INTERACTING FIELDS -A COMPLET ANALYTICAL SOLUTION

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Abstract: *We derive the analytical solutions to the system of Klein-Gordon-Maxwell-Einstein equations for a complex scalar field minimally coupled to a spherically symmetric space-time and the feedback of gravity and electric field on the charged scalar source in a constant potential hypothesis for a particular curved space-time configuration.*

Keywords: *Klein-Gordon equation, Maxwell equation, Einstein equation, curved spacetime*

1. INTRODUCTION

The problem of the deficiency of a consistent probability interpretation for Klein-Gordon fields is nearly as old as quantum mechanics theory. In a previous approach [2] some problems of Dirac equation were presented. For a general problem of Klein-Gordon field interacting with Maxwell electromagnetic one in presence of gravity, difficulties are analytically, exactly solvable models for boson with large self interaction [12] and for boson-fermion systems have been worked out only in low-dimensional gravity. In four dimensions, the bosonic or the mixed fermion - bosonic fields interacting via gravity have been investigated mainly by numerical calculations [3-9,11,13].

2. THEORY

Let us consider a spherically symmetric configuration, expressed in Schwarzchild coordinates

$$ds^2 = e^{2h}(dt)^2 - e^{2f}(dr)^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

where f and h are functions of r and t. We introduce the pseudo-orthonormal tetradic frame $\{e_a\}_{a=1,4}$, with the corresponding dual orthonormal base

$$\omega^1 = e^h dt \qquad \omega^2 = e^f dr$$
$$\omega^3 = r d\theta \qquad \omega^4 = r\sin\theta d\varphi$$

The Lagrangean density in a $SO(3,1)\times U(1)$ gauge invariance interaction of charged boson of mass m_0 with electromagnetic field, is

$$L = \eta^{ab}\overline{\Phi}_{,a}\Phi_{,b} + m_0{}^2\,\overline{\Phi}\Phi + \frac{1}{4}F^{ab}F_{ab} \tag{2}$$

where

$$\Phi_{,a} = \Phi_{|a} - ieA_a\Phi$$

and

$$\overline{\Phi}_{,a} = \overline{\Phi}_{|a} + ieA_a\overline{\Phi} \tag{3}$$

The η^{ab} tensor is the minkowskian kind, defined as

$$\eta^{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The Maxwell tensor

$$F_{ab} = A_{b:a} - A_{a:b}$$

is expressed in the terms of the Levi-Civita covariant derivative of the four-potential $\{A_a\}_{a=\overline{1,4}}$, i.e.

$$A_{a;b} = A_{a|b} - A_c\Gamma^c{}_{ab}$$

where the Christoffel symbols are computed as

$$\Gamma^c{}_{ab} = g^{cd}\,\frac{1}{2}\left(\frac{\partial g_{ba}}{\partial x^d} + \frac{\partial g_{db}}{\partial x^a} - \frac{\partial g_{da}}{\partial x^b}\right)$$

By varying with respect to different fields, it can be computed the Klein - Gordon equation as

$$\Box\Phi - m_0{}^2\Phi = 2ieA^c\Phi_{|c} + e^2A^cA_c\Phi \quad \text{and its h.c.} \tag{4}$$

and Maxwell equations

$$F^{ab}{}_{:c} = -ie\eta^{ab}\left[\overline{\Phi}\left(\Phi_{|b} - ieA_b\Phi\right) - \left(\overline{\Phi}_{|b} - ieA_b\overline{\Phi}\right)\Phi\right] \tag{5}$$

Building up the energy-momentum tensor

$$T_{ab} = \overline{\Phi}_{:a}\Phi_{:a} + \overline{\Phi}_{:a}\Phi_{:a} + F_{ac}F_b{}^c - \eta_{ab}L\,. \tag{6}$$

it can be derived the Einstein equation

$$G_{ab} = kT_{ab}\,, \tag{7}$$

where the tensor G_{ab} have the explicit form expressed using the Ricci tensor R_{ab}

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{8}$$

Working in the minimally symmetric ansatz $A_1 = A_1(r,t)$, $A_4 = A_4(r,t)$, $\Phi = \Phi(r,t)$, the Klein-Gordon-Maxwell equations system (7), (8) turn into

$$\begin{aligned} &e^{-2f}\left\{\Phi_{,rr} + \left[h_{,r} - f_{,r} + \frac{2}{r}\right]\Phi_{,r}\right\} - e^{-2h}\left\{\Phi_{,tt} + \left[f_{,t} - f_{,r}\right]\Phi_{,t}\right\} - {m_0}^2\Phi = \\ &= 2ieA_1 e^{-f}\Phi_{,r} - 2ieA_4 e^{-h}\Phi_{,t} + e^2\left[(A_1)^2 - (A_4)^2\right] \end{aligned} \tag{9}$$

and its h.c, and respectively

$$e^{-h}\left[e^{-f}\left(A_{4,r} + h_{,r}A_4\right) - e^{-h}\left(A_{1,t} + f_{,t}A_1\right)\right]_{,t} = ie\left[e^{-f}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) - 2ieA_1\overline{\Phi}\Phi\right] \tag{10}$$

and

$$\begin{aligned} &e^{-f}\left[e^{-f}\left(A_{4,r} + h_{,r}A_4\right) - e^{-h}\left(A_{1,t} + f_{,t}A_1\right)\right]_{,r} + 2\frac{e^{-f}}{r}\left[e^{-f}\left(A_{4,r} + h_{,r}A_4\right)\right] \\ &- 2\frac{e^{-f}}{r}\left[e^{-h}\left(A_{1,r} + f_{,t}A_1\right)\right] = ie\left[e^{-h}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right) - 2ieA_4\overline{\Phi}\Phi\right] \end{aligned} \tag{11}$$

where $(\cdot)$ stands for usual derivative.

Using the Lorentz condition (13) and the Maxwell equation (11) and (12), we read

$$e^{-f}\left[A_{1,r} + \left(h_{,r} + \frac{2}{r}\right)A_1\right] + e^{-h}\left[e^{-f}\left(A_{4,t} + f_{,t}A_4\right)\right] = 0 \tag{12}$$

The Einstein equation explicitly become

$$2\frac{e^{-2f}}{r}h_{,r} - \frac{1 - e^{-2f}}{r^2} = \kappa\left[\left(\overline{\Phi}_{;1}\Phi_{;1} + \overline{\Phi}_{;4}\Phi_{;4}\right) - m_0^2\overline{\Phi}\Phi - \frac{1}{2}\left(F_{14}\right)^2\right] \tag{13}$$

and

$$\begin{aligned} &e^{-2f}\left[h_{,rr} + \left(h_{,r} - f_{,r}\right)h_{,r} + \frac{1}{r}\left(h_{,r} - f_{,r}\right)\right] - \\ &- e^{-2h}\left[f_{,tt} + \left(f_{,t} - h_{,t}\right)f_{,t}\right] = \kappa\left[\left(\overline{\Phi}_{;1}\Phi_{;1} - \overline{\Phi}_{;4}\Phi_{;4}\right) + m_0^2\overline{\Phi}\Phi - \frac{1}{2}\left(F_{14}\right)^2\right] \end{aligned} \tag{14}$$

respectively

$$2\frac{e^{-2f}}{r}f_{,r}+\frac{1-e^{-2f}}{r^{2}}=\kappa\left[\left(\overline{\Phi}_{;1}\Phi_{;1}-\overline{\Phi}_{;4}\Phi_{;4}\right)+m_0^2\overline{\Phi}\Phi-\frac{1}{2}\left(F_{14}\right)^2\right] \tag{15}$$

$$2\frac{e^{-(f+h)}}{r}f_{,t}=\kappa\left[\overline{\Phi}_{;1}\Phi_{;4}-\overline{\Phi}_{;4}\Phi_{;1}\right] \tag{16}$$

For solving we start with the physically reasonable assumptions that the charged scalar field is the main source of both the electromagnetic and gravitational fields. Neglecting the feedback of gravity in a first order approximation, can write the equation for potential Φ (imposing $f=h=0$):

$$\Phi_{,rr}+\frac{2}{r}\Phi_{,r}-\Phi_{,tt}-em_0{}^2\Phi=0 \qquad \text{and h.c.} \tag{17}$$

with the solutions

$$\Phi=\frac{N}{r}e^{i(\omega t-kr)} \qquad \text{and} \qquad \overline{\Phi}=\frac{N}{r}e^{-i(\omega t-kr)}$$

Using the same conditions, from Maxwell equations it can be read, from (11) and (12)

$$A_{1,rr}+\frac{2}{r}A_{1,r}-\frac{2}{r^2}A_1-A_{1,tt}=-2ek\frac{|N|^2}{r^2} \tag{18}$$

and

$$A_{4,rr}+\frac{2}{r}A_{4,r}-A_{4,tt}=-2e\omega\frac{|N|^2}{r^2} \tag{19}$$

with the Lorentz condition

$$A_{1,r}+\frac{2}{r}A_1-A_{4,t}=0 \tag{20}$$

Considering the particular solutions

$$A_1=ek|N|^2 \tag{21}$$

form Maxwell equations can be found that

$$A_4=2e\omega_k|N|^2\log\left(\frac{r}{a}\right)+2ek\frac{|N|^2}{r}t \tag{22}$$

where a is a constant value.

This solution for the A_4 potential may include an arbitrary function of time $F(t)$.
Introducing this first order perturbative solution in the Einstein equation, can be read, in the hypothesis $f = f(r)$ and $h = h(r)$:

$$\frac{2}{r}h_{,r} - \frac{2}{r^2}f = k\left[\frac{|N|^2}{r^4} - 2e^2 m_0{}^2\frac{|N|^4}{r^2}\right] + k\left[4e^2 m_0{}^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \tag{23}$$

and

$$h_{,rr} + \frac{1}{r}\left(h_{,r} - f_{,r}\right) + f_{,tt} = k\left[-\frac{|N|^2}{r^4} + 2e^2 m_0{}^2\frac{|N|^4}{r^2}\right] + k\left[4e^2 m_0{}^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \tag{24}$$

respectively

$$\frac{2}{r}f_{,r} + \frac{2}{r}f = k\left[2m_0{}^2\frac{|N|^2}{r^2} + \frac{|N|^2}{r^4} + 2e^2 m_0{}^2\frac{|N|^4}{r^2}\right] + k\left[4e^2 m_0{}^2\frac{|N|^4}{r^2}\log\left(\frac{r}{r_0}\right)\right] \tag{25}$$

The analytically solutions of this system equation, can be read as:

$$\begin{aligned} f(r) = &-\frac{1}{4}\frac{\kappa|N|^2}{r^2} + \kappa|N|^2\ln(r)m_0^2 + \kappa e^2|N|^4\ln(r)m_0^2 + \\ &+\kappa e^2|N|^4\ln(r)k^2 + \kappa|N|^2\ln(r)k^2 - \frac{1}{2}\frac{\kappa e^2|N|^4 k^2 t^2}{r^2} + \frac{W_1}{r} + W_2 \end{aligned} \tag{26}$$

and,

$$\mathrm{h(r)} = -\frac{\kappa|\mathrm{N}|^2}{\mathrm{r}^2} - \frac{1}{2}\frac{1}{\mathrm{r}^2}\ln\left(\frac{\mathrm{r}^{\chi-2} + \mathrm{k}^2\mathrm{r}^{-\chi}}{\mathrm{k}^2 + \mathrm{m}_0{}^2}\right) + \frac{4\mathrm{W}_1}{\mathrm{r}} + 4\mathrm{W}_2 \tag{27}$$

with

$$\chi = -2\kappa^2|\mathrm{N}|^2\mathrm{m}_0^2 - 2\kappa^2|\mathrm{N}|^2\mathrm{k}^2$$

and W_1, W_2 are two integration constant.
Defining

$$a = \kappa m_0{}^2|N|^2, \qquad b = \kappa|N|^2, \qquad c = \kappa e^2 m_0{}^2|N|^4$$

These solutions can be written as those from [16].

$$f(r) = -\frac{1}{4}\frac{b}{r^2} + b\ln(r)\omega^2 + \frac{c}{m_0^2}\ln(r)\omega^2 - \frac{1}{2}\frac{c}{m_0^2}\frac{k^2t^2}{r^2} + \frac{W_1}{r} + W_2$$

and respectively

$$h(r) = -\frac{b}{r^2} - \frac{1}{2}\frac{1}{r^2}\ln\left(\frac{r^{\chi-2} + k^2 r^{-\chi}}{\omega^2}\right) + \frac{4W_1}{r} + 4W_2$$

In the particular case of a constant potential

$$\Phi = \frac{N}{r}$$

thus, following the same steps, it can be read $\mathrm{h}(\mathrm{r})$ and $\mathrm{f}(\mathrm{r})$ function as

$$f(r) = -\frac{1}{4}\frac{\kappa|N|^2}{r^2} + \kappa|N|^2\ln(r)m_0^2 + \frac{W_1}{r} \tag{28}$$

and

$$h(r) = \frac{1}{2}\frac{\kappa|N|^4 e^2}{r^2}t^2 - \kappa|N|^4 e^2\ln(r)\left(k^2 + m_0^2\right) + 2\frac{\kappa e^2|N|^4 k\sqrt{k^2 + m_0^2}}{r}t^2 + \frac{W_1}{r} + W_2 \tag{29}$$

3. RESULTS AND DISCUSSION

This results are obtained with a computational symbolic script which is developed in roder to succeed in solving this kind of differential system equations. The script is build in an independent space-time metric structure . It was tested on several space-times cases and will be a subject for a future paper.

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