“Vasile Alecsandri” University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 33 (2023), No. 2, 33 - 62

A STUDY ON GENERALIZED SPACE-MATTER TENSOR

SANJIB KUMAR JANA, KANAK KANTI BAISHYA, BIKIRAN DAS

Abstract. The object of the present study is to determine effect of space-matter tensor in several types of curvature restrictions on Riemannian manifolds like Einstein manifold, quasi Einstein manifold, generalized quasi Einstein manifold and pseudo generalized quasi Einstein manifold.

1. Introduction

A. Z. Petrov [12] in 1949 introduced a tensor $\mathfrak{P}$ of type $(0,4)$ satisfying the equation all the algebraic properties of the Riemannian curvature tensor. It is defined by

$$\mathfrak{P}\text{-}\frac{k}{2}G \wedge T - R + \sigma G = 0, \tag{1}$$

Keywords and phrases: Generalized Space-Matter Tensor, Quasi-Einstein Manifold, Generalized Quasi-Einstein Manifold, Scalar Curvature.
(2020) Mathematics Subject Classification: Primary: 53; Secondary: 53B50; 53C35.

where R is the Riemann curvature tensor of type (0, 4), T is the energy-momentum tensor of type (0, 2), k is a cosmological constant, σ is the energy density (scalar), G is a tensor of type $(0,4)$ given by

$$G(U_1,U_2,U_3,U_4) = G(U_1,U_4)G(U_2,U_3) - G(U_1,U_3)G(U_2,U_4)$$

for all $U_1,U_2,U_3,U_4 \in \chi(M)$, $\chi(M)$ being the Lie algebra of smooth vector fields on M and the Kulkarni-Nomizu product $u \wedge v$ of two (0, 2) tensors u and v is defined by

$$\begin{aligned}(u \wedge v)(U_1,U_2,U_3,U_4) &= u(U_1,U_4)v(U_2,U_3) + u(U_2,U_3)v(U_1,U_4) \\ &- u(U_1,U_3)v(U_2,U_4) - u(U_2,U_4)v(U_1,U_3),\end{aligned}$$

$U_i \in \chi(M)$, $i = 1,\ 2,\ 3,\ 4$. The tensor $\mathfrak{P}$ is known as the space-matter tensor of type (0, 4) of the manifold M. The most important characteristic of space matter tensor is that it satisfies all the algebraic properties of the Riemannian curvature tensor. The first part of the tensor represents the curvature of the space and the second part represents the distribution and motion of the matter.

A tensor field $\mathfrak{P}$ of type $(0,4)$ is said to be generalized space matter tensor if it satisfies the following equation

$$\mathfrak{P} - \mu_1 R - \frac{k}{2}\mu_2\, G \wedge T - \mu_3 G{=}0, \tag{2}$$

where $\mu_1,\ \mu_2,\ \mu_3$ are non-zero scalars. If we take $\mu_1 = 1$, $\mu_2 = 1$ and $\mu_3 = -\sigma$ then generalized space matter tensor becomes simply a space matter tensor.

Einstein's field equation with cosmological constant is given by

$$kT = S + \left(\lambda - \frac{R}{2}\right) G, \tag{3}$$

where λ is a cosmological constant, R is the scalar curvature and S is the Ricci tensor of type (0, 2). By virtue of (3), (2

$$\mathfrak{P} = \mu_1 R + \frac{\mu_2}{2}\, G \wedge S + \left(\mu_3 + \mu_2\lambda - \frac{\mu_2 R}{2}\right) G. \tag{4}$$

Some interesting properties of generalized space-matter tensor $\mathfrak{P}$ satisfying certain curvature conditions have discussed in section 2.

2. Preliminaries

In this section we deal with some fundamental properties of $\mathfrak{P}$ under certain curvature conditions. The Ricci tensor S of type (0, 2) and the scalar curvature R can be obtained from the curvature tensor by the following relations

$$S(X,Y) = G(QX,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i)$$

and

$$R = \sum_{i=1}^{n} S(e_i, e_i) = \sum_{i=1}^{n} G(Qe_i, e_i),$$

where $\{e_i : i = 1, 2, ..., n\}$ be an orthonormal basis of the tangent space at any point of the manifold and Q is the symmetric endomorphism corresponding to the Ricci tensor S.

Differentiating (2) covariantly and then taking cyclic sum with respect to X, Y, Z; we obtain by the view of Bianchi identity that

$$\begin{aligned}
& (\nabla_X \mathfrak{P})(Y,Z,U,V) + (\nabla_Y \mathfrak{P})(Z,X,U,V) + (\nabla_Z \mathfrak{P})(X,Y,U,V) \\
= \ & \frac{k}{2}\mu_2[\{(\nabla_X T)(Z,U) - (\nabla_Z T)(X,U)\}G(Y,V) + \{(\nabla_Y T)(X,U) \\
& -(\nabla_X T)(Y,U)\}G(Z,V) + \{(\nabla_Z T)(Y,U) - (\nabla_Y T)(Z,U)\}G(X,V) \\
& +\{(\nabla_X T)(Y,V) - (\nabla_Y T)(X,V)\}G(Z,U) + \{(\nabla_Z T)(X,V) \\
& -(\nabla_X T)(Z,V)\}G(Y,U) + \{(\nabla_Y T)(Z,V) - (\nabla_Z T)(Y,V)\}G(X,U)] \\
& +\frac{k}{2}\{d\mu_2(X)\, G \wedge T(Y,Z,U,V) + d\mu_2(Y)\, G \wedge T(Z,X,U,V) \\
& +d\mu_2(Z)\, G \wedge T(X,Y,U,V)\} + d\mu_3(X)\{G(Z,U)G(Y,V) - G(Z,V)G(Y,U)\} \\
& +d\mu_3(Y)\{G(Z,V)G(X,U) - G(Z,U)G(X,V)\} \\
& +d\mu_3(Z)\{G(Y,U)G(X,V) - G(X,U)G(Y,V)\}.
\end{aligned}$$

We now list few major results (for proof refer [21]) which will be fundamental in the entire work.

Consider a Riemannian manifold $(M^n,\ g)$ $(n > 3)$, in which the generalized space-matter tensor of type $(0,4)$ vanishes identically. Then equation (4) takes the form

$$aR + \frac{b}{2}g \wedge S + \left(c + b\lambda - \frac{br}{2}\right) G = 0. \tag{5}$$

Contractions of (5) yields

$$uS + [2(n-1)(c+b\lambda) - (n-2)br]g = 0, \tag{6}$$

$$vr + 2n(n-1)(c+b\lambda) = 0, \tag{7}$$

where where $u = 2a + (n-2)b$, $v = 2a - (n-1)(n-2)b$.

Next in a Riemannian manifold (M^n, g) $(n > 3)$ with

$$(\nabla_X \tilde{P})(Y,Z,U,V) = 0. \tag{8}$$

we have

$$\begin{aligned} &2a(\nabla_X R)(Y,Z,U,V) + b[(\nabla_X S)(Y,V)g(Z,U) \\ &+(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) \\ &-(\nabla_X S)(Z,V)g(Y,U)] + 2[dc(X) + \lambda db(X) \\ &-\frac{1}{2}\{bdr(X) + rdb(X)\}]G(Y,Z,U,V) + 2da(X)R(Y,Z,U,V) \\ &+db(X)[S(Y,V)g(Z,U) + S(Z,U)g(Y,V) \\ &-S(Y,U)g(Z,V) - S(Z,V)g(Y,U)] = 0 \end{aligned} \tag{9}$$

whose contraction gives

$$\begin{aligned} u(\nabla_X S)(Z,U) \quad &+ \quad du(X)S(Z,U) + [2(n-1)\{dc(X) + \lambda db(X)\} \\ &- \quad (n-2)\{bdr(X) + rdb(X)\}]g(Z,U) = 0. \end{aligned}$$

Further contraction and then letting $u = 0$ gives

$$dc(X) = -\lambda db(X). \tag{10}$$

$$\begin{aligned} [2a - (n-2)b]dr(Z) + 2du(QZ) \quad &+ \quad 4(n-1)[dc(Z) + \lambda db(Z)] \\ &= \quad 2(n-2)rdb(Z). \end{aligned} \tag{11}$$

$$(n-2)udr(Z) + 2ndu(QZ) = 2du(Z)r. \tag{12}$$

If r is constant then from the above relation we get that either u is also constant or

$$J_1(QX) = \frac{r}{n} J_1(X),$$

which gives

$$S(X, \tau_1) = \frac{r}{n} g(X, \tau_1),$$

where $g(X, \tau_1) = J_1(X) = du(X)$ for all vector fields X. Let us consider that a, b are constants and u is non-zero. Then we have

$$dr(X) = 0 \ \text{ for all } \ X \in \chi(M). \tag{13}$$

$$dc(X) = 0 \ \text{ for all } \ X \in \chi(M). \tag{14}$$

$$(\nabla_X S)(Z,U) = 0 \ \text{ for all } \ X,\ Z,\ U \in \chi(M). \tag{15}$$

$$\nabla R = 0. \tag{16}$$

$$(\nabla_X \tilde{P})(Y,Z,U,V) = dc(X)G(Y,Z,U,V).$$

Again if in a Riemannian manifold (M^n, g) $(n > 3)$ admitting Einstein's field equation, the generalized space-matter tensor $\tilde{P}$ of type $(0, 4)$ is recurrent then we have

$$(\nabla_X \tilde{P})(Y, Z, U, V) = L(X)\tilde{P}(Y, Z, U, V), \tag{17}$$

where L is the non-zero 1-form of recurrence such that $L(X) = g(X, \rho)$ for all vector fields X and ρ be the unit vector field associated with L. By the virtue of the above relation we obtain

$$S(Z, \rho) = \frac{r_0}{2u} g(Z, \rho), \tag{18}$$

where $r_0 = 2(n-1)(n-2)(c + \lambda b) + \{2a - (n-2)(n-3)b\}r$ and $g(Z, \rho) = L(Z)$ for all vector fields Z. Now in the view of (17), (4) converts into

$$\begin{aligned} & u(\nabla_X S)(Z, U) + du(X)S(Z, U) + [2(n-1)\{dc(X) \\ & +\lambda db(X)\} - (n-2)\{bdr(X) + rdb(X)\}]g(Z, U) \\ = \; & L(X)[uS(Z, U) + \{2(n-1)(c + \lambda b) - (n-2)br\}g(Z, U)]. \end{aligned} \tag{19}$$

Taking contraction we get

$$\begin{aligned} vdr(X) + dv(X)r \; + \; & 2n(n-1)[dc(X) + \lambda db(X)] \\ = \; & [vr + 2n(n-1)(c + \lambda b)]L(X). \end{aligned} \tag{20}$$

which reduces to

$$\begin{aligned} vdr(\rho) + dv(\rho)r \; + \; & 2n(n-1)[dc(\rho) + \lambda db(\rho)] \\ = \; & [vr + 2n(n-1)(c + \lambda b)]L(\rho). \end{aligned} \tag{21}$$

by substituting $X = \rho$.

Again $v = 0$ gives

$$L(X) = \frac{dc(X) + \lambda db(X)}{(c + \lambda b)}. \tag{22}$$

Further if $v \neq 0$ and $r,\ a,\ b,\ c$ are constants gives

$$r = -2n(n-1)\frac{(c + \lambda b)}{v}, \tag{23}$$

Also

$$\begin{aligned} & [2a - (n-2)b]dr(Z) + 2du(QZ) + 4(n-1)[dc(Z) \\ & +\lambda db(Z)] - 2(n-2)rdb(Z) \\ = \; & 2uL(QZ) + 2[2(n-1)(c + \lambda b) - (n-2)br]L(Z). \end{aligned} \tag{24}$$

By the virtue of (20) and (24) it follows that

$$(n-2)udr(Z) + 2n[du(QZ) - uL(QZ)] = 2r[du(Z) - uL(Z)]. \tag{25}$$

If r is constant then the last relation yields

$$J_2(QZ) = \frac{r}{n} J_2(Z), \tag{26}$$

which gives

$$S(Z, \tau_2) = \frac{r}{n} g(Z, \tau_2),$$

where $g(Z, \tau_2) = J_2(Z) = du(Z) - uL(Z)$ for all vector fields Z. Let us consider $r,\ a,\ b$ be constants. Then we have from (25)

$$L(QZ) = \frac{r}{n} L(Z), \tag{27}$$

which yields

$$S(Z, \rho) = \frac{r}{n} g(Z, \rho).$$

At last in a Riemannian manifold $(M^n,\ g)$ $(n > 3)$ admitting Einstein's field equation, we take the generalized space-matter tensor $\tilde{P}$ of type (0, 4) satisfying

$$\begin{aligned}(\nabla_X \tilde{P})(Y, Z, U, V) &= A(X)\tilde{P}(Y, Z, U, V) + B(Y)\tilde{P}(X, Z, U, V) \\ &+ B(Z)\tilde{P}(Y, X, U, V) + E(U)\tilde{P}(Y, Z, X, V) \\ &+ E(V)\tilde{P}(Y, Z, U, X),\end{aligned} \tag{28}$$

where A, B and E are 1-forms (not simultaneously zero) such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$, $E(X) = g(X, \rho_3)$ for all vector fields X and ρ_1, ρ_2, ρ_3 be the unit vector fields associated with A, B, E respectively. In the view of (28), the equation (**??**) takes the form

$$J_3(QX) = \frac{r_0}{2u} J_3(X) \tag{29}$$

where $g(X, \tau_3) = J_3(X)$ for all vector fields X.

Now in the view of (28), we obtain

$$
\begin{aligned}
(30)\quad & (\nabla_X S)(Z,U) + du(X)S(Z,U) + [2(n-1)\{dc(X) + \lambda db(X)\} \\
& -(n-2)\{bdr(X) + rdb(X)\}]g(Z,U) \\
= \ & A(X)[uS(Z,U) + \{2(n-1)(c+\lambda b) \\
& -(n-2)br\}g(Z,U)] + [2aR(X,Z,U,\rho_2) + b\{B(QX)g(Z,U) \\
& +B(X)S(Z,U) - B(Z)S(X,U) - B(QZ)g(X,U)\} + 2(c+\lambda b \\
& -\frac{br}{2})G(X,Z,U,\rho_2)] + B(Z)[2aS(X,U) + b\{rg(X,U) \\
& +(n-2)S(X,U)\} + 2(n-1)(c+\lambda b - \frac{br}{2})g(X,U)] \\
& +E(U)[2aS(X,Z) + b\{rg(X,Z) + (n-2)S(X,Z)\} + 2(n-1)(c+\lambda b \\
& -\frac{br}{2})g(X,Z)] + [2aR(\rho_3,Z,U,X) + b\{E(QX)g(Z,U) \\
& +E(X)S(Z,U) - E(U)S(X,Z) - E(QU)g(X,Z)\} \\
& +2(c+\lambda b - \frac{br}{2})G(\rho_3,Z,U,X)].
\end{aligned}
$$

Setting $Z = U = e_i$ in (**??**) and taking summation over i, $1 \leq i \leq n$, we find

$$
\begin{aligned}
(31)\quad vdr(X) \ & + \ rdv(X) + 2n(n-1)[dc(X) + \lambda db(X)] \\
& = \ [vr + 2n(n-1)(c+\lambda b)]A(X) + 2[uJ_4(QX) \\
& + \ \{br + 2(n-1)(c+\lambda b - \frac{br}{2})\}J_4(X)],
\end{aligned}
$$

where $J_4(X) = B(X) + E(X)$ for all vector fields X. Contracting (**??**) with respect to X and U, we have

$$
\begin{aligned}
(32)\qquad & \frac{1}{2}[2a-(n-2)b]dr(Z) + du(QZ) \\
& +2(n-1)[dc(Z) + \lambda db(Z)] - (n-2)rdb(Z) \\
= \ & uA(QZ) + [2(n-1)(c+\lambda b) - (n-2)br]A(Z) \\
& +[\{2a + (2n-3)b\}r + 2(n-1)^2(c+\lambda b - \frac{br}{2})]B(Z) \\
& -uJ_5(QZ) + [br + 2(n-1)(c+\lambda b - \frac{br}{2})]E(Z),
\end{aligned}
$$

where $J_5(Z) = B(Z) - E(Z)$ for all vector fields Z. Finally contracting (**??**) with respect to X, Z and replacing U by Z, we get

$$\begin{aligned}
(33)\qquad & \frac{1}{2}[2a-(n-2)b]dr(Z)+du(QZ) \\
& +2(n-1)[dc(Z)+\lambda db(Z)]-(n-2)rdb(Z) \\
= \; & uA(QZ)+[2(n-1)(c+\lambda b)-(n-2)br]A(Z) \\
& +uJ_5(QZ)+[br+2(n-1)(c+\lambda b-\frac{br}{2})]B(Z) \\
& +[\{2a+(2n-3)b\}r+2(n-1)^2(c+\lambda b-\frac{br}{2})]E(Z).
\end{aligned}$$

Now (32) and (33) yield

$$(34)\quad J_5(QX) = \frac{1}{u}[\{a+(n-2)b\}r+(n-1)(n-2)\{c-(r-2\lambda)\frac{b}{2}\}]J_5(X),$$

provided that u is non-zero. Which gives

$$S(X,\tau_5) = \frac{1}{u}[\{a+(n-2)b\}r+(n-1)(n-2)\{c-(r-2\lambda)\frac{b}{2}\}]g(X,\tau_5),$$

where $g(X,\tau_5) = J_5(X) = B(X) - E(X)$ for all vector fields X. And if $u = 0$, then we can obtain

$$(35)\qquad \text{either} \quad r = \frac{2(n-1)(c+\lambda b)}{(n-2)b} \quad \text{or} \quad B(X) = E(X).$$

Again we have

$$\begin{aligned}
(36)\qquad & [2a-(n-2)b]dr(Z)+2du(QZ) \\
& +4(n-1)[dc(Z)+\lambda db(Z)]-2(n-2)rdb(Z) \\
= \; & 2uA(QZ)+2[2(n-1)(c+\lambda b)-(n-2)br]A(Z) \\
& +2[\{a+(n-1)b\}r+n(n-1)(c+\lambda b-\frac{br}{2})]J_4(Z),
\end{aligned}$$

$$\begin{aligned}
(37)\quad (n-2)udr(Z) \;&+\; 2ndu(QZ)-2du(Z)r \\
&=\; 2u[nA(QZ)-rA(Z)-2J_4(QZ)] \\
&+\; 4[\{2na-(n-2)(n^2-n-4)b\}r \\
&+\; 2(n+2)(n-1)(n-2)(c+\lambda b)]J_4(Z).
\end{aligned}$$

Suppose $u = 0$. Then from the above relation we get

$$(38)\qquad \text{either} \quad r = \frac{2(n-1)(c+\lambda b)}{(n-2)b} \quad \text{or} \quad B(X) = -E(X).$$

Hence from (35) and (38) we have, if $u = 0$ then the only possible case is $r = \frac{2(n-1)(c+\lambda b)}{(n-2)b}$. Now by the virtue of (31) and (36), we have

$$\begin{aligned} & [2da(Z) - (n-2)(n-3)db(Z)]r \\ & +2(n-1)(n-3)[dc(Z) + \lambda db(Z)] \\ & -(n-2)^2 bdr(Z) - [2ar - (n-2) \\ & \cdot\{(n-3)br - 2(n-1)(c+\lambda b)\}]J_6(Z) \\ = & \ 2du(QZ) - 2uJ_6(QZ), \end{aligned} \tag{39}$$

where $J_6(Z) = A(Z) - B(Z) - E(Z)$ for all vector fields Z. If $r,\ a,\ b,\ c$ are constants and u is non-zero then the above relation becomes

$$J_6(QZ) = \frac{1}{2u}[\{2a - (n-3)(n-2)b\}r + 2(n-2)(n-1)(c+\lambda b)]J_6(Z),$$

which implies

$$S(Z, \tau_6) = \frac{1}{2u}[\{2a - (n-3)(n-2)b\}r + 2(n-2)(n-1)(c+\lambda b)]g(Z, \tau_6),$$

where $g(Z, \tau_6) = J_6(Z) = A(Z) - B(Z) - E(Z)$ for all vector fields Z. Further using (31) and (36), we can obtain

$$\begin{aligned} & [4a - n(n-2)b]dr(Z) + 2du(QZ) \\ & +[2da(Z) - (n+1)(n-2)db(Z)]r \\ & +2(n+2)(n-1)[dc(Z) + \lambda db(Z)] \\ = & \ 2uJ_7(QZ) + [2ar - (n+1)(n-2)br \\ & +2(n+2)(n-1)(c+\lambda b)]J_7(Z), \end{aligned} \tag{40}$$

where $J_7(Z) = A(Z) + B(Z) + E(Z)$ for all vector fields Z.

3. Einstein manifold with generalized space-matter tensor

In the present section we study about some interesting properties of Einstein manifold admitting generalized space-matter tensor. If a Riemannian manifold is Einstein manifold then its Ricci tensor satisfies the following condition

$$S(X, Y) = \frac{R}{n}G(X, Y) \tag{41}$$

from which it follows that

$$(\nabla_Z S)(X, Y) = 0 \text{ and } dR(X) = 0 \quad \text{for all } X,\ Y,\ Z \in \chi(M). \tag{42}$$

By the virtue of (41), we find from (18) and also from (23) that

$$R = -2n(n-1)\frac{(\mu_3 + \lambda\mu_2)}{v}, \tag{43}$$

which leads to the following:

Theorem 1. *In an Einstein manifold (M^n, G) $(n > 3)$ admitting Einstein's field equation and recurrent generalized space-matter tensor, if μ_2, μ_3 are constants then the scalar curvature is given by the relation (43), provided that either μ_1 is also constant or the energy-momentum tensor is of Codazzi type.*

Let us assume that μ_1, μ_2, μ_3 be constants then by the virtue of (41), (42) and (34) it follows that

$$R = -2n(n-1)\frac{(\mu_3 + \lambda\mu_2)}{v}. \tag{44}$$

Hence we can state the following:

Theorem 2. *In an Einstein manifold (M^n, G) $(n > 3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if μ_1, μ_2, μ_3 are constants, then always μ_1, μ_2 are connected by the relation $2\mu_1 + (n-2)\mu_2 = 0$ and the scalar curvature R is given by the relation (44).*

4. Quasi-Einstein manifold with generalized space-matter tensor

This section is concerned about Quasi-Einstein manifold admitting generalized space-matter tensor. A Riemannian manifold (M^n, G) $(n > 3)$ is said to be quasi-Einstein ([2], [4], [5], [6], [7], [8]) if its Ricci tensor S is not identically zero and satisfies the following relation

$$S = \alpha_1 G + \alpha_2 \pi \otimes \pi, \tag{45}$$

where $\alpha_1, \alpha_2 (\neq 0)$ are associated scalars and π is a non-zero 1-form which is defined by $G(X, \varsigma) = \pi(X)$ for any vector field X; ς being a unit vector field, called the generator of the manifold. Such type of manifold is denoted by $(QE)_n$ $(n > 3)$. The relation (45) implies

$$S(\varsigma, \varsigma) = \alpha_1 + \alpha_2 \quad \text{and} \quad R = n\alpha_1 + \alpha_2. \tag{46}$$

Differentiating covariantly the equation (45) with respect to X, we obtain

$$(47)\qquad \begin{aligned}(\nabla_X S)(Y,Z) &= d\alpha_1(X)G(Y,Z) + d\alpha_2(X)\pi(Y)\pi(Z)\\ &+ \alpha_2[(\nabla_X)\pi(Y)\pi(Z) + (\nabla_X)\pi(Z)\pi(Y)]\end{aligned}$$

Replacing X and Y by ς in (6) and using (46), we can obtain

$$(48)\qquad v\alpha_1 + 2\mu_1\alpha_2 + 2(n-1)(\mu_3 + \mu_2\lambda) = 0.$$

Again by (46) and (7), we have

$$(49)\qquad v(n\alpha_1 + \alpha_2) + 2n(n-1)(\mu_3 + \mu_2\lambda) = 0.$$

So from (48) and (49), we obtain $u\alpha_2 = 0$. Since α_2 is non-zero, therefore $u = 0$ i.e. $2\mu_1 + (n-2)\mu_2 = 0$. Therefore from (49) we have

$$(n-2)\mu_2(n\alpha_1 + \alpha_2) = 2(n-1)(\mu_3 + \mu_2\lambda).$$

Hence we can state the following:

Theorem 3. *In a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with vanishing generalized space-matter tensor; $\alpha_1,\ \alpha_2,\ \mu_2,\ \mu_3$ are connected by the relation $(n-2)\mu_2(n\alpha_1+\alpha_2) = 2(n-1)(\mu_3+\mu_2\lambda)$ whenever $\mu_1,\ \mu_2$ are connected by the relation $2\mu_1 + (n-2)\mu_2 = 0$.*

Let us consider $n\alpha_1 + \alpha_2$ be constant. Then by the virtue of (46) and (12) it follows that either u is also constant or

$$(50)\qquad J_1(QX) = \frac{n\alpha_1 + \alpha_2}{n} J_1(X),$$

which gives $S(X, \tau_1) = \frac{n\alpha_1+\alpha_2}{n} G(X, \tau_1)$. This leads to the following:

Theorem 4. *If in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor; $n\alpha_1 + \alpha_2$ is constant then either u is also constant or $\frac{n\alpha_1+\alpha_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_1 defined by $G(X, \tau_1) = J_1(X) = du(X)$ for all $X \in \chi(M)$.*

Again if $\mu_1,\ \mu_2$ are constants and $u \neq 0$ then also by the virtue of (46), (12) reduces to $nd\alpha_1(X) + d\alpha_2(X) = 0$ and consequently from (15) it can be shown that $d\alpha_1(X) + d\alpha_2(X) = 0$. Therefore we get the following equation

$$(51)\qquad d\alpha_1(X) = 0 \quad \text{and} \quad d\alpha_2(X) = 0.$$

Now in the view of (15) and (51), (47) takes the form

$$(52)\qquad (\nabla_X)\pi(Y)\pi(Z) + (\nabla_X)\pi(Z)\pi(Y) = 0.$$

Putting $Z = \varsigma$ in the above relation we obtain

$$(\nabla_X)\pi(Y) = 0. \tag{53}$$

Hence we can state the following:

Theorem 5. *If in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor; $\mu_1,\ \mu_2$ are constants and u is non-zero then $\alpha_1,\ \alpha_2$ and the 1-form π all are constants.*

Applying (45) in (18), we have

$$L(QX) = \frac{R_1}{2u}L(X),$$

which implies

$$S(X,\rho) = \frac{R_1}{2u}G(X,\rho),$$

where $R_1 = 2(n-1)(n-2)(\mu_3+\lambda\mu_2)+\{2\mu_1-(n-2)(n-3)\mu_2\}(n\alpha_1+\alpha_2)$. This leads to the following:

Theorem 6. *If in a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation with recurrent generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_1}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $G(X,\rho) = L(X)$ for all $X \in \chi(M)$, whenever $\mu_2,\ \mu_3$ are constants.*

Now setting $X = \rho$ in (20) and using (46), we get

$$\begin{aligned} v[nd\alpha_1(\rho) + d\alpha_2(\rho)] \;&+\; dv(\rho)(n\alpha_1 + \alpha_2) + 2n(n-1)[d\mu_3(\rho) + \lambda d\mu_2(\rho)] \\ &=\; v(n\alpha_1 + \alpha_2) + 2n(n-1)(\mu_3 + \lambda\mu_2)L(\rho). \end{aligned}$$

Therefore we gate the following:

Theorem 7. *In a $(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with recurrent generalized space-matter tensor, the generator of recurrence ρ is given by the relation* (**??**).

Suppose $n\alpha_1 + \alpha_2$ is constant. Then by the virtue of (46), (20) we get that

$$v(n\alpha_1 + \alpha_2) + 2n(n-1)(\mu_3 + \lambda\mu_2) = 0, \tag{54}$$

provided that $\mu_1,\ \mu_2,\ \mu_3$ are constants and again by the virtue of (46) and (25), we also get that

$$J_2(QX) = \frac{n\alpha_1 + \alpha_2}{n}J_2(X), \tag{55}$$

which gives $S(X,\tau_2) = \frac{n\alpha_1+\alpha_2}{n} G(X,\tau_2)$ and

$$L(QX) = \frac{n\alpha_1+\alpha_2}{n} L(X), \tag{56}$$

which implies $S(X,\rho) = \frac{n\alpha_1+\alpha_2}{n} G(X,\rho)$, provided that $\mu_1,\ \mu_2$ are constants. Thus we can state the following:

Theorem 8. *In a $(QE)_n$ $(n>3)$ admitting Einstein's field equation and recurrent generalized space-matter tensor, if $n\alpha_1+\alpha_2$ is constant then,*
(i) $\frac{n\alpha_1+\alpha_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_2, defined by $G(X,\tau_2) = J_2(X) = du(X) - uL(X)$ for all $X \in \chi(M)$;
(ii) $\frac{n\alpha_1+\alpha_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ, defined by $L(X) = G(X,\rho)$ for all $X \in \chi(M)$, whenever $\mu_1,\ \mu_2$ are constants.
(iii) $\alpha_1,\ \alpha_2,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation (54), whenever $\mu_1,\ \mu_2,\ \mu_3$ are constants.

Again applying (45) in (29), we find

$$J_3(QX) = \frac{R_1}{2u} J_3(X), \tag{57}$$

which implies

$$S(X,\tau_3) = \frac{R_1}{2u} G(X,\tau_3). \tag{58}$$

Hence we have the following:

Theorem 9. *If in a $(QE)_n$, $(n>3)$ admitting Einstein's field equation with weakly symmetric generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_1}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector τ_3, defined by $G(X,\tau_3) = J_3(X) = \mu_1(X) - 2\mu_2(X)$ for all $X \in \chi(M)$, whenever $\mu_2,\ \mu_3$ are constants.*

Now (46), (32) and (33) yield

$$\begin{aligned} J_5(QX) &= \frac{1}{u}[(n\alpha_1+\alpha_2)\{\mu_1+(n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3 - (n\alpha_1+\alpha_2-2\lambda)\frac{\mu_2}{2}\}]J_5(X), \end{aligned} \tag{59}$$

provided that u is non-zero, which in turn gives

$$\begin{aligned} S(X,\tau_5) &= \frac{1}{u}[(n\alpha_1+\alpha_2)\{\mu_1+(n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3-(n\alpha_1+\alpha_2-2\lambda)\frac{\mu_2}{2}\}]G(X,\tau_5). \end{aligned} \tag{60}$$

And if $u=0$, then

$$(n-2)(n\alpha_1+\alpha_2)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2).$$

If $n\alpha_1+\alpha_2,\ \mu_1,\ \mu_2,\ \mu_3$ are constants then by the virtue of (46), (**??**) and (40); we obtain

$$\begin{aligned} J_6(QX) &= \frac{1}{2u}[(n\alpha_1+\alpha_2)\{2\mu_1 \\ &\quad -(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]J_6(X), \end{aligned} \tag{61}$$

which implies

$$\begin{aligned} S(X,\tau_6) &= \frac{1}{2u} \\ &\quad \cdot[(n\alpha_1+\alpha_2)\{2\mu_1-(n-3)(n-2)\mu_2\} \\ &\quad +2(n-2)(n-1)(\mu_3+\lambda\mu_2)]G(X,\tau_6) \end{aligned} \tag{62}$$

and

$$\begin{aligned} J_7(QX) &= \frac{-1}{2u}[(n\alpha_1+\alpha_2)\{2\mu_1-(n+1)(n-2)\mu_2\} \\ &\quad +2(n+2)(n-1)(\mu_3+\lambda\mu_2)]J_7(X), \end{aligned} \tag{63}$$

which implies

$$\begin{aligned} S(X,\tau_7) &= -\frac{1}{2u} \\ &\quad \cdot[(n\alpha_1+\alpha_2)\,\{2\mu_1-(n+1)(n-2)\mu_2\} \\ &\quad +2(n+2)(n-1)(\mu_3+\lambda\mu_2)]G(X,\tau_7) \end{aligned} \tag{64}$$

where $J_6(X)=\mu_1(X)-\mu_2(X)-E(X)$
and
$J_7(X)=\mu_1(X)+\mu_2(X)+E(X)$
provided that u is non-zero unless it follows that

$$(n-2)(n\alpha_1+\alpha_2)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2). \tag{65}$$

This leads to the following:

Theorem 10. *In a $(QE)_n$ $(n>3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if u is zero then the scalars $\alpha_1,\ \alpha_2,\ \mu_2,\ \mu_3$ are connected by the relation $(n-$*

$2)(n\alpha_1+\alpha_2)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2)$ *otherwise,*
(i) $\frac{1}{u}[(n\alpha_1+\alpha_2)\{\mu_1+(n-2)\mu_2\}+(n-1)(n-2)\{\mu_3-(n\alpha_1+\alpha_2-2\lambda)\frac{\mu_2}{2}\}]$ *is an eigen value of the Ricci tensor* S *corresponding to the eigen vector* τ_5*, defined by* $G(X,\tau_5) = J_5(X) = \mu_2(X) - E(X)$ *for all* $X \in \chi(M)$*;*
(ii) $\frac{1}{2u}[(n\alpha_1+\alpha_2)\{2\mu_1-(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]$ *and* $\frac{-1}{2u}[(n\alpha_1+\alpha_2)\{2\mu_1-(n+1)(n-2)\mu_2\}+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]$ *are eigen values of the Ricci tensor* S *corresponding to the eigen vector* τ_6*, defined by* $G(X,\tau_6) = J_6(X) = \mu_1(X) - \mu_2(X) - E(X)$ *for all* $X \in \chi(M)$ *and the eigen vector* τ_7*, defined by* $G(X,\tau_7) = J_7(X) = \mu_1(X)+\mu_2(X)+E(X)$ *for all* $X \in \chi(M)$ *respectively, whenever* $n\alpha_1+\alpha_2,\ \mu_1,\ \mu_2,\ \mu_3$ *are constants.*

5. Generalized quasi-Einstein manifold(U. C. De) with generalized space-matter tensor

This section deals with the study of generalized quasi-Einstein manifold admitting generalized space-matter tensor. A Riemannian manifold (M^n, G) $(n > 3)$ is said to be generalized quasi-Einstein manifold [] if its Ricci tensor is not identically zero and satisfies the following relation

$$S(X,Y) = \beta_1 G(X,Y) + \beta_2\mu(X)\mu(Y) + \beta_3\varphi(X)\varphi(Y). \tag{66}$$

where $\beta_1,\ \beta_2(\neq 0),\ \beta_3(\neq 0)$ are nonzero scalars and μ, φ are non-zero 1-forms such that $\mu(X) = G(X,\varsigma_1)$, $\varphi(X) = G(X,\varsigma_2)$ for all vector fields X and ς_1, ς_2 are the unit vector fields. This type of manifold of dimension n is denoted by $G(QE)_n$.
From (66) we have
(67)
$$S(\varsigma_1,\varsigma_1) = \beta_1+\beta_2,\ S(\varsigma_2,\varsigma_2) = \beta_1+\beta_3,\ S(\varsigma_1,\varsigma_2) = 0;\ R = n\beta_1+\beta_2+\beta_3.$$

By the virtue of (66), (5) becomes to the following equation

$$2\mu_1 R = [\mu_2 R - 2(\lambda\mu_2+\mu_3+\beta_1)]G - \mu_2(\beta_2 G\wedge\mu' + \beta_3 G\wedge\varphi'), \tag{68}$$

where $\mu'(X,Y) = \mu(X)\mu(Y)$ and $\varphi'(X,Y) = \varphi(X)\varphi(Y)$. The contraction of (68) gives

$$\begin{aligned} 2\mu_1 S(Z,U) &= [(n-1)\{\mu_2 R - 2(\lambda\mu_2+\mu_3+\beta_1)\} - \mu_2(\beta_2+\beta_3)]G(Z,U) \\ &- (n-2)\mu_2[\beta_2\mu'(Z,U) + \beta_3\varphi'(Z,U). \end{aligned}$$

Setting $Z = U = \varsigma_1$ in, we get
(69)
$$2\mu_1 S(\varsigma_1,\varsigma_1) = (n-1)\{\mu_2 R - 2(\lambda\mu_2+\mu_3+\beta_1)\} - \mu_2(\beta_2+\beta_3) - (n-2)\mu_2\beta_2.$$

Further setting $Z = U = \varsigma_2$ we have

(70)
$$2\mu_1 S(\varsigma_2,\varsigma_2) = (n-1)\{\mu_2 R - 2(\lambda\mu_2+\mu_3+\beta_1)\} - \mu_2(\beta_2+\beta_3) - (n-2)\mu_2\beta_3.$$

In the view of (67), (69) and (70) it follows that

$$S(\varsigma_1,\varsigma_1) = \beta_1+\beta_2 = S(\varsigma_2,\varsigma_2), \text{ provided that } u \neq 0 \tag{71}$$

This gives the following:

Theorem 11. *In a $G(QE)_n$ $(n > 3)$ manifold admitting Einstein's field equation and with vanishing generalized space matter tensor, the scalar $\beta_1+\beta_2$ is the Ricci curvature in the directions of both the generators ς_1 and ς_2, whenever $u \neq 0$.*

Differentiating covariantly the equation (66) with respect to X, we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) &= d\beta_1(X)G(Y,Z) + d\beta_2(X)\mu(Y)\mu(Z) \\ &+ d\beta_3(X)\varphi(Y)\varphi(Z) \\ &+ \beta_2[(\nabla_X)\mu(Y)\mu(Z) + (\nabla_X)\mu(Z)\mu(Y)] \\ &+ \beta_3[(\nabla_X)\varphi(Y)\varphi(Z) + (\nabla_X)\varphi(Z)\varphi(Y)].\end{aligned} \tag{72}$$

By the virtue of (67), (7) takes the form

$$v(n\beta_1+\beta_2+\beta_3) + 2n(n-1)(\mu_3+\mu_2\lambda) = 0. \tag{73}$$

Hence we have the following:

Theorem 12. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with vanishing generalized space-matter tensor; $\beta_1,\ \beta_2,\ \beta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation (73).*

Suppose $n\beta_1+\beta_2+\beta_3$ is constant. Then using the equations (67) and (12), we find that either u is also constant or

$$J_1(QX) = \frac{n\beta_1+\beta_2+\beta_3}{n} J_1(X),$$

which gives $S(X,\tau_1) = \frac{n\beta_1+\beta_2+\beta_3}{n} G(X,\tau_1)$. This leads to the following:

Theorem 13. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor, the scalar curvature $n\beta_1+\beta_2+\beta_3$ is constant then either u is also constant or $\frac{n\beta_1+\beta_2+\beta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_1, defined by $G(X,\tau_1) = J_1(X) = du(X)$ for all vector fields X.*

Applying (67) in (18), we have

$$L(QX) = \frac{R_2}{2u} L(X),$$

which implies

$$S(X, \rho) = \frac{R_2}{2u} G(X, \rho),$$

where $R_2 = 2(n-1)(n-2)(\mu_3 + \lambda\mu_2) + \{2\mu_1 - (n-2)(n-3)\mu_2\}(n\beta_1 + \beta_2 + \beta_3)$. Thus we can state the following:

Theorem 14. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation with recurrent generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_2}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $G(X, \rho) = L(X)$ for all vector fields X, whenever μ_2, μ_3 are constants.*

Now setting $X = \rho$ in (20) and using (67), (72); we have

$$\begin{aligned} (74) \qquad & v[nd\beta_1(\rho) + d\beta_2(\rho) + d\beta_3(\rho)] + 2n(n-1)[d\mu_3(\rho) + \lambda d\mu_2(\rho)] \\ = \quad & \{v - dv(\rho)\}(n\beta_1 + \beta_2 + \beta_3) + 2n(n-1)(\mu_3 + \lambda\mu_2). \end{aligned}$$

Hence we have the following:

Theorem 15. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with recurrent generalized space-matter tensor, the generator of recurrence ρ is given by the relation* (74).

Let us consider $n\beta_1 + \beta_2 + \beta_3$ be constant. Then by the virtue of (67), (20) reduces to

$$(75) \qquad v(n\beta_1 + \beta_2 + \beta_3) + 2n(n-1)(\mu_3 + \lambda\mu_2) = 0,$$

provided that μ_1, μ_2, μ_3 are constants and by the virtue of (67), (25) it also reduces to

$$J_2(QX) = \frac{n\beta_1 + \beta_2 + \beta_3}{n} J_2(X),$$

which gives $S(X, \tau_2) = \frac{n\beta_1+\beta_2+\beta_3}{n} G(X, \tau_2)$ and

$$L(QX) = \frac{n\beta_1 + \beta_2 + \beta_3}{n} L(X),$$

which implies $S(X, \rho) = \frac{n\beta_1+\beta_2+\beta_3}{n} G(X, \rho)$, provided that μ_1, μ_2 are constants. Thus we can state the following:

Theorem 16. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent generalized space-matter tensor, if $n\beta_1 + \beta_2 + \beta_3$ is constant then,*
(i) $\frac{n\beta_1+\beta_2+\beta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_2, defined by $G(X, \tau_2) = J_2(X) = du(X) - uL(X)$ for all $X \in \chi(M)$;
(ii) $\frac{n\beta_1+\beta_2+\beta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ, defined by $L(X) = G(X, \rho)$ for all $X \in \chi(M)$, whenever $\mu_1,\ \mu_2$ are constants;
(iii) $\beta_1,\ \beta_2,\ \beta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation (75), whenever $\mu_1,\ \mu_2,\ \mu_3$ are constants.

Again applying (67) in (29), we get

$$J_3(QX) = \frac{R_2}{2u} J_3(X),$$

which implies

$$S(X, \tau_3) = \frac{R_2}{2u} G(X, \tau_3).$$

This gives the following:

Theorem 17. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation with weakly symmetric generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_2}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector τ_3, defined by $G(X, \tau_3) = J_3(X) = \mu_1(X) - 2\mu_2(X)$ for all $X \in \chi(M)$, whenever $\mu_2,\ \mu_3$ are constants.*

Now in the view of (67), (32) and (33) it follows that

$$\begin{aligned} J_5(QX) &= \frac{1}{u}[(n\beta_1 + \beta_2 + \beta_3)\{\mu_1 + (n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3 - (n\beta_1 + \beta_2 + \beta_3 - 2\lambda)\frac{\mu_2}{2}\}]J_5(X), \end{aligned}$$

which gives

$$\begin{aligned} S(X, \tau_5) &= \frac{1}{u}[(n\beta_1 + \beta_2 + \beta_3)\{\mu_1 + (n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3 - (n\beta_1 + \beta_2 + \beta_3 - 2\lambda)\frac{\mu_2}{2}\}]G(X, \tau_5), \end{aligned}$$

provided that u is non-zero. And if $u = 0$, then we can obtain

$$(n-2)(n\beta_1 + \beta_2 + \beta_3)\mu_2 = 2(n-1)(\mu_3 + \lambda\mu_2).$$

If $\beta_1,\ \beta_2,\ \beta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are constants then by the virtue of (67), (**??**) and (40); we have

$$\begin{aligned} J_6(QX) &= \frac{1}{2u}[(n\beta_1+\beta_2+\beta_3)\{2\mu_1-(n-3)(n-2)\mu_2\} \\ &\quad +2(n-2)(n-1)(\mu_3+\lambda\mu_2)]J_6(X), \end{aligned} \tag{76}$$

which implies

$$\begin{aligned} S(X,\tau_6) &= \frac{1}{2u} \\ &\quad \cdot[(n\beta_1+\beta_2+\beta_3) \\ &\quad \{2\mu_1-(n-3)(n-2)\mu_2\} \\ &\quad +2(n-2)(n-1)(\mu_3+\lambda\mu_2)]G(X,\tau_6) \end{aligned} \tag{77}$$

and

$$\begin{aligned} J_7(QX) &= \frac{-1}{2u}[(n\beta_1+\beta_2+\beta_3)\{2\mu_1-(n+1)(n-2)\mu_2\} \\ &\quad +2(n+2)(n-1)(\mu_3+\lambda\mu_2)]J_7(X), \end{aligned} \tag{78}$$

which implies

$$\begin{aligned} S(X,\tau_7) &= \frac{-1}{2u} \\ &\quad \cdot[(n\beta_1+\beta_2+\beta_3)\{2\mu_1-(n+1)(n-2)\mu_2\} \\ &\quad fr+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]G(X,\tau_7). \end{aligned} \tag{79}$$

And if $u=0$, then we have

$$(n-2)(n\beta_1+\beta_2+\beta_3)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2).$$

Thus we have the following:

Theorem 18. *In a $G(QE)_n$ $(n>3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if u is zero then $\beta_1,\ \beta_2,\ \beta_3,\ \mu_2,\ \mu_3$ are connected by the relation $(n-2)(n\beta_1+\beta_2+\beta_3)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2)$ otherwise,*
(i) $\frac{1}{u}[(n\beta_1+\beta_2+\beta_3)\{\mu_1+(n-2)\mu_2\}+(n-1)(n-2)\{\mu_3-(n\beta_1+\beta_2+\beta_3-2\lambda)\frac{\mu_2}{2}\}]$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_5, defined by $G(X,\tau_5)=J_5(X)=\mu_2(X)-E(X)$ for all $X\in\chi(M)$;
(ii) $\frac{1}{2u}[(n\beta_1+\beta_2+\beta_3)\{2\mu_1-(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]$ and $\frac{-1}{2u}[(n\beta_1+\beta_2+\beta_3)\{2\mu_1-(n+1)(n-2)\mu_2\}+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]$ are eigen values of the Ricci tensor S corresponding to the eigen vector τ_6, defined by $G(X,\tau_6)=J_6(X)=\mu_1(X)-\mu_2(X)-E(X)$ for all $X\in\chi(M)$ and the eigen vector τ_7, defined by $G(X,\tau_7)=J_7(X)=$

$\mu_1(X) + \mu_2(X) + E(X)$ for all $X \in \chi(M)$ respectively, whenever $n\beta_1 + \beta_2 + \beta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are constants.

6. Generalized quasi-Einstein manifold(M. C. Chaki) with generalized space-matter tensor

In this section we study about generalized quasi-Einstein manifold, which is introduced by M. C. Chaki [3], admitting generalized space-matter tensor. A Riemannian manifold $(M^n,\ G)$ $(n > 3)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor is not identically zero and satisfies the following condition

$$S(X, Y) = \gamma_1 G(X, Y) + \gamma_2 \vartheta(X)\vartheta(Y) + \gamma_3[\vartheta(X)\nu(Y) + \vartheta(Y)\nu(X)], \tag{80}$$

where $\gamma_1,\ \gamma_2(\neq 0),\ \gamma_3$ are nonzero scalars and ϑ, ν are non-zero 1-forms such that $\vartheta(X) = G(X, \varsigma_3)$, $\nu(X) = G(X, \varsigma_4)$ for all vector fields X and ς_3, ς_4 are the unit vector fields. This type of manifold of dimension n is denoted by $G(QE)_n$.
Now (80) gives

$$S(\varsigma_3, \varsigma_3) = \gamma_1 + \gamma_2,\ S(\varsigma_4, \varsigma_4) = \gamma_1,\ S(\varsigma_3, \varsigma_4) = \gamma_3;\ R = n\gamma_1 + \gamma_2. \tag{81}$$

By the equation (80), (5) takes the form

$$2\mu_1 R = [\mu_2 R - 2(\lambda\mu_2 + \mu_3 + \gamma_1)]G - \mu_2(\gamma_2 G \wedge \vartheta' + \gamma_3 G \wedge \nu'), \tag{82}$$

where $\vartheta'(X, Y) = \vartheta(X)\vartheta(Y)$ and $\nu'(X, Y) = \vartheta(X)\nu(Y) + \vartheta(Y)\nu(X)$.
Contracting (82), we obtain

$$\begin{aligned} 2\mu_1 S(Z, U) &= [(n-1)\{\mu_2 R - 2(\lambda\mu_2 + \mu_3 + \gamma_1)\} - \mu_2(\gamma_2 + \gamma_3)]G(Z, U) \\ &- (n-2)\mu_2[\gamma_2\vartheta'(Z, U) + \gamma_3\nu'(Z, U). \end{aligned}$$

Replacing Z and U by ς_3, we have

$$2\mu_1 S(\varsigma_3, \varsigma_3) = (n-1)\{\mu_2 R - 2(\lambda\mu_2 + \mu_3 + \gamma_1)\} - \mu_2(\gamma_2 + \gamma_3) - (n-2)\mu_2\gamma_2. \tag{83}$$

Again replacing Z and U by ς_4 , we have

$$2\mu_1 S(\varsigma_4, \varsigma_4) = (n-1)\{\mu_2 R - 2(\lambda\mu_2 + \mu_3 + \gamma_1)\} - \mu_2(\gamma_2 + \gamma_3). \tag{84}$$

Now using (81) in (83) and (84), we find that $u = 0$ i.e. $2\mu_1 = -(n-2)\mu_2$, since $\gamma_3 \neq 0$. Thus we can state the following:

Theorem 19. *In a $G(QE)_n$ $(n > 3)$ manifold admitting Einstein's field equation and with vanishing generalized space matter tensor; $\mu_1,\ \mu_2$ are always connected by the relation $2\mu_1 = -(n-2)\mu_2$.*

Further we obtain by covariant differentiation of the equation (80) with respect to X

$$
\begin{aligned}
(\nabla_X S)(Y,Z) &= d\gamma_1(X)G(Y,Z) + d\gamma_2(X)\vartheta(Y)\vartheta(Z) \\
&+ d\gamma_3(X)\nu(Y)\nu(Z) \\
&+ \gamma_2[(\nabla_X)\vartheta(Y)\vartheta(Z) + (\nabla_X)\vartheta(Z)\vartheta(Y)] \\
&+ \gamma_3[(\nabla_X)\vartheta(Y)\nu(Z) + (\nabla_X)\vartheta(Z)\nu(Y) \\
&+ (\nabla_X)\nu(Y)\vartheta(Z) + (\nabla_X)\nu(Z)\vartheta(Y)].
\end{aligned} \tag{85}
$$

In the view of (81), (7) takes the form

$$v(n\gamma_1 + \gamma_2) + 2n(n-1)(\mu_3 + \mu_2\lambda) = 0. \tag{86}$$

Hence we have the following:

Theorem 20. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with vanishing generalized space-matter tensor; $\gamma_1, \gamma_2, \mu_1, \mu_2, \mu_3$ are connected by the relation* (86).

Suppose $n\gamma_1 + \gamma_2$ is constant. Then in the view of (81) and (12) it follows that either u is also constant or

$$J_1(QX) = \frac{n\gamma_1 + \gamma_2}{n} J_1(X),$$

which gives $S(X,\tau_1) = \frac{n\gamma_1+\gamma_2}{n}G(X,\tau_1)$. This leads to the following:

Theorem 21. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor; $n\gamma_1 + \gamma_2$ is constant then either u is also constant or $\frac{n\gamma_1+\gamma_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_1 defined by $G(X,\tau_1) = J_1(X) = du(X)$ for all $X \in \chi(M)$.*

By the virtue of (81), (18) reduces to the following relation

$$L(QX) = \frac{R_3}{2u} L(X),$$

which implies

$$S(X,\rho) = \frac{R_3}{2u} G(X,\rho),$$

where $R_3 = 2(n-1)(n-2)(\mu_3 + \lambda\mu_2) + \{2\mu_1 - (n-2)(n-3)\mu_2\}(n\gamma_1 + \gamma_2)$. This gives the following:

Theorem 22. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation with recurrent generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_3}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $G(X,\rho) = L(X)$ for all $X \in \chi(M)$, whenever μ_2, μ_3 are constants.*

Again setting $X = \rho$ in (20) and using (81), (85); we have

$$\begin{aligned} (87) \qquad & v[nd\gamma_1(\rho) + d\gamma_2(\rho)] + 2n(n-1)[d\mu_3(\rho) + \lambda d\mu_2(\rho)] \\ = \ & \{v - dv(\rho)\}(n\gamma_1 + \gamma_2) + 2n(n-1)(\mu_3 + \lambda\mu_2), \end{aligned}$$

which leads to the following:

Theorem 23. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with recurrent generalized space-matter tensor, the generator of recurrence ρ is given by the relation* (87).

Now let $n\gamma_1 + \gamma_2$ be constant. Then by the application of (81), (20) we have

$$(88) \qquad v(n\gamma_1 + \gamma_2) + 2n(n-1)(\mu_3 + \lambda\mu_2) = 0,$$

provided that $\mu_1,\ \mu_2,\ \mu_3$ are constants and again by the application of (81), (25) we also have

$$J_2(QX) = \frac{n\gamma_1 + \gamma_2}{n} J_2(X),$$

which gives $S(X, \tau_2) = \frac{n\gamma_1+\gamma_2}{n} G(X, \tau_2)$ and

$$L(QX) = \frac{n\gamma_1 + \gamma_2}{n} L(X),$$

which gives $S(X, \rho) = \frac{n\gamma_1+\gamma_2}{n} G(X, \rho)$, provided that $\mu_1,\ \mu_2$ are constants. Thus we can state the following:

Theorem 24. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent generalized space-matter tensor, if $n\gamma_1 + \gamma_2$ is constant then,*
(i) $\frac{n\gamma_1+\gamma_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_2, defined by $G(X, \tau_2) = J_2(X) = du(X) - uL(X)$ for all $X \in \chi(M)$;
(ii) $\frac{n\gamma_1+\gamma_2}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ, defined by $L(X) = G(X, \rho)$ for all $X \in \chi(M)$, whenever $\mu_1,\ \mu_2$ are constants;
(iii) $\gamma_1,\ \gamma_2,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation (88), *whenever $\mu_1,\ \mu_2,\ \mu_3$ are constants.*

In the view of (81), (29) yields

$$J_3(QX) = \frac{R_3}{2u} J_3(X),$$

which implies

$$S(X, \tau_3) = \frac{R_3}{2u} G(X, \tau_3).$$

Hence we have the following:

Theorem 25. *If in a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation with weakly symmetric generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_3}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector τ_3, defined by $G(X, \tau_3) = J_3(X) = \mu_1(X) - 2\mu_2(X)$ for all $X \in \chi(M)$, whenever μ_2, μ_3 are constants.*

By the relations (81), (32) and (33); we get

$$\begin{aligned} J_5(QX) &= \frac{1}{u}[(n\gamma_1 + \gamma_2)\{\mu_1 + (n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3 - (n\gamma_1 + \gamma_2 - 2\lambda)\frac{\mu_2}{2}\}]J_5(X), \end{aligned} \tag{89}$$

which implies

$$\begin{aligned} S(X, \tau_5) &= \frac{1}{u}[(n\gamma_1 + \gamma_2)\{\mu_1 + (n-2)\mu_2\} \\ &\quad +(n-1)(n-2)\{\mu_3 - (n\gamma_1 + \gamma_2 - 2\lambda)\frac{\mu_2}{2}\}]G(X, \tau_5), \end{aligned} \tag{90}$$

provided that u is non-zero, otherwise

$$(n-2)(n\gamma_1 + \gamma_2)\mu_2 = 2(n-1)(\mu_3 + \lambda\mu_2).$$

Again if $n\gamma_1 + \gamma_2$, μ_1, μ_2, μ_3 are constants and u is non-zero then using (81) we also obtain

$$\begin{aligned} J_6(QX) &= \frac{1}{2u}[(n\gamma_1 + \gamma_2)\{2\mu_1 - (n-3)(n-2)\mu_2\} \\ &\quad +2(n-2)(n-1)(\mu_3 + \lambda\mu_2)]J_6(X), \end{aligned} \tag{91}$$

which implies

$$\begin{aligned} &S(X, \tau_6) = \frac{1}{2u} \\ &\cdot[(n\gamma_1 + \gamma_2)\{2\mu_1 - (n-3)(n-2)\mu_2\} + 2(n-2)(n-1) \\ &(\mu_3 + \lambda\mu_2)]G(X, \tau_6) \end{aligned} \tag{92}$$

and

$$\begin{aligned} J_7(QX) &= \frac{-1}{2u}[(n\gamma_1 + \gamma_2)\{2\mu_1 - (n+1)(n-2)\mu_2\} \\ &\quad +2(n+2)(n-1)(\mu_3 + \lambda\mu_2)]J_7(X), \end{aligned} \tag{93}$$

which implies

$$S(X,\tau_7) = \frac{-1}{2u} \cdot [(n\gamma_1 + \gamma_2)\{2\mu_1 - (n+1)(n-2)\mu_2\} +2(n+2)(n-1)(\mu_3 + \lambda\mu_2)]G(X,\tau_7). \tag{94}$$

And if $u = 0$, then we get

$$n\gamma_1 + \gamma_2 = \frac{2(n-1)(n-2)(\mu_3 + \lambda\mu_2)}{(n-2)(n-3)\mu_2 - 2\mu_1} = \frac{2(n-1)(\mu_3 + \lambda\mu_2)}{(n-2)\mu_2}.$$

Hence we have the following:

Theorem 26. *In a $G(QE)_n$ $(n > 3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if u is zero then γ_1, γ_2, μ_2, μ_3 are connected by the relation $(n-2)(n\gamma_1+\gamma_2)\mu_2 = 2(n-1)(\mu_3+\lambda\mu_2)$ otherwise,*
(i) $\frac{1}{u}[(n\gamma_1+\gamma_2)\{\mu_1+(n-2)\mu_2\}+(n-1)(n-2)\{\mu_3-(n\gamma_1+\gamma_2-2\lambda)\frac{\mu_2}{2}\}]$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_5, defined by $G(X,\tau_5) = J_5(X) = \mu_2(X) - E(X)$ for all $X \in \chi(M)$;
(ii) $\frac{1}{2u}[(n\gamma_1+\gamma_2)\{2\mu_1-(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]$ and $\frac{-1}{2u}[(n\gamma_1+\gamma_2)\{2\mu_1-(n+1)(n-2)\mu_2\}+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]$ are eigen values of the Ricci tensor S corresponding to the eigen vector τ_6, defined by $G(X,\tau_6) = J_6(X) = \mu_1(X) - \mu_2(X) - E(X)$ for all $X \in \chi(M)$ and the eigen vector τ_7, defined by $G(X,\tau_7) = J_7(X) = \mu_1(X) + \mu_2(X) + E(X)$ for all $X \in \chi(M)$ respectively, whenever $n\gamma_1+\gamma_2$, μ_1, μ_2, μ_3 are constants.

7. Pseudo generalized quasi-Einstein manifold with generalized space-matter tensor

This section concerned about the study of pseudo generalized quasi-Einstein manifold admitting generalized space-matter tensor. A Riemannian manifold (M^n, G) $(n > 3)$ is said to be pseudo generalized quasi-Einstein manifold [14] if its Ricci tensor is not identically zero and satisfies the following relation

$$S(X,Y) = \delta_1 G(X,Y) + \delta_2 H(X)H(Y) + \delta_3 F(X)F(Y) + \delta_4 D(X,Y), \tag{95}$$

where δ_1, δ_2, δ_3, δ_4 are nonzero scalars and H, F are non-zero 1-forms such that $H(X) = G(X,\varsigma_5)$, $F(X) = G(X,\varsigma_6)$ for all vector fields X and ς_5, ς_6 are the unit vector fields; D is a symmetric (0, 2) tensor, with zero trace, which satisfies the condition $D(X,\varsigma_5) = 0$ for all vector fields X. This type of manifold is denoted by $\mathfrak{P}(GQE)_n$.

Some types of recurrence can also be found in [17], [18], [19] and [20]. Now we get from (95)

$$\left\{ \begin{array}{c} S(\varsigma_5,\varsigma_5)=\delta_1+\delta_2, \quad S(\varsigma_6,\varsigma_6)=\delta_1+\delta_3+D(\varsigma_6,\varsigma_6), \\ S(\varsigma_5,\varsigma_6)=0; \; R=n\delta_1+\delta_2+\delta_3. \end{array} \right. \tag{96}$$

In the view of (95), (5) reduces to the following equation

$$2\mu_1 R=[\mu_2 R-2(\lambda\mu_2+\mu_3+\delta_1)]G-\mu_2(\delta_2 G\wedge H'+\delta_3 G\wedge F'+\delta_4 G\wedge D), \tag{97}$$

where $H'(X,Y)=H(X)H(Y)$ and $F'(X,Y)=F(X)F(Y)$. Taking contraction we have

$$\begin{aligned} 2\mu_1 S(Z,U) &= [(n-1)\{\mu_2 R-2(\lambda\mu_2+\mu_3+\delta_1)\}-\mu_2(\delta_2+\delta_3)]G(Z,U) \\ &- (n-2)\mu_2[\delta_2 H'(Z,U)+\delta_3 F'(Z,U)+\delta D(Z,U)]. \end{aligned}$$

Let us set $Z=U=\varsigma_5$ to get

$$2\mu_1 S(\varsigma_5,\varsigma_5)=(n-1)\{\mu_2 R-2(\lambda\mu_2+\mu_3+\delta_1)\}-\mu_2(\delta_2+\delta_3)-(n-2)\mu_2\delta_2. \tag{98}$$

Further setting $Z=U=\varsigma_6$ we also get

$$2\mu_1 S(\varsigma_6,\varsigma_6)=(n-1)\{\mu_2 R-2(\lambda\mu_2+\mu_3+\delta_1)\}-\mu_2(\delta_2+\delta_3)-(n-2)\mu_2[\delta_3+\delta D(\varsigma_6,\varsigma_6)]. \tag{99}$$

By the virtue of (96), (98) and (99); we have

$$S(\varsigma_5,\varsigma_5)=\delta_1+\delta_2=S(\varsigma_6,\varsigma_6), \text{ provided that } u\neq 0$$

This leads to the following:

Theorem 27. *In a $\mathfrak{P}(GQE)_n$ $(n>3)$ manifold admitting Einstein's field equation and with vanishing space matter tensor, the scalar $\delta_1+\delta_2$ is the Ricci curvature in the directions of both the generators ς_5 and ς_6, whenever $u\neq 0$.*

Taking covariant differentiation of the equation (95) with respect to X, we obtain

$$\begin{aligned} (\nabla_X S)(Y,Z) &= d\delta_1(X)G(Y,Z)+d\delta_2(X)H(Y)H(Z) \\ &+ d\delta_3(X)F(Y)F(Z)+d\delta_4 D(Y,Z) \\ &+ \delta_2[(\nabla_X)H(Y)H(Z)+(\nabla_X)H(Z)H(Y)] \\ &+ \delta_3[(\nabla_X)F(Y)F(Z)+(\nabla_X)F(Z)F(Y)] \\ &+ \delta_4(\nabla_X)D(Y,Z). \end{aligned} \tag{100}$$

In the view of (96), (7) yields

$$v(n\delta_1+\delta_2+\delta_3)+2n(n-1)(\mu_3+\mu_2\lambda)=0. \tag{101}$$

Hence we have the following:

Theorem 28. *In a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation and with vanishing generalized space-matter tensor; $\delta_1,\ \delta_2,\ \delta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation* (101).

Suppose $n\delta_1 + \delta_2 + \delta_3$ is constant. Then in the view of (96) and (12) it follows that either u is also constant or

$$J_1(QX) = \frac{n\delta_1 + \delta_2 + \delta_3}{n} J_1(X),$$

which gives $S(X, \tau_1) = \frac{n\delta_1+\delta_2+\delta_3}{n} G(X, \tau_1)$. This leads to the following:

Theorem 29. *If in a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation and with symmetric generalized space-matter tensor; $n\delta_1 + \delta_2 + \delta_3$ is constant then either u is also constant or $\frac{n\delta_1+\delta_2+\delta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_1 define by $G(X, \tau_1) = J_1(X) = du(X)$ for all $X \in \chi(M)$.*

Using (96) in (18), we have

$$L(QX) = \frac{R_4}{2u} L(X),$$

which implies

$$S(X, \rho) = \frac{R_2}{2u} G(X, \rho),$$

where $R_4 = 2(n-1)(n-2)(\mu_3 + \lambda\mu_2) + \{2\mu_1 - (n-2)(n-3)\mu_2\}(n\delta_1 + \delta_2 + \delta_3)$. Therefore we have the following:

Theorem 30. *If in a Riemannian manifold (M^n, G) $(n > 3)$ admitting Einstein's field equation with recurrent generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_4}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ, defined by $G(X, \rho) = L(X)$ for all vector fields X, whenever $\mu_2,\ \mu_3$ are constants.*

Now setting $X = \rho$ in (20) and by the relations (96), (100); we have

$$\begin{aligned} (102) \quad & v[nd\delta_1(\rho) + d\delta_2(\rho) + d\delta_3(\rho)] + 2n(n-1)[d\mu_3(\rho) + \lambda d\mu_2(\rho)] \\ = \quad & \{v - dv(\rho)\}(n\delta_1 + \delta_2 + \delta_3) + 2n(n-1)(\mu_3 + \lambda\mu_2). \end{aligned}$$

Hence we get the following:

Theorem 31. *In a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation and with recurrent generalized space-matter tensor, the generator of recurrence ρ is given by the relation* (102).

Let us consider $n\delta_1 + \delta_2 + \delta_3$ as constant. Then in the view of (96), (20) takes the form

$$v(n\delta_1 + \delta_2 + \delta_3) + 2n(n-1)(\mu_3 + \lambda\mu_2) = 0, \tag{103}$$

provided that $\mu_1,\ \mu_2,\ \mu_3$ are constants and in the view of (96), (25) also takes the form

$$J_2(QX) = \frac{n\delta_1 + \delta_2 + \delta_3}{n} J_2(X),$$

which gives $S(X, \tau_2) = \frac{n\delta_1+\delta_2+\delta_3}{n} G(X, \tau_2)$ and

$$L(QX) = \frac{n\delta_1 + \delta_2 + \delta_3}{n} L(X),$$

which implies $S(X, \rho) = \frac{n\delta_1+\delta_2+\delta_3}{n} G(X, \rho)$, provided that $\mu_1,\ \mu_2$ are constants. Thus we can state the following:

Theorem 32. *In a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation and recurrent generalized space-matter tensor, if $n\delta_1 + \delta_2 + \delta_3$ is constant then,*
(i) $\frac{n\delta_1+\delta_2+\delta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_2, defined by $G(X, \tau_2) = J_2(X) = du(X) - uL(X)$ for all $X \in \chi(M)$;
(ii) $\frac{n\delta_1+\delta_2+\delta_3}{n}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ, defined by $L(X) = G(X, \rho)$ for all $X \in \chi(M)$, whenever $\mu_1,\ \mu_2$ are constants;
(iii) $\delta_1,\ \delta_2,\ \delta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are connected by the relation (103)*, whenever $\mu_1,\ \mu_2,\ \mu_3$ are constants.*

Using (96) in (29), we obtain

$$J_3(QX) = \frac{R_4}{2u} J_3(X),$$

which implies

$$S(X, \tau_3) = \frac{R_4}{2u} G(X, \tau_3).$$

Thus we can state the following:

Theorem 33. *If in a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation with weakly symmetric generalized space-matter tensor, the energy-momentum tensor is of Codazzi type then $\frac{R_4}{2u}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector τ_3, defined by $G(X, \tau_3) = J_3(X) = \mu_1(X) - 2\mu_2(X)$ for all $X \in \chi(M)$, whenever $\mu_2,\ \mu_3$ are constants.*

Again if u is non-zero then by the virtue of (96), we find

$$J_5(QX) = \frac{1}{u}[(n\delta_1 + \delta_2 + \delta_3)\{\mu_1 + (n-2)\mu_2\} + (n-1)(n-2)\{\mu_3 - (n\delta_1 + \delta_2 + \delta_3 - 2\lambda)\frac{\mu_2}{2}\}]J_5(X), \tag{104}$$

which gives
$S(X, \tau_5) = \frac{1}{u}[(n\delta_1 + \delta_2 + \delta_3)\{\mu_1 + (n-2)\mu_2\} + (n-1)(n-2)\{\mu_3 - (n\delta_1 + \delta_2 + \delta_3 - 2\lambda)\frac{\mu_2}{2}\}]G(X, \tau_5)$.
And if $u = 0$, then we have

$$(n-2)(n\delta_1 + \delta_2 + \delta_3)\mu_2 = 2(n-1)(\mu_3 + \lambda\mu_2).$$

Further if $\delta_1,\ \delta_2,\ \delta_3,\ \mu_1,\ \mu_2,\ \mu_3$ are constants and u is non-zero then by the virtue of (96) and (40); we get

$$J_6(QX) = \frac{1}{2u}[(n\delta_1+\delta_2+\delta_3)\{2\mu_1-(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]J_6(X),$$

which implies
$S(X, \tau_6) = \frac{1}{2u}[(n\delta_1 + \delta_2 + \delta_3)\{2\mu_1 - (n-3)(n-2)\mu_2\} + 2(n-2)(n-1)(\mu_3 + \lambda\mu_2)]G(X, \tau_6)$
and

$$J_7(QX) = \frac{-1}{2u}[(n\delta_1+\delta_2+\delta_3)\{2\mu_1-(n+1)(n-2)\mu_2\}+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]J_7(X),$$

which implies $S(X, \tau_7) = \frac{-1}{2u}[(n\delta_1 + \delta_2 + \delta_3)\{2\mu_1 - (n+1)(n-2)\mu_2\} + 2(n+2)(n-1)(\mu_3 + \lambda\mu_2)]G(X, \tau_7)$. And if $u = 0$, then it follows that

$$(n-2)(n\delta_1 + \delta_2 + \delta_3)\mu_2 = 2(n-1)(\mu_3 + \lambda\mu_2).$$

This leads to the following:

Theorem 34. *In a $\mathfrak{P}(GQE)_n$ $(n > 3)$ admitting Einstein's field equation and with weakly symmetric generalized space-matter tensor if u is zero then $\delta_1,\ \delta_2,\ \delta_3,\ \mu_2,\ \mu_3$ are connected by the relation $(n-2)(n\delta_1 + \delta_2 + \delta_3)\mu_2 = 2(n-1)(\mu_3 + \lambda\mu_2)$ otherwise,*
(i) $\frac{1}{u}[(n\delta_1 + \delta_2 + \delta_3)\{\mu_1 + (n-2)\mu_2\} + (n-1)(n-2)\{\mu_3 - (n\delta_1 + \delta_2 + \delta_3 - 2\lambda)\frac{\mu_2}{2}\}]$ is an eigen value of the Ricci tensor S corresponding to the eigen vector τ_5, defined by $G(X, \tau_5) = J_5(X) = \mu_2(X) - E(X)$ for all $X \in \chi(M)$;
(ii) $\frac{1}{2u}[(n\delta_1+\delta_2+\delta_3)\{2\mu_1-(n-3)(n-2)\mu_2\}+2(n-2)(n-1)(\mu_3+\lambda\mu_2)]$ and $\frac{-1}{2u}[(n\delta_1+\delta_2+\delta_3)\{2\mu_1-(n+1)(n-2)\mu_2\}+2(n+2)(n-1)(\mu_3+\lambda\mu_2)]$ are eigen values of the Ricci tensor S corresponding to the eigen vector τ_6, defined by $G(X, \tau_6) = J_6(X) = \mu_1(X) - \mu_2(X) - E(X)$ for all $X \in \chi(M)$and the eigen vector τ_7, defined by $G(X, \tau_7) = J_7(X) =$

$\mu_1(X) + \mu_2(X) + E(X)$ for all $X \in \chi(M)$ respectively, whenever $n\delta_1 + \delta_2 + \delta_3$, μ_1, μ_2, μ_3 are constants.

References

[1] T.Q. Binh, **On weakly symmetric Riemannian spaces**, Publ. Math. Debrecen., 42(1993), 103-107.

[2] M.C. Chaki, and R.K. Maity, **On quasi-Einstein manifolds**, Publ. Math. Debrecen, 57(2000), 297-306 .

[3] M.C. Chaki: **On generaliszed quasi-Einstein manifolds**, Publ. Math. Debrecen, 58/4(2001), 683-691.

[4] F. Defever, R.Deszcz, M. Hotloś, M. Kucharski and Z. Sentürk, **Generalisations of Robertson-Walker spaces**, Annales Univ. Sci. Budapest. Eötvös Sect. Math., 43(2000), 13-24.

[5] R.Deszcz, M. Hotloś, **On some pseudosymmetry type curvature condition**, Tsukuba J. Math., 27(2003), 13-30 .

[6] F. Defever, R.Deszcz, M. Hotloś, Z. Sentürk, **Quasi-Einstein hypersurfaces in semi-Riemannian space forms**, Colloq. Math., 81(2001), 81-97.

[7] R.Deszcz, M. Hotloś, Z. Sentürk, **On curvature properties of quasi-Einstein hypersurfaces in semi-Euclidean spaces**, Soochow J. Math., 27(4)(2001), 375-389.

[8] R.Deszcz, P. Verheyen, and L. Verstraelen, **On some generalized Einstein metric conditions**, Publ. Inst. Math. (Beograd),60:74(1996), 108-120.

[9] D. Ferus, **A remark on Codazzi tensors on constant curvature space, Lecture Note in Math.**, Global Differential Geometry and Global Analysis, Springer-Verlag, New York 838(1981).

[10] A. Ghosh, **On the non-existence of certain types of weakly symmetric manifold**, Sarajevo Journal of Mathematics, 2(15)(2006), 223-230 .

[11] S. K. Jana and A.A. Shaikh, **On quasi-Einstein spacetime with space-matter tensor**,Lobachevski Journal of Mathematics, 33(3)(2012), 255-261.

[12] A.Z. Petrov, **Einstein Spaces**, Pergamon Press, Oxford, (1949).

[13] J.A. Schouten, **Ricci-Calculus**, Springer-Verlag, Berlin (1954).

[14] A.A. Shaikh and S.K. JanaANA, **On pseudo generalized quasi-Einstein manifolds**, Tamkang J. of Mathematics, 39(1)(2008), 9-24.

[15] L. Tamássy, T.Q. Binh, **On weakly symmetric and weakly projective symmetric Rimannian manifolds**, Coll. Math. Soc., J. Bolyai, 50(1989), 663-670.

[16] K. Yano, S. Sawaki, **Riemannian manifolds admitting a conformal transformation group**, J. Diff. Geom. 2(1968), 161-184.

[17] J. Meikes, **On Geodesic Mappings of 2-Ricci Symmetric Riemannian Spaces**, Math. Notes, 28 (1980), 622–624.

[18] J. Meikes, **Projective-Symmetric and Projective-Recurrent Affine Connection Spaces**, Tr. Geom. Semin. 13 (1981), 61–62.

[19] J. Meikes, A. Vanžurová, I. Hinterleitner, **Geodesic Mappings and Some Generalizations**, Palacky Univ. Press, Olomouc 2009.

[20] J. Meikes, et al, **Differential Geometry of Special Mappings**, Palacky Univ. Press, Olomouc (2015).

[21] S.K. Ghosh, S.K. Jana, B. Das, K.K. BaishyaD, **On generalized space matter tensor**, Journal of the Indonesian Mathematical Society, submitted.

Vidyasagar College, University of Calcutta,
Faculty, Department of Mathematics,
Kolkata-700 006, Kolkata-India, India
e-mail:skjana23@rediffmail.com

Kurseong College, University of North Bengal,
Faculty, Department of Mathematics,
Kurseong-734203, Darjeeling, India
e-mail:kanakkanti.kc@gmail.com

Salesian College (Autonomous) Siliguri, University of North Bengal,
Faculty, Department of Mathematics,
Siliguri-734001, Darjeeling, India
e-mail:bikiran.math@gmail.com