $\Rightarrow$ (f). Let $A \in I^X$. Then $m_1 - cl(f(A))$ is a fuzzy m_1-closed set in Y (as m_1 satisfies M-condition). By (d), $f^{-1}(m_1 - cl(f(A))) \in FmS^*C(X)$ containing A. Therefore, $m - s^*cl\,(A) \leq m - s^*cl(f^{-1}(m_1 - cl(f(A)))) = f^{-1}(m_1 - cl(f(A)))$, hence $f(m - s^*clA)$

$\leq m_1 - cl(f(A))$.
(f) $\Rightarrow$ (d). Let B be a fuzzy m_1-closed set in Y. Then $f^{-1}(B) \in I^X$. By (f), as m_1 satisfies M-condition we get $f(m - s^*cl(f^{-1}(B))) \leq m_1 - cl(f(f^{-1}(B))) \leq m_1 - cl\,(B) = B$, therefore $m - s^*cl(f^{-1}(B)) \leq f^{-1}(B)$, hence $f^{-1}(B) \in FmS^*C(X)$.
(f) $\Rightarrow$ (g). Let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. By (f),
$f(m - s^*cl(f^{-1}(B))) \leq m_1 - cl(f(f^{-1}(B))) \leq m_1 - cl\,(B)$, hence
$m - s^*cl(f^{-1}(B)) \leq f^{-1}(m_1 - cl\,(B))$.
(g) $\Rightarrow$ (f). Let $A \in I^X$. Let $B = f(A)$. Then $B \in I^Y$. By (g), $m - s^*cl\,(A) = m - s^*cl(f^{-1}(B)) \leq f^{-1}(m_1 - cl\,(B)) = f^{-1}(m_1 - cl(f(A)))$, therefore $f(m - s^*cl\,(A)) \leq m_1 - cl(f(A))$.
(b) $\Rightarrow$ (h). Let $B \in I^Y$. Then $m - int\,(B)$ is a fuzzy m_1-open set in Y (as m_1 satisfies M-condition). By (b), $f^{-1}(m_1 - int\,(B))$
$\leq m - int(m - scl(f^{-1}(m_1 - int\,(B))))$, hence $f^{-1}(m_1 - int\,(B)) \in FmS^*O(X)$, which implies $f^{-1}(m_1 - int\,(B)) = m - s^*int(f^{-1}(m_1 - int\,(B))) \leq m - s^*int(f^{-1}(B))$.
(h) $\Rightarrow$ (b). Let A be any fuzzy m_1-open set in Y. Then $f^{-1}(A) = f^{-1}(m_1 - int\,(A))$ (as m_1 satisfies M-condition) $\leq m - s^*int(f^{-1}(A))$ (by (h)), hence $f^{-1}(A) \in FmS^*O(X)$.

Theorem 4.5. *A function $f : (X, m) \to (Y, m_1)$ is fuzzy almost (m, m_1)-s-continuous if and only if for each fuzzy point x_α in X and each fuzzy m_1-open q-nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy $m - s^*$-open set W in X with $x_\alpha qW$ such that $f(W) \leq V$.*

Proof. Let f be fuzzy almost (m, m_1)-s-continuous function and x_α be a fuzzy point in X and V be a fuzzy m_1-open set in Y with $f(x_\alpha)qV$. Let $f(x) = y$. Then $V(y) + \alpha > 1$, i.e. $V(y) > 1 - \alpha$, hence $V(y) > \beta > 1 - \alpha$, for some real number β. Then V is a fuzzy m_1-open nbd of y_β. By Theorem 4.4 (a)$\Rightarrow$(c), there exists $W \in FmS^*O(X)$ containing x_β, i.e., $W(x) \geq \beta$ such that $f(W) \leq V$. Then $W(x) \geq \beta > 1 - \alpha$, hence $x_\alpha qW$ and $f(W) \leq V$.
Conversely, let the given condition hold and let V be a fuzzy m_1-open set in Y. Put $W = f^{-1}(V)$. If $W = 0_X$, then we are done. Suppose $W \neq 0_X$. Then for any $x \in W_0$, let $y = f(x)$. Then $W(x) = [f^{-1}(V)](x) = V(f(x)) = V(y)$. Let us choose $m \in \mathcal{N}$ where $\mathcal{N}$ is the set of all natural numbers such that $1/m \leq W(x)$. Put $\alpha_n = 1 + 1/n - W(x)$, for all $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$ and $n \geq m$, hence $1 + 1/n \leq 1 + 1/m$, therefore $\alpha_n = 1 + 1/n - W(x) \leq 1 + 1/m - W(x) \leq 1$. Again $\alpha_n > 0$, for all $n \in \mathcal{N}$, hence $0 < \alpha_n \leq 1$ so that $V(y) + \alpha_n > 1$, therefore $y_{\alpha_n} qV$ and then V is a fuzzy m_1-open q-nbd of y_{α_n}. By the given condition, there exists

$U_n^x \in FmS^*O(X)$ such that $x_{\alpha_n} q U_n^x$ and $f(U_n^x) \leq V$, for all $n \geq m$. Let $U^x = \bigvee\{U_n^x : n \in \mathcal{N}, n \geq m\}$. Then $U^x \in FmS^*O(X)$ (by Result 3.10) and $f(U^x) \leq V$. Now $n \geq m$ implies $U_n^x(x) + \alpha_n > 1$, therefore $U_n^x(x) + 1 + 1/n - W(x) > 1$, hence $U_n^x(x) > W(x) - 1/n$, therefore $U_n^x(x) \geq W(x)$, for each $x \in W_0$. Then $W \leq U_n^x$, for all $n \geq m$ and for all $x \in W_0$, hence $W \leq U^x$, for all $x \in W_0 \Rightarrow W \leq \bigvee_{x \in W_0} U^x = U$ (say) (1)

and $f(U^x) \leq V$, for all $x \in W$ implies $f(U) \leq V$, hence $U \leq f^{-1}(f(U)) \leq f^{-1}(V) = W$ (2).

By (1) and (2), $U = W = f^{-1}(V)$, therefore $f^{-1}(V) \in FmS^*O(X)$. Hence by Theorem 4.2, f is fuzzy almost (m, m_1)-s-continuous.

Remark 4.6. Let $f : (X, m) \to (Y, m_1)$ be fuzzy almost (m, m_1)-s-continuous function where m_1 satisfies M-condition. Since every fuzzy m_1-regular open set is fuzzy m_1-open set in Y, by Remark 4.3, we can easily see that the inverse image of fuzzy m_1-regular open set under fuzzy almost (m, m_1)-s-continuous function is fuzzy $m - s^*$-open set in X. But the converse may not be true, as it seen from the following example.

Example 4.7. Let $X = \{a, b\}$, $m = \{0_X, 1_X, A, B\}$, $m_1 = \{0_X, 1_X\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55$. Then (X, m) and (X, m_1) are fuzzy m-spaces. Consider the identity function $i : (X, m) \to (X, m_1)$. Clearly every fuzzy set in (X, m_1) is fuzzy m_1-s^*-open set in (X, m_1). Consider the fuzzy set C defined by $C(a) = C(b) = 0.5$. Then $C \in Fm_1S^*O(X)$. Now $i^{-1}(C) = C$ which is not fuzzy m-regular open set in (X, m), though i is clearly fuzzy almost (m, m_1)-s-continuous function.

Remark 4.8. The inverse image of a fuzzy m_1-semiopen set under fuzzy almost (m, m_1)-s-continuous function may not be fuzzy m-semiopen as well as fuzzy $m - s^*$-open, as follows from the following example.

Example 4.9. Let $X = \{a, b\}$, $m = \{0_X, 1_X, C\}$, $m_1 = \{0_X, 1_X, A, B\}$, where $A(a) = 0.5, A(b) = 0.3, B(a) = 0.5, B(b) = 0.4, C(a) = 0.5, C(b) = 0.4$. Then (X, m) and (X, m_1) are fuzzy m-spaces. Consider the identity function $i : (X, m) \to (X, m_1)$. Clearly i is a fuzzy almost $(m, m_1) - s$-continuous function. Indeed, $FmSO(X) = \{0_X, 1_X, U\}$ where $C \leq U \leq 1_X \setminus C$ and $FmSC(X) = \{0_X, 1_X, 1_X \setminus U\}$ where $C \leq 1_X \setminus U \leq 1_X \setminus C$. Then $i^{-1}(A) = A = m - int(m - scl(i^{-1}(A))) = m - int(m - scl(A)) = m - int(C) = C \geq A$, hence $A \in FmS^*O(X)$. Also $i^{-1}(B) = B$,

$m - int(m - sclB) = m - intC = C = B$, therefore $B \in FmS^*O(X)$. Consider a fuzzy set D in X defined by $D(a) = 0.5, D(b) = 0.35$. Now $m_1 - cl(m_1 - intD) = m_1 - clA = 1_X \setminus B \geq D$, hence $D \in Fm_1SO(X)$. Now $i^{-1}(D) = D$. $m - cl(m - intD) = m - cl0_X = 0_X$, therefore $D \notin FmSO(X)$. Hence the result.

Consider the fuzzy set E defined by $E(a) = E(b) = 0.5$. Then $m_1 - cl(m_1 - intE) = m_1 - clB = 1_X \setminus B \geq E$, therefore $E \in Fm_1SO(X)$. Now $i^{-1}(E) = E$, $m - int(m - sclE) = m - intE = C \not\geq E$, therefore $E \notin FmS^*O(X)$.

Remark 4.10. Composition of two fuzzy almost (m, m_1)-s-continuous functions may not be so, as it seen from the next example.

Example 4.11. Let $X = \{a, b\}$, $m = \{0_X, 1_X, A, B\}$, $m_1 = \{0_X, 1_X\}$, $m_2 = \{0_X, 1_X, C\}$ where $A(a) = 0.5, A(b) = 0.4, B(a) = 0.5, B(b) = 0.55, C(a) = 0.5, C(b) = 0.43$. Then (X, m), (X, m_1) and (X, m_2) are fuzzy m-spaces. Consider two identity functions $i_1 : (X, m) \to (X, m_1)$ and $i_2 : (X, m_1) \to (X, m_2)$. Clearly i_1 and i_2 are fuzzy almost $(m, m_1) - s$-continuous, respectively fuzzy almost (m_1, m_2)-s-continuous. Let $i_3 = i_1 \circ i_2$. Then $i_3 : (X, m) \to (X, m_2)$. Now $C \in m_2, i_3^{-1}(C) = C \not\leq m - int(m - sclC) = A$, hence $C \notin FmS^*O(X)$ and therefore i_3 is not a fuzzy almost $(m, m_2) - s$-continuous function.

5. Fuzzy Almost (m, m_1)-s^*-Continuous Function: Some Characterizations

In this section we introduce fuzzy almost (m, m_1)-s^*-continuous function which is fuzzy almost s-continuous and the converse is true only under certain condition.

Definition 5.1. A function $f : (X, m) \to (Y, m_1)$ is called fuzzy almost (m, m_1)-s^*-continuous if the inverse image of every fuzzy m_1-s^*-open set in Y is fuzzy $m - s^*$-open in X.

Theorem 5.2. *For a function $f : (X, m) \to (Y, m_1)$ where m_1 satisfies M-condition, the following statements are equivalent :*

(a) f is fuzzy almost (m, m_1)-s^*-continuous,

(b) for each fuzzy point x_α in X and each fuzzy m_1-s^*-open nbd V of $f(x_\alpha)$ in Y, there exists a fuzzy $m - s^*$-open nbd U of x_α in X such that $f(U) \leq V$,

(c) $f^{-1}(F) \in FmS^*C(X)$, for all $F \in Fm_1S^*C(Y)$,

(d) for each fuzzy point x_α in X, the inverse image under f of every fuzzy m_1-s^*-open nbd of $f(x_\alpha)$ in Y is a fuzzy $m - s^*$-open nbd of x_α in X,

(e) $f(m - s^*clA) \leq m_1 - s^*cl(f(A))$, *for all* $A \in I^X$,
(f) $m - s^*cl(f^{-1}(B)) \leq f^{-1}(m_1 - s^*cl\,(B))$, *for all* $B \in I^Y$,
(g) $f^{-1}(m_1 - s^*int\,(B)) \leq m - s^*int(f^{-1}(B))$, *for all* $B \in I^Y$.

Proof. The proof is similar to that of Theorem 4.4 and hence is omitted.

Theorem 5.3. *A function* $f : (X, m) \to (Y, m_1)$ *is fuzzy almost* $(m, m_1) - s^*$*-continuous if and only if for each fuzzy point* x_α *in* X *and corresponding to any fuzzy* $m_1 - s^*$*-open q-nbd* V *of* $f(x_\alpha)$ *in* Y, *there exists a fuzzy* $m - s^*$*-open q-nbd* W *of* x_α *in* X *such that* $f(W) \leq V$.

Proof. The proof is similar to that of Theorem 4.5 and hence is omitted.

Remark 5.4. Clearly, the composition of two fuzzy almost (m, m_1)-s^*-continuous functions is fuzzy almost $(m, m_1) - s^*$-continuous.

Theorem 5.5. *If* $f : (X, m) \to (Y, m_1)$ *is fuzzy almost* $(m, m_1) - s^*$*-continuous and* $g : (Y, m_1) \to (Z, m_2)$ *is fuzzy almost* $(m_1, m_2) - s$*-continuous, then* $g \circ f : (X, m) \to (Z, m_2)$ *is fuzzy almost* $(m, m_2) - s$*-continuous.*

Proof. Obvious.

Remark 5.6. Every fuzzy almost $(m, m_1) - s^*$-continuous function is fuzzy almost $(m, m_1) - s$-continuous, but the converse is not true, in general, as follows from the following example.

Example 5.7. There exists a fuzzy almost $(m, m_1) - s$-continuous function which is not fuzzy almost $(m, m_1) - s^*$-continuous.
Let $X = \{a, b\}$, $m = \{0_X, 1_X, A\}$, $m_1 = \{0_X, 1_X, B\}$ where $A(a) = 0.33, A(b) = 0.67, B(a) = B(b) = 0.4$. Then (X, m) and (X, m_1) are fuzzy m-spaces. Now $FmSO(X) = \{0_X, 1_X, U\}$ where $U \geq A$ and $FmSC(X) = \{0_X, 1_X, 1_X \setminus U\}$ where $1_X \setminus U \leq 1_X \setminus A$. $Fm_1SO(X) = \{0_X, 1_X, V\}$ where $B \leq V \leq 1_X \setminus B$ and $Fm_1SC(X) = \{0_X, 1_X, 1_X \setminus V\}$ where $B \leq 1_X \setminus V \leq 1_X \setminus B$. Consider the identity function $i : (X, m) \to (X, m_1)$. Now $i^{-1}(B) = B, m - int(m - scl\,(B)) = m - int 1_X = 1_X > B$, hence i is fuzzy almost (m, m_1)-s-continuous. Let D be a fuzzy set in X, defined by $D(a) = D(b) = 0.3$. Now $m_1 - int(m_1 scl\,(D)) = B > D$, hence $D \in Fm_1S^*O(X)$. Then $i^{-1}(D) = D$. But $m - int(m - scl(i^{-1}(D))) = m - int(m - scl\,(D)) = m - int\,(D) = 0_X < D$, therefore $D \notin FmS^*O(X)$, which implies that i is not fuzzy almost (m, m_1)-s^*-continuous.

To achieve the converse of Remark 5.6, we have to introduce the following concept.

Definition 5.8. A function $f : (X, m) \to (Y, m_1)$ is said to be fuzzy

(m, m_1)-semiopen if $f(U)$ is fuzzy m_1-semiopen in Y for every fuzzy m-semiopen set U in X.

Lemma 5.9. *If $f : (X, m) \to (Y, m_1)$ is a fuzzy (m, m_1)-semiopen function, then $f^{-1}(m_1 - sclU) \leq m - scl(f^{-1}(U))$, for any fuzzy set U in Y.*

Proof. Let $x_\alpha \notin m - scl(f^{-1}(U))$ for some fuzzy set U in Y. Then there exists $W \in FmSO(X)$ such that $x_\alpha qW$, $W \not{q} f^{-1}(U)$, hence $f(W) \not{q} U$. As f is fuzzy (m, m_1)-semiopen function, $f(W) \in Fm_1SO(Y)$. Now $x_\alpha qW$, hence $f(x_\alpha)qf(W)$, therefore $f(W)$ is a fuzzy m_1-semiopen q-nbd of $f(x_\alpha)$ in Y, but $f(W) \not{q} U$ implies $f(x_\alpha) \notin m_1 sclU$, whence $x_\alpha \notin f^{-1}(m_1 sclU)$.

Theorem 5.10. *If $f : (X, m) \to (Y, m_1)$ is fuzzy almost (m, m_1)-s-continuous and fuzzy (m, m_1)-semiopen function, then f is a fuzzy almost (m, m_1)-s^*-continuous function.*

Proof. Let $V \in Fm_1S^*O(Y)$. Then $V \leq m_1 - int(m_1 sclV)$. Since f is fuzzy almost (m, m_1)-s-continuous and we have Theorem 4.4 ((a)$\Leftrightarrow$ (b)), it follows that $f^{-1}(V) \leq f^{-1}(m_1 - int(m_1 - sclV)) \leq m - int(m - scl(f^{-1}(m_1 - int(m_1 - sclV)))) \leq m - int(m - scl(f^{-1}(m_1 sclV))) \leq m - int(m - scl(m - scl(f^{-1}(V))))$ (by Lemma 5.9) $= m - int(m - scl(f^{-1}(V)))$, that implies $f^{-1}(V) \in FmS^*O(X)$, therefore f is a fuzzy almost $(m, m_1) - s^*$-continuous function.

6. Fuzzy $m - s^*$-Regular Space

In this section a new type of fuzzy regularity, viz, fuzzy $m - s^*$-regularity is introduced in which fuzzy m-closed (resp., fuzzy m-open) set and fuzzy $m - s^*$-closed (resp., fuzzy $m - s^*$-open) set coincide.

Definition 6.1. A fuzzy m-space (X, m) is said to be fuzzy $m - s^*$-regular if for each fuzzy $m - s^*$-closed set F in X and each fuzzy point x_α in X with $x_\alpha q(1_X \setminus F)$, there exist a fuzzy m-open set U in X and a fuzzy $m - s^*$-open set V in X such that $x_\alpha qU$, $F \leq V$ and $U \not{q} V$.

Theorem 6.2. *For a fuzzy m-space (X, m) where m satisfies the M-condition, the following statements are equivalent:*
(a) X is fuzzy $m - s^*$-regular,
(b) for each fuzzy point x_α in X and each fuzzy $m - s^*$-open set U in X with $x_\alpha qU$, there exists a fuzzy m-open set V in X such that $x_\alpha qV \leq m - s^*clV \leq U$,
(c) for each fuzzy $m - s^*$-closed set F in X we have
$\bigcap\{m - cl\,(V) : F \leq V, V \in FmS^*O(X)\} = F$,
(d) for each fuzzy set G in X and each fuzzy $m - s^*$-open set U in X such that GqU, there exists a fuzzy m-open set V in X such that

GqV and $m - s^*clV \leq U$.

Proof (a)$\Rightarrow$(b). Let x_α be a fuzzy point in X and U, a fuzzy $m-s^*$-open set in X with $x_\alpha qU$. By (a), there exist a fuzzy m-open set V and a fuzzy $m - s^*$-open set W in X such that $x_\alpha qV$, $1_X \setminus U \leq W$, $V \not{q} W$. Then $x_\alpha qV \leq 1_X \setminus W \leq U$, hence $x_\alpha qV$ and $m - s^*cl\,(V) \leq m - s^*cl(1_X \setminus W) = 1_X \setminus W \leq U$, therefore $x_\alpha qV \leq m - s^*clV \leq U$.
(b)$\Rightarrow$(a). Let F be a fuzzy $m - s^*$-closed set in X and x_α be a fuzzy point in X with $x_\alpha q(1_X \setminus F)$. Then $1_X \setminus F \in FmS^*O(X)$. By (b), there exists a fuzzy m-open set V in X such that $x_\alpha qV \leq m-s^*clV \leq 1_X \setminus F$. Put $U = 1_X \setminus m - s^*clV$. Then $U \in FmS^*O(X)$ (as m satisfies M-condition) and $x_\alpha qV$, $F \leq U$ and $U \not{q} V$.
(b)$\Rightarrow$(c). Let F be fuzzy $m - s^*$-closed set in X. It is clear that $F \leq \bigcap\{m - cl\,(V) : F \leq V, V \in FmS^*O(X)\}$.
Conversely, let $x_\alpha \notin F$. Then $F(x) < \alpha$ implies $x_\alpha q(1_X \setminus F)$ where $1_X \setminus F \in FmS^*O(X)$. By (b), there exists a fuzzy m-open set U in X such that $x_\alpha qU \leq m-s^*clU \leq 1_X \setminus F$. Put $V = 1_X \setminus m-s^*clU$. Then $F \leq V$ and $U \not{q} V$, hence $x_\alpha \notin m - cl\,(V)$, therefore $\bigcap\{m - cl\,(V) : F \leq V, V \in FmS^*O(X)\} \leq F$.
(c)$\Rightarrow$(b). Let V be any fuzzy $m - s^*$-open set in X and x_α be any fuzzy point in X with $x_\alpha qV$. Then $V(x) + \alpha > 1$, hence $x_\alpha \notin (1_X \setminus V)$ where $1_X \setminus V \in FmS^*C(X)$. By (c), there exists $G \in FmS^*O(X)$ such that $1_X \setminus V \leq G$ and $x_\alpha \notin m - cl\,(G)$. Then there exists a fuzzy m-open set U in X with $x_\alpha qU$, $U \not{q} G$, hence $U \leq 1_X \setminus G \leq V$, therefore $x_\alpha qU \leq m - s^*clU \leq m - s^*cl(1_X \setminus G) = 1_X \setminus G \leq V$.
(c)$\Rightarrow$(d). Let G be any fuzzy set in X and U be any fuzzy $m - s^*$-open set in X with GqU. Then there exists $x \in X$ such that $G(x) + U(x) > 1$. Let $G(x) = \alpha$. Then $x_\alpha qU$ implies $x_\alpha \notin 1_X \setminus U$ where $1_X \setminus U \in FmS^*C(X)$. By (c), there exists $W \in FmS^*O(X)$ such that $1_X \setminus U \leq W$ and $x_\alpha \notin m - cl\,(W)$, hence $(m - cl\,(W))(x) < \alpha$, therefore $x_\alpha q(1_X \setminus m - cl\,(W))$. Let $V = 1_X \setminus m - cl\,(W)$. Then V is fuzzy m-open in X (as m satisfies the M-condition) and $V(x)+\alpha > 1$, hence $V(x) + G(x) > 1$, therefore VqG and
$m-s^*cl\,(V) = m-s^*cl(1_X \setminus m-clW) \leq m-s^*cl(1_X \setminus W) = 1_X \setminus W \leq U$.
(d)$\Rightarrow$(b). Obvious.

Note 6.3. It is clear from Theorem 6.2 that in a fuzzy $m - s^*$-regular space, every fuzzy $m - s^*$-closed set is fuzzy m-closed and hence every fuzzy $m - s^*$-open set is fuzzy m-open. As a result, in a fuzzy $m - s^*$-regular space, the collection of all fuzzy m-closed (resp., fuzzy m-open) sets and fuzzy $m - s^*$-closed (resp., fuzzy $m - s^*$-open) sets coincide.

Theorem 6.4. *If* $f : (X, m) \to (Y, m_1)$ *is a fuzzy almost* (m, m_1)-*s-continuous function and* Y *is a fuzzy* $m - s^*$*-regular space, then* f *is a fuzzy almost* $(m, m_1) - s^*$*-continuous function.*

Proof. Let x_α be a fuzzy point in X and V be any fuzzy m_1-s^*-open q-nbd of $f(x_\alpha)$ in Y where Y is fuzzy $m - s^*$-regular space. By Theorem 6.2 (a)⇒(b), there exists a fuzzy m_1-open set W in Y such that $f(x_\alpha)qW \leq m_1 s^* clW \leq V$. Since f is fuzzy almost (m, m_1)-s-continuous, by Theorem 4.5, there exists $U \in FmS^*O(X)$ with $x_\alpha qU$ and $f(U) \leq W \leq V$. By Theorem 5.3, f is fuzzy almost (m, m_1)-s^*-continuous function.

We recall the following definitions from [3] for ready references.

Definition 6.5 [3]. A collection $\mathcal{U}$ of fuzzy sets in a fuzzy minimal space (X, m) is said to be a fuzzy cover of X if $\bigcup \mathcal{U} = 1_X$. If, in addition, every member of $\mathcal{U}$ is fuzzy m-open, then $\mathcal{U}$ is called a fuzzy m-open cover of X.

Definition 6.6 [3]. A fuzzy cover $\mathcal{U}$ of a fuzzy minimal space (X, m) is said to have a finite subcover $\mathcal{U}_0$ if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 1_X$.

Definition 6.7 [3]. A fuzzy m-space (X, m) is said to be fuzzy almost m-compact if every fuzzy m-open cover $\mathcal{U}$ of X has a finite proximate subcover, i.e., there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{m - cl\,(U) : U \in \mathcal{U}_0\}$ is also a fuzzy cover of X.

Theorem 6.8. *If* $f : (X, m) \to (Y, m_1)$ *is a fuzzy almost* (m, m_1)-*s-continuous, surjective function and* X *is a fuzzy* $m - s^*$*-regular and fuzzy almost* m*-compact space, then* Y *is a fuzzy almost* m_1*-compact space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy m_1-open cover of Y. Then as f is a fuzzy almost $(m, m_1) - s$-continuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy $m - s^*$-open cover and hence a fuzzy m-open cover of X, as X is fuzzy $m - s^*$-regular space. Since X is fuzzy almost m-compact, there are finitely many members $U_1, U_2, ..., U_n$ of $\mathcal{U}$ such that $\bigcup_{i=1}^{n} m - cl(f^{-1}(U_i)) = 1_X$. Since X is fuzzy $m - s^*$-regular, by Theorem 6.2, $m - cl\,(A) = m - s^* cl\,(A)$ and so $1_X = \bigcup_{i=1}^{n} m - s^* cl(f^{-1}(U_i))$, hence

$$1_Y = f(\bigcup_{i=1}^{n} m - s^* cl(f^{-1}(U_i))) = \bigcup_{i=1}^{n} f(m - s^* cl(f^{-1}(U_i)))$$

$\leq$(by Theorem 4.4 (a)$\Rightarrow$(f)) $\bigcup\limits_{i=1}^{n} m_1 - cl(f(f^{-1}(U_i))) \leq \bigcup\limits_{i=1}^{n} m_1 - cl(U_i)$, hence $\bigcup\limits_{i=1}^{n} m_1 - cl(U_i) = 1_Y$, which implies that Y is a fuzzy almost m-compact space.

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