"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE FUNCTIONS RELATING TO (α, β, γ)-ORDER AND (α, β, γ)-TYPE

TANMAY BISWAS, CHINMAY BISWAS, SARMILA BHATTACHARYYA

Abstract. In this paper, we establish some growth properties of composite entire functions on the basis of their (α, β, γ)-order and (α, β, γ)-type.

1. Introduction

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f = \sum\limits_{n=0}^{+\infty} a_n z^n$ be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of f on $|z| = r$ is defined as $M_f = \max\limits_{|z|=r} |f(z)|$. We use the standard notations and definitions of the theory of entire functions which are available in [7, 8] and therefore we do not explain those in details.

First of all, let L be a class of continuous non-negative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

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Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, m (≥ 2) is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1-t) \cdot 0) \\ &\geq t\alpha(x) + (1-t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned} \alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$. Throughout the present paper we take $\alpha, \alpha_1, \alpha_2, \alpha_3 \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [3] have introduced a new concept of φ-order of entire function considering φ as subadditive function. For details one may see [3]. Later on Belaïdi et al. [1] have extended the idea and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order of an entire function f, which are as follows:

Definition 1. [1] *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of an entire function f are defined as:*

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)} \\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f)))}{\beta\left(\log(\gamma(r))\right)}. \end{aligned}$$

Remark 2. *Let $\alpha(r) = \log^{[p]} r$, $(p \geq 0)$, $\beta(r) = \log^{[q]} r$, $(q \geq 0)$ and $\gamma(r) = r$, where $\log^{[k]} x = \log(\log^{[k-1]} x)$ $(k \geq 1)$, with convention that $\log^{[0]} x = x$. If $p = 0$ and $q = 0$, i.e., $\alpha(r) = \beta(r) = r$, Definition 1* *coincides with the usual order and lower order, when* $\alpha(r) = \log^{[p-1]} r$, $(p \geq 1)$, $\beta(r) = r$, *we obtain the iterated* p*-order and iterated lower* p*-order (see* [6]*), moreover when* $\alpha(r) = \log^{[p-1]} r$ *and* $\beta(r) = \log^{[q-1]} r$, $(p \geq q \geq 1)$, *we get the* (p, q)*-order and lower* (p, q)*-order (see* [4, 5]*).*

Belaïdi et al. [2] have also introduced the definition of another growth indicator, called (α, β, γ)-type of an entire function f in the following way:

Definition 3. [2] *The* (α, β, γ)*-type denoted by* $\sigma_{(\alpha,\beta,\gamma)}[f]$, *of an entire function* f *having finite positive* (α, β, γ)*-order* $\left(0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ *is defined as:*

$$\sigma_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]}(M(r,f))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

In this line, further one may introduce the definition of (α, β, γ)*-lower type of an entire function* f *which is as follows:*

The (α, β, γ)*-lower type denoted by* $\overline{\sigma}_{(\alpha,\beta,\gamma)}[f]$ *of an entire function* f *having finite positive* (α, β, γ)*-order* $\left(0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ *is defined as:*

$$\overline{\sigma}_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]}(M(r,f))))}{(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta,\gamma)}[f] \leq \sigma_{(\alpha,\beta,\gamma)}[f] \leq +\infty$.

Analogously, to determine the relative growth of two entire functions having same non-zero finite (α, β, γ)-lower order, one can introduce the definitions of (α, β, γ)-weak type and (α, β, γ)-upper weak type of an entire function f of finite positive (α, β, γ)-lower order which are as follows:

Definition 4. *The* (α, β, γ)*-weak type denoted by* $\tau_{(\alpha,\beta,\gamma)}[f]$ *and* (α, β, γ)*-upper weak type denoted by* $\overline{\tau}_{(\alpha,\beta,\gamma)}[f]$ *of an entire function* f *having finite positive* (α, β, γ)*-lower order* $\left(0 < \lambda_{(\alpha,\beta,\gamma)}[f] < +\infty\right)$ *are defined as:*

$$\begin{aligned} \overline{\tau}_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]}(M(r,f))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}} \\ \text{and } \tau_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]}(M(r,f))))}{(\exp(\beta(\log(\gamma(r)))))^{\lambda_{(\alpha,\beta,\gamma)}[f]}}. \end{aligned}$$

It is obvious that $0 \leq \tau_{(\alpha,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha,\beta,\gamma)}[f] \leq +\infty$.

In this paper we study some growth properties relating to the composition of two entire functions on the basis of (α, β, γ)-order, (α, β, γ)-type and (α, β, γ)-weak type as compared to the growth of their corresponding left and right factors.

2. Main results

In this section, the main results of the paper are presented.

Theorem 5. *Let f and g be two entire functions such that $0 < \lambda_{(\alpha_1,\beta,\gamma)}[f(g)] \leq \rho_{(\alpha_1,\beta,\gamma)}[f(g)] < +\infty$ and $0 < \lambda_{(\alpha_2,\beta,\gamma)}[f] \leq \rho_{(\alpha_2,\beta,\gamma)}[f] < +\infty$. Then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)}\\ &\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Proof. From the definitions of $\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]$, $\rho_{(\alpha_1,\beta,\gamma)}[f(g)]$, $\lambda_{(\alpha_2,\beta,\gamma)}[f]$ and $\rho_{(\alpha_2,\beta,\gamma)}[f]$, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha_1\left(\log^{[2]}(M(r,f(g)))\right) \geqslant \left(\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon\right)\beta(\log(\gamma(r))), \tag{1}$$

$$\alpha_1\left(\log^{[2]}(M(r,f(g)))\right) \leq \left(\rho_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon\right)\beta(\log(\gamma(r))), \tag{2}$$

$$\alpha_2\left(\log^{[2]}(M(r,f))\right) \geqslant \left(\lambda_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon\right)\beta(\log(\gamma(r))) \tag{3}$$

$$\text{and } \alpha_2\left(\log^{[2]}(M(r,f))\right) \leq \left(\rho_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon\right)\beta(\log(\gamma(r))). \tag{4}$$

Again for a sequence of values of r tending to infinity,

$$\alpha_1\left(\log^{[2]}(M(r,f(g)))\right) \leq \left(\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon\right)\beta(\log(\gamma(r))), \tag{5}$$

$$\alpha_1\left(\log^{[2]}(M(r,f(g)))\right) \geqslant \left(\rho_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{6}$$

$$\alpha_2\left(\log^{[2]}(M(r,f))\right) \leq \left(\lambda_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)\beta(\log(\gamma(r))) \tag{7}$$

$$\text{and } \alpha_2\left(\log^{[2]}(M(r,f))\right) \geqslant \left(\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)\beta(\log(\gamma(r))). \tag{8}$$

Now from (1) and (4) it follows for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{9}$$

Combining (3) and (5) , we have for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{10}$$

Again from (1) and (7), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{11}$$

It follows from (2) and (3) , for all sufficiently large values of r that

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\lambda_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\lambda_{(\alpha_2,\beta,\gamma)}[f]}. \tag{12}$$

Now from (2) and (8) , it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{13}$$

Combining (4) and (6) , we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\rho_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1\left(\log^{[2]}(M(r,f(g)))\right)}{\alpha_2\left(\log^{[2]}(M(r,f))\right)} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{14}$$

Thus the theorem follows from (9) , (10) , (11), (12) , (13) and (14) . ∎

Remark 6. *If we take " $0<\lambda_{(\alpha_3,\beta,\gamma)}[g]\leq\rho_{(\alpha_3,\beta,\gamma)}[g]<+\infty$ " instead of " $0<\lambda_{(\alpha_2,\beta,\gamma)}[f]\leq\rho_{(\alpha_2,\beta,\gamma)}[f]<+\infty$ " and other conditions remain same, the conclusion of Theorem 5 remains true with "$\lambda_{(\alpha_3,\beta,\gamma)}[g]$", "$\rho_{(\alpha_3,\beta,\gamma)}[g]$" and "$\alpha_3\left(\log^{[2]}(M(r,g))\right)$" in replace of "$\lambda_{(\alpha_2,\beta,\gamma)}[f]$", "$\rho_{(\alpha_2,\beta,\gamma)}[f]$" and "$\alpha_2\left(\log^{[2]}(M(r,f))\right)$" respectively in the denominators.*

Theorem 7. *Let f and g be two non-constant entire functions such that $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f(g)] = +\infty$. Then*

$$\lim_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f(g))))}{\alpha(\log^{[2]}(M(r,f)))} = +\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log^{[2]}(M(r,f(g)))) \leq \Delta \cdot \alpha(\log^{[2]}(M(r,f))). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\alpha(\log^{[2]}(M(r,f))) \leq (\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(r))). \tag{16}$$

From (15) and (16), for a sequence of values of r tending to $+\infty$, we have

$$\alpha(\log^{[2]}(M(r,f(g)))) \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon)\beta(\log(\gamma(r))),$$

$$i.e., \ \frac{\alpha(\log^{[2]}(M(r,f(g))))}{\beta(\log(\gamma(r)))} \leq \Delta(\rho_{(\alpha,\beta,\gamma)}[f] + \epsilon),$$

$$i.e., \ \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]}(M(r,f(g))))}{\beta(\log(\gamma(r)))} < +\infty,$$

$$i.e., \ \lambda_{(\alpha,\beta,\gamma)}[f(g)] < +\infty.$$

This is a contradiction.

Thus the theorem follows. ∎

Remark 8. *If we take " $0 < \lambda_{(\alpha,\beta,\gamma)}[g] \leq \rho_{(\alpha,\beta,\gamma)}[g] < +\infty$" instead of " $0 < \lambda_{(\alpha,\beta,\gamma)}[f] \leq \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$" and other conditions remain same, the conclusion of Theorem 7 remains true with "$\alpha(\log^{[2]}(M(r,g)))$" in replace of "$\alpha(\log^{[2]}(M(r,f)))$" in the denominators.*

Remark 9. *Theorem 7 and Remark 8 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f(g)] = +\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f(g)] = +\infty$" and the other conditions remain the same.*

Theorem 10. *Let f and g be two entire functions such that $0 < \overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] \leq \sigma_{(\alpha_1,\beta,\gamma)}[f(g)] < +\infty$, $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f]$*

$< +\infty$ *and* $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$. *Then*

$$\begin{aligned}
\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]} &\le \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \\
&\le \min\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\le \max\left\{\frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\
&\le \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \le \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}.
\end{aligned}$$

Proof. From the definition of $\sigma_{(\alpha_2,\beta,\gamma)}[f]$, $\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$, $\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]$ and $\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]$, we have for arbitrary $\varepsilon(>0)$ and for all sufficiently large values of r that

$$\begin{aligned} &\exp(\alpha_1(\log^{[2]}(M(r,f(g))))) \\ \le\ &\left(\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}, \end{aligned} \tag{17}$$

$$\begin{aligned} &\exp(\alpha_1(\log^{[2]}(M(r,f(g))))) \\ \ge\ &\left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}, \end{aligned} \tag{18}$$

$$\exp(\alpha_2(\log^{[2]}(M(r,f)))) \le \left(\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{19}$$

$$\exp(\alpha_2(\log^{[2]}(M(r,f)))) \ge \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{20}$$

Again for a sequence of values of r tending to infinity, we get that

$$\begin{aligned} &\exp(\alpha_1(\log^{[2]}(M(r,f(g))))) \\ \geqslant\ &(\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}, \end{aligned} \tag{21}$$

$$\begin{aligned} &\exp(\alpha_1(\log^{[2]}(M(r,f(g))))) \\ \le\ &\left(\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_1,\beta,\gamma)}[f(g)]}, \end{aligned} \tag{22}$$

$$\exp(\alpha_2(\log^{[2]}(M(r,f)))) \le \left(\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}, \tag{23}$$

$$\exp(\alpha_2(\log^{[2]}(M(r,f)))) \ge \left(\sigma_{(\alpha_2,\beta,\gamma)}[f]-\varepsilon\right)(\exp(\beta(\log(\gamma(r)))))^{\rho_{(\alpha_2,\beta,\gamma)}[f]}. \tag{24}$$

Now from (18), (19) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \geqslant \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{25}$$

Combining (20) and (22) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{26}$$

Again from (18), (23) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, we obtain for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \geq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)] - \varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] + \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \geq \frac{\overline{\sigma}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{27}$$

In view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows from (17) and (20) for all sufficiently large values of r that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]}. \tag{28}$$

From (17), (24) and the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, it follows for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)] + \varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f] - \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))}\leq\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{29}$$

Combining (19) and (21) and in view of the condition $\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$, we get for a sequence of values of r tending to infinity that

$$\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]-\varepsilon}{\sigma_{(\alpha_2,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))}\geqslant\frac{\sigma_{(\alpha_1,\beta,\gamma)}[f(g)]}{\sigma_{(\alpha_2,\beta,\gamma)}[f]}. \tag{30}$$

Thus the theorem follows from (25) , (26) , (27), (28) , (29) and (30) . ∎

Remark 11. *If we take " $0<\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]\leq\sigma_{(\alpha_3,\beta,\gamma)}[g]<+\infty$"and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_3,\beta,\gamma)}[g]$" instead of " $0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\leq\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 10 remain true with "$\sigma_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$" and"$\exp(\alpha_3(\log^{[2]}(M(r,g))))$" instead of "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log^{[2]}(M(r,f))))$" respectively in the denominators.*

Remark 12. *If we take " $0<\tau_{(\alpha_2,\beta,\gamma)}[f]\leq\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" instead of " $0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\leq\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 10 remain true with "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" and "$\tau_{(\alpha_2,\beta,\gamma)}[f]$" in place of "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" and "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$ " respectively in the denominators.*

Remark 13. *If we take " $0<\tau_{(\alpha_3,\beta,\gamma)}[g]\leq\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]<+\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_3,\beta,\gamma)}[g]$" instead of " $0<\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]\leq\sigma_{(\alpha_2,\beta,\gamma)}[f]<+\infty$" and "$\rho_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" and other conditions remain same, the results of Theorem 10 remain true with "$\tau_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$" and "$\exp(\alpha_3(\log^{[2]}(M(r,g))))$" in place of "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$", "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" and "$\exp(\alpha_2(\log^{[2]}(M(r,f))))$" respectively in the denominators.*

Now in the line of Theorem 10 , one can easily prove the following theorem using the notions of (α,β,γ)-weak type and (α,β,γ)-upper weak type and so the proof is omitted.

Theorem 14. *Let f and g be two entire functions such that* $0 < \tau_{(\alpha_1,\beta,\gamma)}[f(g)] \leq \overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)] < +\infty$, $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$ *and* $\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$. *Then*

$$\begin{aligned}\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]} &\leq \liminf_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \\ &\leq \min\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\ &\leq \max\left\{\frac{\tau_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}, \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]}\right\} \\ &\leq \limsup_{r\to+\infty} \frac{\exp(\alpha_1(\log^{[2]}(M(r,f(g)))))}{\exp(\alpha_2(\log^{[2]}(M(r,f))))} \leq \frac{\overline{\tau}_{(\alpha_1,\beta,\gamma)}[f(g)]}{\tau_{(\alpha_2,\beta,\gamma)}[f]}.\end{aligned}$$

Remark 15. *If we take* " $0 < \tau_{(\alpha_3,\beta,\gamma)}[g] \leq \overline{\tau}_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_3,\beta,\gamma)}[g]$" *instead of* " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" *and other conditions remain same, the results of Theorem 14 remain true with* "$\tau_{(\alpha_3,\beta,\gamma)}[g]$", "$\overline{\tau}_{(\alpha_3,\beta,\gamma)}[g]$" *and* "$\exp(\alpha_3(\log^{[2]}(M(r,g))))$" *in place of* "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" *and* "$\exp(\alpha_2(\log^{[2]}(M(r,f))))$" *respectively in the denominators.*

Remark 16. *If we take* " $0 < \overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f] \leq \sigma_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_2,\beta,\gamma)}[f]$" *instead of* " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" *and other conditions remain same, the results of Theorem 14 remain true with* "$\overline{\sigma}_{(\alpha_2,\beta,\gamma)}[f]$" *and* "$\sigma_{(\alpha_2,\beta,\gamma)}[f]$" *in place of* "$\tau_{(\alpha_2,\beta,\gamma)}[f]$" *and* "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" *respectively in the denominators.*

Remark 17. *If we take* " $0 < \overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g] \leq \sigma_{(\alpha_3,\beta,\gamma)}[g] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \rho_{(\alpha_3,\beta,\gamma)}[g]$" *instead of* " $0 < \tau_{(\alpha_2,\beta,\gamma)}[f] \leq \overline{\tau}_{(\alpha_2,\beta,\gamma)}[f] < +\infty$" *and* "$\lambda_{(\alpha_1,\beta,\gamma)}[f(g)] = \lambda_{(\alpha_2,\beta,\gamma)}[f]$" *and other conditions remain same, the results of Theorem 14 remain true with* "$\overline{\sigma}_{(\alpha_3,\beta,\gamma)}[g]$", "$\sigma_{(\alpha_3,\beta,\gamma)}[g]$" *and* "$\exp(\alpha_3(\log^{[2]}(M(r,g))))$" *in place of* "$\tau_{(\alpha_2,\beta,\gamma)}[f]$", "$\overline{\tau}_{(\alpha_2,\beta,\gamma)}[f]$" *and* "$\exp(\alpha_2(\log^{[2]}(M(r,f))))$" *respectively in the denominators.*

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Tanmay Biswas
Rajbari, Rabindrapally, R. N. Tagore Road
P.O.-Krishnagar, P.S.-Kotwali, Dist-Nadia,
PIN- 741101, West Bengal, India.
e-mail: tanmaybiswas_math@rediffmail.com

Chinmay Biswas
Department of Mathematics, Nabadwip Vidyasagar College,
P.O.-Nabadwip, P.S.-Nabadwip, Dist.- Nadia,
PIN-741302, West Bengal, India.
e-mail: chinmay.shib@gmail.com

Sarmila Bhattacharyya
Department of Mathematics, Netaji Mahavidyalaya,
P.O.- Arambagh, P.S.- Arambagh, Dist.-Hooghly,
PIN-712601, West Bengal, India.
e-mail: sarmilabhattacharyya1982@gmail.com