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# A GENERALIZATION OF VUORINEN'S DISTANCE RATIO METRIC IN METRIC SPACES AND BI-LIPSCHITZ EQUIVALENT HYPERBOLIC-TYPE METRICS

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**Abstract.** We prove in the setting of a general metric space $(X, d)$ the bi-Lipschitz equivalence of generalized versions of Vuorinen's distance ratio metric, Gehring-Osgood metric, Dovgoshey-Hariri-Vuorinen metric, Nikolov-Andreev metric and Ibragimov metric. For the generalized Vuorinen's distance ratio metric $j$ on the complement of a nonempty closed subset $M$ of $X$ we show that the identity of $X \setminus M$ between $(X \setminus M, d)$ and $(X \setminus M, j)$ is $1-$quasiconformal. We also provide sufficient conditions for the completeness of $(X \setminus M, j)$, that is equivalent to the completeness of $X \setminus M$ with each of the above mentioned metrics.

## 1. Introduction

Hyperbolic-type metrics have a long history, but the interest for their study is constantly renewed, partly due to applications of these metrics to various fields of mathematics, such as geometry, group theory, geometric function theory, dynamical system theory etc., as well as to theoretical computer science. In the setting of Euclidean space $\mathbb{R}^n$, with $n \geq 3$, the counterparts for the classical hyperbolic metrics are defined only on the unit ball $\mathbb{B}^n$ and on the upper half-plane $\mathbb{H}^n$.

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The introduction of quasihyperbolic metric as a counterpart for the hyperbolic metric, in an arbitrary proper subdomain of $\mathbb{R}^n$, by Gehring and Palka [11], was motivated by the study of quasiconformal mappings in $n-$dimensional Euclidean spaces and opened the way for advances of geometric function theory, in particular in the theory of quasiregular mappings [30]. A thorough study of hyperbolic-type metrics and their applications to the theory of quasiregular mappings can be found in the monograph of Hariri, Klén and Vuorinen [13]. The study of hyperbolic-type metrics has been extended by the introduction of new metrics, by Gehring and Osgood (1979 [10]), Vuorinen (1985 [29]), Ferrand (1988 [7]), Beardon (1998 [1]), Seittenranta (1999 [26]), Hästö (2002 [14]), Hästö and Lindén (2004 [15]), Ibragimov (2009 [18] and 2011 [17]), Dovgoshey, Hariri and Vuorinen (2016 [6]), Boskoff and Suceavă (2017 [3]), Nikolov and Andreev (2017 [24]), Rainio and Vuorinen (2022 [25]), Song and Wang (2024 [27]), Chen and Zhang (2024 [4]) etc. Very recently, in 2024, results on hyperbolic-type metrics have been obtained by Luo, Rasila, Wang, Zhuo [22], Zhou, Zheng, Ponnusamy and Guan [32], Fujimura and Vuorinen [9].

The metrics of hyperbolic type in a proper open subset $G$ of the Euclidean space $\mathbb{R}^n$ are intrinsic metrics, in the sense that their formulas take into account, besides the Euclidean distance between two points, the position of the points with respect to the boundary $\partial G$.

There is a long tradition in studying the hyperbolicity of general metric spaces, with the remarkable case of geodesic spaces, starting with Gromov (1987 [12]), see also the survey of Väisälä (2005 [28]) and, to mention only a few, the paper of Ibragimov (2011 [17]), the studies on hyperbolic-type metrics on Ptolemy spaces by Zhang and Xiao (2019 [31]) and by Chen and Zhang (2024 [4]). In [17], the role of the boundary of $G$ is played by a nonempty proper closed subset $M$ of a metric space $X$. Note that in [24] some preliminary estimates for the Nikolov-Andreev metric involved a general positive $1-$Lipschitz function $F$ on $X \setminus M$ instead of the distance from the points in $G$ to the boundary $\partial G$.

Following ideas from [17] and [24], we considered in [23], in the setting of metric spaces, some generalized versions of Gehring-Osgood metric, Dovgoshey-Hariri-Vuorinen metric, Nikolov-Andreev metric and Ibragimov metric. In the following, $(X, d)$ is an arbitrary metric space and $M \subset X$ is a nonempty closed set. For each generalization $\rho$ of the hyperbolic-type metrics mentioned above we prove that $(X \setminus M, \rho)$ is a Gromov hyperbolic space and that the identity map between $(X \setminus M, d)$ and $(X \setminus M, \rho)$ is quasiconformal. Some improvements for the Gromov constants for the Gehring-Osgood metric and the Nikolov-Andreev metric, as well for the quasiconformality constant (in the sense of metric definition) of the identity map of $X \setminus M$ with the Ibragimov metric have been obtained.

In the present paper, using the same setting as in [23], we consider an analogous generalization of distance ratio metric $j$ introduced by Vuorinen [29], as a more manageable replacement of Gehring-Osgood metric. It turns out that each of the following metrics: Gehring-Osgood, Dovgoshey-Hariri-Vuorinen, Nikolov-Andreev and Ibragimov, is bi-Lipschitz equivalent to the distance ratio metric, that plays therefore a pivotal role. It is proved that the identity map between $(X \setminus M, d)$ and $(X \setminus M, j)$ is quasiconformal. Moreover, if $F$ has a continuous extension $\widetilde{F}$ to $(X, d)$ with $\widetilde{F}(x) = 0$ for every $x \in M$ and $(X, d)$ is complete, then $(X \setminus M, \rho)$ is also a complete metric space, for each of the five above mentioned metrics.

We will show that the distance ratio metric $j$ does not change the quasiconformal geometry of the space, using the below metric definition of quasiconformal maps between metric spaces [16]. Given a homeomorphism $f$ from a metric space $(X, d)$ to a metric space $(Y, \rho)$, then for $x \in X$ and $r > 0$ we set

$$H_f(x, r) = \frac{\sup\{\rho(f(x), f(y)) : d(x, y) \le r\}}{\inf\{\rho(f(x), f(y)) : d(x, y) \ge r\}}.$$

A homeomorphism $f : (X, d) \to (Y, \rho)$ is called $H-$quasiconformal, with a nonnegative constant $H < \infty$, if $\limsup_{r \to 0} H_f(x, r) \le H$ for every $x \in X$.

## 2. A generalization of Vuorinen's distance ratio metric

Let $(X, d)$ be a metric space and let $G \subset X$ be an open set with non-empty boundary. For every $x \in X$ denote the distance from $x$ to the boundary of $G$ by $d_G(x) = \operatorname{dist}(x, \partial G) = \inf\{d(x, y) : y \in \partial G\}$. It is well-known that $d_G(\cdot)$ is a $1-$Lipschitz function on $(X, d)$.

The distance ratio metric introduced by Vuorinen [29] is defined by $j_G(x, y) = \log\left(1 + \frac{d(x,y)}{\min\{d_G(x), d_G(y)\}}\right)$, where $x, y \in G$ and $G \subset \mathbb{R}^n$ is an open set with non-empty boundary.

Now we consider a generalization of Vuorinen's distance ratio metric and investigate its properties.

**Theorem 2.1.** *Let* $(X,d)$ *be a metric space and* $M$ *be a nonempty closed proper subset of* $X$. *Let* $F:(X\setminus M,d)\to(0,\infty)$ *be a* $1-$*Lipschitz function. Define* $j(x,y)=\log\left(1+\frac{d(x,y)}{\min\{F(x),F(y)\}}\right)$ *for* $x,y\in X\setminus M$. *Then:*

*(1)* $j$ *is a metric on* $X\setminus M$;

*(2) The identity map* $1_{X\setminus M}:(X\setminus M,d)\to(X\setminus M,j)$ *is* $1-$*quasiconformal;*

*(3) If* $F$ *has a continuous extension* $\widetilde{F}$ *to* $(X,d)$ *with* $\widetilde{F}(x)=0$ *for every* $x\in M$ *and* $(X,d)$ *is complete, then* $(X\setminus M,j)$ *is also a complete metric space.*

*Proof.* As an immediate consequence of the definition of $j$, for every $x,y\in X\setminus M$ we get

$$j(x,y)\geq\log\left(1+\frac{d(x,y)}{F(x)}\right). \tag{2.1}$$

Since $F$ is $1-$Lipschitz, for every $x,y\in X\setminus M$ we have $\max\{F(x),F(y)\}-\min\{F(x),F(y)\}\leq d(x,y)$, hence

$$j(x,y)\geq\log\frac{\max\{F(x),F(y)\}}{\min\{F(x),F(y)\}}=\left|\log\frac{F(x)}{F(y)}\right| \tag{2.2}$$

and $\min\{F(x),F(y)\}\geq F(x)-d(x,y)$, therefore if $d(x,y)<F(x)$, then

$$j(x,y)\leq\log\frac{F(x)}{F(x)-d(x,y)}. \tag{2.3}$$

(1) Clearly, $j(x,y)\geq0$ and $j(x,y)=j(y,x)$ for every $x,y\in X\setminus M$ and $j(x,y)=0$ if and only if $x=y$. We will prove the triangle inequality $j(x,y)\leq j(x,z)+j(z,y)$ for every $x,y,z\in X\setminus M$. The latter inequality is equivalent to

$$\begin{aligned}\frac{d(x,y)}{\min\{F(x),F(y)\}}&\leq\frac{d(x,z)}{\min\{F(x),F(z)\}}+\frac{d(z,y)}{\min\{F(z),F(y)\}}\\&+\frac{d(x,z)}{\min\{F(x),F(z)\}}\cdot\frac{d(z,y)}{\min\{F(z),F(y)\}}.\end{aligned} \tag{2.4}$$

If the following inequality is true, then the above inequality follows by the triangle inequality for the metric $d$:

$$\begin{aligned}\frac{d(x,z)+d(z,y)}{\min\{F(x),F(y)\}}&\leq\frac{d(x,z)}{\min\{F(x),F(z)\}}+\frac{d(z,y)}{\min\{F(z),F(y)\}}\\&+\frac{d(x,z)}{\min\{F(x),F(z)\}}\cdot\frac{d(z,y)}{\min\{F(z),F(y)\}}.\end{aligned} \tag{2.5}$$

We may assume by symmetry that $F(x) \leq F(y)$.

**Case 1.** Assume that $F(z) \leq \min\{F(x), F(y)\} = F(x)$. Then $\frac{d(x,z)}{\min\{F(x),F(z)\}} + \frac{d(z,y)}{\min\{F(z),F(y)\}} = \frac{d(x,z)+d(z,y)}{F(z)} \geq \frac{d(x,z)+d(z,y)}{\min\{F(x),F(y)\}}$ and the inequality (2.5) follows.

**Case 2.** Assume that $F(z) \geq \max\{F(x), F(y)\} = F(y)$. The inequality (2.4) writes in this case as $\frac{d(x,y)}{F(x)} \leq \frac{d(x,z)}{F(x)} + \frac{d(z,y)}{F(y)} + \frac{d(x,z)}{F(x)} \cdot \frac{d(z,y)}{F(y)}$, which is equivalent to

$$(d(x,y) - d(x,z)) F(y) \leq d(z,y) (F(x) + d(x,z)).$$

Note that $F(x) + d(x,z) \geq F(z)$, since $F$ is $1-$Lipschitz. But $F(z) \geq F(y)$ in this case, therefore $F(x) + d(x,z) \geq F(y)$. Then $d(z,y)(F(x) + d(x,z)) \geq d(z,y) F(y) \geq (d(x,y) - d(x,z)) \cdot F(y)$.

**Case 3.** It remains to consider the case where $F(x) \leq F(z) \leq F(y)$. The inequality (2.5) writes as $\frac{d(x,z)+d(z,y)}{F(x)} \leq \frac{d(x,z)}{F(x)} + \frac{d(z,y)}{F(z)} + \frac{d(x,z)}{F(x)} \cdot \frac{d(z,y)}{F(z)}$, which is equivalent to

$$d(z,y)(F(z) - F(x) - d(x,z)) \leq 0.$$

Since $F$ is $1-$Lipschitz, $F(z) - F(x) \leq d(x,z)$ and the above inequality follows.

(2)

a) Assume that $\lim_{n\to\infty} d(x_n, x) = 0$, where $x \in X \setminus M$ and $x_n \in X \setminus M$ for every $n \geq 1$. Then $\lim_{n\to\infty} F(x_n) = F(x) > 0$, hence $\lim_{n\to\infty} j(x_n, x) = \lim_{n\to\infty} \log\left(1 + \frac{d(x_n,x)}{\min\{F(x_n),F(x)\}}\right) = 0$. We proved that the identity map $1_{X\setminus M} : (X \setminus M, d) \to (X \setminus M, j)$ is *continuous*.

b) Assume that $\lim_{n\to\infty} j(x_n, x) = 0$, where $x \in X \setminus M$ and $x_n \in X \setminus M$ for every $n \geq 1$. By (2.1), $d(x_n, x) \leq F(x)\left(e^{j(x_n,x)} - 1\right)$ for every $n \geq 1$. It follows that $\lim_{n\to\infty} d(x_n, x) = 0$, therefore the identity map $1_{X\setminus M} : (X \setminus M, d) \to (X \setminus M, j)$ is *open*.

c) We proved that the identity map $1_{X\setminus M} : (X \setminus M, d) \to (X \setminus M, j)$ is a *homeomorphism*. For $f = 1_{X\setminus M}$ we have, whenever $x \in X$ and $r > 0$,

$$H_f(x,r) = \frac{\sup\{j(x,y) : d(x,y) \leq r,\ y \in X \setminus M\}}{\inf\{j(x,y) : d(x,y) \geq r,\ y \in X \setminus M\}}.$$

Let $x \in X \setminus M$.

By (2.1), it follows that $\inf\{j(x,y) : d(x,y) \geq r\} \geq \log\left(1 + \frac{r}{F(x)}\right)$.

Using (2.3) for $d(x,y) \leq r < F(x)$ and the fact that the function $t \mapsto \frac{a}{a-t}$ with $a > 0$ is increasing on $[0,a)$, we infer that $\sup\{j(x,y) : d(x,y) \leq r\} \leq \log \frac{F(x)}{F(x)-r}$.

Then $H_f(x,r) \leq \frac{\log F(x) - \log(F(x)-r)}{\log(F(x)+r) - \log F(x)}$ for every $r \in (0, F(x))$. It follows that

$$\limsup_{r\to 0} H_f(x,r) \leq \limsup_{r\to 0} \frac{\log F(x) - \log(F(x)-r)}{\log(F(x)+r) - \log F(x)}.$$

But $\lim_{r\to 0} \frac{\log F(x)-\log(F(x)-r)}{\log(F(x)+r)-\log F(x)} = \lim_{r\to 0} \frac{\frac{1}{F(x)-r}}{\frac{1}{F(x)+r}} = 1$, hence $\limsup_{r\to 0} H_f(x,r) \leq 1$.

(3) Assuming that $F$ has a continuous extension $\widetilde{F}$ to $X$ with $\widetilde{F}(x) = 0$ for every $x \in M$ and that $(X,d)$ is complete, we prove that $(X \setminus M, j)$ is complete.

Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $(X \setminus M, j)$. By (2.2), $(\log F(x_n))_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Then $(\log F(x_n))_{n\geq 1}$ is bounded: $\alpha = \inf_{n\geq 1} \log F(x_n) \in \mathbb{R}$ and $\beta = \sup_{n\geq 1} \log F(x_n) \in \mathbb{R}$. Then $e^\alpha = \inf_{n\geq 1} F(x_n) \leq \sup_{n\geq 1} F(x_n) = e^\beta$. As (2.1) implies $d(x_n, x) \leq F(x)\left(e^{j(x_n,x)} - 1\right) \leq e^\beta \left(e^{j(x_n,x)} - 1\right)$ for all $x, y \in X \setminus M$, it follows that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $(X,d)$. Let $x \in X$ be such that $\lim_{n\to\infty} d(x_n, x) = 0$. Since $\widetilde{F}$ is continuous on $X$, $\widetilde{F}(x) = \lim_{n\to\infty} \widetilde{F}(x_n) = \lim_{n\to\infty} F(x_n) \geq e^\alpha > 0$, hence $x \in X \setminus M$ and $\widetilde{F}(x) = F(x)$. Finally, $\lim_{n\to\infty} j(x_n, x) = \lim_{n\to\infty} \log\left(1 + \frac{d(x_n,x)}{\min\{F(x_n),F(x)\}}\right) = 0$, hence $(x_n)_{n\geq 1}$ converges to $x$ in $(X \setminus M, j)$. ∎

## 3. Comparison results and completeness of hyperbolic-type metrics

In the following, we assume that $(X,d)$ is a metric space, $M$ is a closed nonempty proper subset of $X$ and $F : (X \setminus M, d) \to (0,\infty)$ is a $1-$Lipschitz function.

In order to generalize known metrics, we considered in [23] the following functions defined on $(X \setminus M) \times (X \setminus M)$:

1) The generalized Gehring-Osgood metric

$\widetilde{j}(x,y) = \frac{1}{2} \log\left(1 + \frac{d(x,y)}{F(x)}\right)\left(1 + \frac{d(x,y)}{F(y)}\right)$.

2) The generalized Dovgoshey-Hariri-Vuorinen metric

$h_c(x,y) = \log\left(1 + c\frac{d(x,y)}{\sqrt{F(x)F(y)}}\right)$, where $c \geq 2$ is a constant.

3) The generalized Nikolov-Andreev metric
$i\left(x,y\right)=2\log\frac{F(x)+F(y)+d(x,y)}{2\sqrt{F(x)F(y)}}$.

4) The generalized Ibragimov metric
$v\left(x,y\right)=2\log\frac{d(x,y)+\max\{F(x),F(y)\}}{\sqrt{F(x)F(y)}}$.

We proved in [23] that these functions are indeed metrics, that are Gromov hyperbolic. Note that for the generalized Ibragimov metric the function $F:(X\setminus M,d)\to(0,\infty)$ is not assumed to be $1-$Lipschitz in order to obtain the triangle inequality and Gromov hyperbolicity.

For each $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$ we obtained a Gromov constant $\delta(\rho)$ of $\rho$, as follows:

1) $\delta\left(\widetilde{j}\right)=\frac{1}{4}\log 24$; 2) $\delta\left(h_c\right)=\log\left(2+\frac{1}{c}\right)$;
3) $\delta\left(i\right)=\log 9$; 4) $\delta\left(v\right)=\log 4$.

In addition, for each $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$ it is proved in [23] that the identity map $1_{X\setminus M}:(X\setminus M,d)\to(X\setminus M,\rho)$ is $H-$quasiconformal in the sense of the metric definition of quasiconformality of a homeomorphism, with a constant $H$ as follows:

1) $H=1$ if $\rho=\widetilde{j}$; 2) $H=1$ if $\rho=h_c$; 3) $H=3$ if $\rho=i$; 4) $H=\frac{5}{2}$ if $\rho=v$ (where it is assumed that $F$ is $1-$Lipschitz).

Moreover, if $F$ has a continuous extension $\widetilde{F}$ to $(X,d)$ with $\widetilde{F}\left(x\right)=0$ for every $x\in M$ and $(X,d)$ is complete, then $(X\setminus M,v)$ is also a complete metric space [23].

We will show, under the assumptions that $F:(X\setminus M,d)\to(0,\infty)$ is $1-$Lipschitz and has a continuous extension $\widetilde{F}$ to $(X,d)$ with $\widetilde{F}\left(x\right)=0$ for every $x\in M$, that the completeness of $(X,d)$ guarantees the completeness of $(X\setminus M,\rho)$ for each $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$. This follows from the completeness of $(X\setminus M,j)$, provided by Theorem 2.1 and from the bi-Lipschitz equivalence of $\rho$ and $j$, where $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$.

We recall that the completeness of every Euclidean domain with nonempty boundary with respect to the Dovgoshey-Hariri-Vuorinen metric $h_c$ has been proved in [32].

It remains to prove that every metric $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$ is bi-Lipschitz equivalent to the distance ratio metric $j$.

We recall that two metrics $\rho_1,\rho_2$ on a nonempty set $Y$ are said to be bi-Lipschitz equivalent if there exist some positive real constants

$a \leq b$ such that

$$a\rho_1(x,y) \leq \rho_2(x,y) \leq b\rho_1(x,y)$$

for all $x, y \in Y$. In other words, two metrics $\rho_1, \rho_2$ on $Y$ are bi-Lipschitz equivalent if and only if the identity map $1_Y : (Y, \rho_1) \to (Y, \rho_2)$ is bi-Lipschitz.

The following lemma is well-known, we note this here for the sake of the reader.

**Lemma 3.1.** *Let $Y$ be a nonempty set and $\rho_1, \rho_2$ be two bi-Lipschitz equivalent metrics on $Y$. Then $(Y, \rho_1)$ is complete if and only if $(Y, \rho_2)$ is complete.*

*Proof.* By symmetry, it suffices to check that the completeness of $(Y, \rho_1)$ implies the completeness of $(Y, \rho_2)$.

Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $(Y, \rho_2)$. Since $\rho_1 \leq \frac{1}{a}\rho_1$ on $Y$, we see that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $(Y, \rho_1)$. Since $(Y, \rho_1)$ is a complete metric space, there exists $x \in Y$ such that $\lim\limits_{n\to\infty} \rho_1(x_n, x) = 0$, which implies $\lim\limits_{n\to\infty} \rho_2(x_n, x) = 0$, as $\rho_2 \leq b\rho_1$ on $Y$. ∎

The following comparison result showing the bi-Lipschitz equivalence of Gehring-Osgood metric and Vuorinen's distance ratio metric is well-known in the classical case.

**Lemma 3.2.** *The metrics defined by* $\widetilde{j}(x,y) = \frac{1}{2}\log\left(1+\frac{d(x,y)}{F(x)}\right)\left(1+\frac{d(x,y)}{F(y)}\right)$ *and* $j(x,y) = \log\left(1+\frac{d(x,y)}{\min\{F(x),F(y)\}}\right)$ *for $x, y \in X \backslash M$ satisfy the inequalities $\widetilde{j} \leq j \leq 2\widetilde{j}$ on $(X \setminus M) \times (X \setminus M)$.*

*Proof.* For every $a \geq 0$ and $b, c \geq 0$ we have $\max\left\{\left(1+\frac{a}{b}\right), \left(1+\frac{a}{c}\right)\right\} \leq 1+\frac{a}{\min\{b,c\}}$, hence $\log\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right) \leq 2\log\left(1+\frac{a}{\min\{b,c\}}\right)$, and, obviously, $\log\left(1+\frac{a}{\min\{b,c\}}\right) \leq \log\left(1+\frac{a}{b}\right)\left(1+\frac{a}{c}\right)$. For $a = d(x,y)$, $b = F(x)$ and $c = F(y)$ it follows that $\widetilde{j}(x,y) \leq j(x,y) \leq 2\widetilde{j}(x,y)$ for every $x, y \in X \setminus M$. ∎

The following partial generalization of [6, Lemma 4.4] shows that the generalized versions of distance ratio metric and Dovgoshey-Hariri-Vuorinen metric are bi-Lipschitz equivalent.

**Lemma 3.3.** *For every $c \geq 1$ and all $x, y \in X \setminus M$ we have*

$$\frac{1}{2}j(x,y) \leq h_c(x,y) \leq cj(x,y).$$

*Proof.* Fix $x, y \in X \setminus M$. Since $\sqrt{F(x)F(y)} \geq \min\{F(x), F(y)\}$, using Bernoulli's inequality $(1+t)^r \geq 1 + rt$ for real numbers $r \geq 1$ and $t \geq -1$, we get $h_c(x,y) \leq \log\left(1 + c\frac{d(x,y)}{\min\{F(x),F(y)\}}\right) \leq cj(x,y)$.

By symmetry, we may assume that $F(x) \leq F(y)$.

Since $F$ is 1–Lipschitz, $F(y) \leq F(x) + d(x,y)$, hence $\frac{d(x,y)}{\sqrt{F(x)F(y)}} \geq \frac{d(x,y)}{\sqrt{F(x)(F(x)+d(x,y))}} = \frac{a}{\sqrt{1+a}}$, where we denoted $a = \frac{d(x,y)}{F(x)}$.

We have $h_c(x,y) \geq \log\left(1 + c\frac{a}{\sqrt{1+a}}\right)$ and $j(x,y) = \log(1+a)$.

For $c \geq 1$ and $t \geq 0$ we have

$$\begin{aligned}\left(1 + c\frac{t}{\sqrt{1+t}}\right)^2 - (1+t) &= t\left(\frac{2c}{\sqrt{1+t}} + c^2\frac{t}{1+t} - 1\right) \\ &\geq t\left(\frac{2}{\sqrt{1+t}} + \frac{t}{1+t} - 1\right) = t\frac{2\sqrt{1+t}-1}{1+t} \geq 0.\end{aligned}$$

Then

$$\log\left(1 + c\frac{t}{\sqrt{1+t}}\right) \geq \frac{1}{2}\log(1+t) \tag{3.1}$$

for every $c \geq 1$ and $t \geq 0$. Since $\lim_{t\to\infty} \frac{\log\left(1+c\frac{t}{\sqrt{1+t}}\right)}{\log(1+t)} = \frac{1}{2}$ for every $c > 0$, we see that $\inf\left\{\frac{\log\left(1+c\frac{t}{\sqrt{1+t}}\right)}{\log(1+t)} : t \geq 0\right\} = \frac{1}{2}$ whenever $c \geq 1$.

In particular, inequality (3.1) implies $\frac{1}{2}j(x,y) \leq h_c(x,y)$. ∎

The study of some estimates of Dovgoshey-Hariri-Vuorinen metric in the Euclidean setting have led to the introduction of a new intrinsic metric in [8]. Let $F_c(t) = \log\left(1 + 2c\sinh\frac{t}{2}\right)$, where $c, t \geq 0$. It was proved in [8, Lemma 3.4] that $\frac{F_c(t)}{t}$ is decreasing from $(0,\infty)$ onto $\left(\frac{1}{2}, c\right)$ if (and only if) $c \geq 1$. Then $F_c$ is subadditive on $[0,\infty)$, since $F_c(0) = 0$ and for every $s, t > 0$ we have

$$F_c(s+t) = s\frac{F_c(s+t)}{s+t} + t\frac{F_c(s+t)}{s+t} \leq s\frac{F_c(s)}{s} + t\frac{F_c(t)}{t}.$$

Therefore, as shown in Theorem 1.1 from [8], given any metric space $(X, \rho)$ and a constant $c \geq 1$, the function defined by $W_c(x,y) = \log\left(1 + 2c\sinh\frac{\rho(x,y)}{2}\right)$ for $x, y \in X$ is a metric. This result and Theorem 2.1 imply the following:

**Proposition 3.4.** *Let $(X,d)$ be a metric space and $M$ be a nonempty closed proper subset of $X$. Assume that $F:(X\setminus M,d)\to(0,\infty)$ is a $1-$Lipschitz function. Let $j(x,y)=\log\left(1+\frac{d(x,y)}{\min\{F(x),F(y)\}}\right)$ for $x,y\in X\setminus M$ and $c\geq 1$. Then the function defined by $W_c(x,y)=\log\left(1+2c\sinh\frac{j(x,y)}{2}\right)$ for $x,y\in X$ is a metric satisfying the inequalities*

$$\frac{1}{2}j(x,y)<W_c(x,y)<j(x,y),$$

*whenever $x,y\in X\setminus M$ are distinct points. In addition, $W_c(x,y)\leq h_c(x,y)$ for all $x,y\in X\setminus M$.*

*Proof.* It remains to check that $W_c(x,y)\leq h_c(x,y)$ for all $x,y\in X\setminus M$, a inequality that was proved in the Euclidean setting in [6, Lemma 4.4], for all $c>0$. If $t=j(x,y)$, then

$$2\sinh\frac{t}{2}=\frac{e^t-1}{e^{t/2}}=\frac{d(x,y)}{\sqrt{\min\{F(x),F(y)\}\left(\min\{F(x),F(y)\}+d(x,y)\right)}}.$$

Since $F:(X\setminus M,d)\to(0,\infty)$ is a $1-$Lipschitz function, $\min\{F(x),F(y)\}+d(x,y)\geq\max\{F(x),F(y)\}$. Then $2c\sinh\frac{j(x,y)}{2}\geq c\frac{d(x,y)}{\sqrt{F(x)F(y)}}$ and the claim follows. ∎

**Corollary 3.5.** *For every $c\geq 1$ and all $x,y\in X\setminus M$ we have $\frac{1}{2}j(x,y)\leq W_c(x,y)\leq\min\{j(x,y),h_c(x,y)\}\leq cj(x,y)$.*

The generalized Nikolov-Andreev metric is bi-Lipschitz equivalent to the generalized distance ratio metric, as shown below.

**Lemma 3.6.** *For all $x,y\in X\setminus M$ we have*

$$\frac{1}{2}j(x,y)\leq i(x,y)\leq 2j(x,y).$$

*Proof.* Let $x,y\in X\setminus M$. Due to mean inequality,

$$i(x,y)\geq\log\left(1+\frac{1}{2}\frac{d(x,y)}{\sqrt{F(x)F(y)}}\right)^2=\log\left(1+\frac{d(x,y)}{\sqrt{F(x)F(y)}}+\frac{d^2(x,y)}{F(x)F(y)}\right).$$

Using Lemma 3.3, we see that $\log\left(1+\frac{d(x,y)}{\sqrt{F(x)F(y)}}+\frac{d^2(x,y)}{F(x)F(y)}\right)\geq\log\left(1+\frac{d(x,y)}{\sqrt{F(x)F(y)}}\right)=h_1(x,y)\geq\frac{1}{2}j(x,y)$. Then $\frac{1}{2}j(x,y)\leq i(x,y)$.

By symmetry, we may assume that $F(x) \le F(y)$.
Then $j(x,y) = \log\left(1 + \frac{d(x,y)}{F(x)}\right)$ and

$$\begin{aligned} i(x,y) &= 2\log\left(\frac{1}{2}\sqrt{\frac{F(x)}{F(y)}} + \frac{1}{2}\sqrt{\frac{F(y)}{F(x)}} + \frac{1}{2}\sqrt{\frac{F(x)}{F(y)}}\frac{d(x,y)}{F(x)}\right) \\ &\le 2\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{F(y)}{F(x)}} + \frac{1}{2}\frac{d(x,y)}{F(x)}\right). \end{aligned}$$

Since $F$ is $1-$Lipschitz, $\frac{F(y)}{F(x)} \le 1 + \frac{d(x,y)}{F(x)}$. The latter two inequalities imply

$$i(x,y) \le 2\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{d(x,y)}{F(x)}} + \frac{1}{2}\frac{d(x,y)}{F(x)}\right).$$

Since $1 + \sqrt{1+t} + t \le 2(1+t)$ for every $t \ge 0$, the above inequality implies $i(x,y) \le 2j(x,y)$. ∎

Now we check that the generalized Ibragimov metric is bi-Lipschitz equivalent to the generalized distance ratio metric.

**Lemma 3.7.** *For all $x, y \in X \setminus M$ we have $j(x,y) \le v(x,y) \le 3j(x,y)$.*

*Proof.* Let $x, y \in X \setminus M$. By symmetry, we may assume that $F(x) \le F(y)$.

Then $j(x,y) = \log\left(1 + \frac{d(x,y)}{F(x)}\right)$ and

$$\begin{aligned} v(x,y) &= 2\log\left(\sqrt{\frac{F(y)}{F(x)}} + \sqrt{\frac{F(x)}{F(y)}}\frac{d(x,y)}{F(x)}\right) \\ &= \log\left(\frac{F(y)}{F(x)} + 2\frac{d(x,y)}{F(x)} + \frac{F(x)}{F(y)}\frac{d^2(x,y)}{F^2(x)}\right) \ge \log\left(1 + 2\frac{d(x,y)}{F(x)}\right), \end{aligned}$$

hence $j(x,y) \le v(x,y)$.
On the other hand,

$$v(x,y) \le 2\log\left(\sqrt{\frac{F(y)}{F(x)}} + \frac{d(x,y)}{F(x)}\right) \le 2\log\left(\sqrt{1 + \frac{d(x,y)}{F(x)}} + \frac{d(x,y)}{F(x)}\right).$$

The inequality $\log\left(\sqrt{1+t} + t\right) \le \frac{3}{2}\log(1+t)$, where $t \ge 0$ is equivalent to $(1+t)^3 - \left(\sqrt{1+t} + t\right)^2 \ge 0$, $t \ge 0$. The latter inequality is equivalent to $t^2\left(t+\frac{2\sqrt{t+1}}{\sqrt{t+1}+1}\right)\geq 0$, $t\geq 0$, that is obviously true. We conclude that $v(x,y)\leq 3j(x,y)$. ∎

**Theorem 3.8.** *Let $(X,d)$ be a metric space and $M$ be a nonempty closed proper subset of $X$. Let $F:(X\setminus M,d)\to(0,\infty)$ be a $1-$Lipschitz function, that has a continuous extension $\widetilde{F}$ to $(X,d)$ with $\widetilde{F}(x)=0$ for every $x\in M$. If the metric space $(X,d)$ is complete, then $(X\setminus M,\rho)$ is also a complete metric space, for every $\rho\in\left\{j,\widetilde{j},h_c,i,v\right\}$. Here $c\geq 2$.*

*Proof.* For $\rho=j$, the distance ratio metric, the claim follows from Theorem 2.1.

Each metric $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$is bi-Lipschitz equivalent to the distance ratio metric on $X\setminus M$, by Lemmas 3.2, 3.3, 3.6 and 3.7, respectively.

The using Lemma 3.1, the completeness of $(X\setminus M,j)$ implies the completeness of $(X\setminus M,\rho)$, for each $\rho\in\left\{\widetilde{j},h_c,i,v\right\}$. ∎

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