ENERGY – MOMENTUM TENSOR IN CURVED SPACE - TIME

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Abstract: The aim of this paper is to study the energy momentum tensor for scalar and electromagnetic fields for symmetric curved space-time configuration. The specific results are considered for a consecrated space –time configuration.

Keywords: Dirac field, computer algebra

1. INTRODUCTION

Systems of interacting boson fields are an interesting subject in the last years [1,2]. From the problem of dark matter to boson stars' study, boson interacting fields are involved [3-6]. For a general problem of Klein-Gordon field interacting with Maxwell electromagnetic one, in presence of gravity, many problems are still opened [7]. Analytically, exactly solvable models for boson with large self interaction and for boson-fermion systems have been worked out only in low-dimensional gravity.
We started from the necessity of studying a larger number of space-time configurations and, from the observation that the building of the fields' equations system request a huge volume of computations. These reasons are leaded to build, check and propose a new MAPLE set of procedures in order to succeed in this purpose. This paper is a result of this new MAPLE package.

2. FIELD EQUATION

Considering a Lagrangean density in a $SO(3,1)\times U(1)$ gauge invariance interaction of charged boson of mass m_0 with electromagnetic field, of the form

$$L = \eta^{ab}\overline{\Phi}_{,a}\Phi_{,b} + m_0{}^2\,\overline{\Phi}\Phi + \frac{1}{4}F^{ab}F_{ab} \tag{1}$$

where

$$\Phi_{,a} = \Phi_{|a} - ieA_a\Phi$$

and

$$\overline{\Phi}_{,a} = \overline{\Phi}_{|a} + ieA_a\overline{\Phi} \tag{2}$$

The electromganetic Maxwell tensor is

$$F_{ab} = A_{b:a} - A_{a:b}$$

is expressed in the terms of the Levi-Civita covariant derivative of the four-potential $\{A_a\}_{a=\overline{1,4}}$, i.e.

$$A_{a;b} = A_{a|b} - A_c \Gamma^c{}_{ab}$$

where the Christoffel symbols are computed as

$$\Gamma^c{}_{ab} = g^{cd} \frac{1}{2}\left(\frac{\partial g_{ba}}{\partial x^d} + \frac{\partial g_{db}}{\partial x^a} - \frac{\partial g_{da}}{\partial x^b} \right)$$

Building up the energy-momentum tensor

$$T_{ab} = \overline{\Phi}_{:a}\Phi_{:a} + \overline{\Phi}_{:a}\Phi_{:a} + F_{ac}F_b{}^c - \eta_{ab}L \,. \tag{3}$$

it can be derived the Einstein equation

$$G_{ab} = kT_{ab}\,, \tag{4}$$

where the tensor G_{ab} have the explicit form expressed using the Ricci tensor R_{ab}

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \tag{5}$$

3. REVIEW OF MAPLE ROUTINES

The main advantage of the present procedures set is coming from the new approaching strategy [11]. The anterior algorithms [12] was developed in a linear manner, being designed to succeed in writing down the energy – momentum tensor for electromagnetic, scalar and gravitational field, using explicit scalar filed covariant derivative implementation.
A different strategy is adopted now, using an Euler – Lagrange procedure in order to obtain the Klein - Gordon and Maxwell equations. This variational approach needs a preliminary structure. Let us describe briefly the algorithm, pointing only the difference of previous procedure sets.
The first part of the program starts, after initializing the main used package, with a set of definitions for the entire set of necessary objects nd global variables. For example, for the particular case of using polar coordinates, it could be written

```
>coord:=[r,theta,phi,t];
>Mg:=array(1..4,1..4,symmetric,
[(1,1)=g11(coord[1],coord[2],coord[3],coord[4])),
(1,2)=g12(coord[1],coord[2],coord[3],coord[4]),....,
(2,2)=g22(coord[1],coord[2],coord[3],coord[4])),(....,
(3,3)=g33(coord[1],coord[2],coord[3],coord[4])),....,
(4,4)=g44(coord[1],coord[2],coord[3],coord[4])])
>g:=create([-1,-1],op(Mg));
```

Now, in the second level, one is able to compute the first and second order derivatives of the metric tensor, the Christoffel symbols, the Riemann and Ricci tensors and finally, the Einstein tensor components, as it follows:

```
> D1g:=d1metric(g,coord);
> D2g:=d2metric(D1g,coord);
> Cf1:=Christoffel1(D1g);
> Cf2:=Christoffel2(g_inv,Cf1);
> Rmn:=Riemann(g_inv,D2g,Cf1);
> Rc:=Ricci(g_inv,Rmn);
> Rscalar:=Ricciscalar(g_inv,Rc);
> Rsc:=Rscalar[compts];
> Einstein1:=lin_com(1,Rc,1/2*Rsc,g);
```

The next level of the present script is dedicated to the Lagrangian symbolic definitions. This point is essential and represents the gate in order to introduce the considered specifications. The central idea was to declare for each involved field, a symbolic variable which will help in defining the Lagrangian density' terms.

```
> Phi_d_symbolic:=create([-1],array(1..4,
[(1)=Phi_d[1](coord[1],coord[2],coord[3],coord[4]),
(2)=Phi_d[2](coord[1],coord[2],coord[3],coord[4]), .....,
(4)=Phi_d[4](coord[1],coord[2],coord[3],coord[4])]));
> Theta_d_symbolic:=create([-1],array(1..4,
[(1)=Theta_d[1](coord[1],coord[2],coord[3],coord[4]),
 ........,
> Ad_symbol:=array(1..4,1..4);
> for i from 1 to 4 do : for j from 1 to 4 do
 Ad_symbol[i,j]:=Ad[i,j](coord[1],.......,coord[4])
end do : end do;
> .......
> A_d_symbol_symmetric:=permute_indices(A_d_symbol,[2,1]);
> F_ab_symbol:=lin_com(1,A_d_symbol,-1,A_d_symbol_sim);
```

In order to implement the Euler - Lagrange procedure for obtaining the Maxwell' system of equations, it has to differentiate with respect to the symbolic defined variables. The explicit form for the Maxwell system is finally written down, using a multi-level recursive replacing procedure of the symbolic variables with the specific ones. In a different block instruction, is built the necessary Lorentz condition.

```
> Current_A:=array(1..4);
  Field_A:=A_gauge[compts];
> for i from 1 to 4 do CoordC=Field_A[i] :
  L1A:=subs(Field_A[i]=CoordC,L):
  .......;
  end do;
> Momentum _rigid_A:=array(1..4,1..4);
  Force_rigid_A:=array(1..4);
>.....
> for l from 1 to 4 do
  for m from 1 to 4 do
  Ltemp[0]:=Momentum_At[l,m]:
  for j from 1 to 4 do :
  for k from 1 to 4 do
  Ltemp[(j-1)*4+k]:=subs(Ad[j,k](coord[1],coord[2],coord[3],coord[4])=Ac1[j,k],
  Ltemp[(j-1)*4+k-1])  end do :
  end do:
```

```
for k from 0 to 16 do
tempq[k]:=expand(Ltemp[k]) end do:
Momentum_Att[l,m]:=expand(tempq[16]):
 end do end do;
```

Finally, in order to obtain a set of simpler equations, it has to impose specific ansatz conditions. Further, considering supplementary hypotheses could be derived solutions for the fields' equations system.

4. SPECIFIC RESULTS

Let us consider a spherically symmetric configuration describe by a metric tensor of static conformal type, expressed in Schwarzchild coordinates as

$$ds^2 = e^{2(H(r))}dr^2 + r^2 d\theta^2 + e^{2(G(r))}d\varphi^2 - e^{2(F(r))}dt^2 \tag{6}$$

The Christoffel symbols derived in this frame are

$$\begin{aligned} \Gamma^2{}_{12} &= -\Gamma^1 22 = \frac{1}{r}e^{-(H(r))} \\ \Gamma^3{}_{13} &= -\Gamma^1{}_{33} = G'(r)e^{-(H(r))} \\ \Gamma^4{}_{14} &= \Gamma^1{}_{44} = F'(r)e^{-(H(r))} \end{aligned} \tag{7}$$

where we used

$$F'(r) = \frac{dF(r)}{dr} \qquad \text{and} \qquad G'(r) = \frac{dG(r)}{dr}$$

The Einstein tensor G_{ab} has the following non-vanishing components

$$G_{11} = \frac{1}{r}\left[G'(r) + F'(r) + rF'(r)G'(r)\right]e^{-2(H(r))}$$

$$\begin{aligned} G_{22} = \big[&G''(r) + G'(r)^2 - H'(r)G'(r) + \\ &+ F''(r) + F'(r)^2 - H'(r)F'(r) + F'(r)G'(r)\big]e^{-2(H(r))} \end{aligned}$$

$$\begin{aligned} G_{33} = \frac{e^{-2(H(r))}}{r}\big[&-H'(r) + rF''(r)^2 - \\ &- rH'(r)F'(r) + F'(r) + rF'(r)^2\big] \end{aligned}$$

$$\begin{aligned} G_{44} = \frac{e^{-2(H(r))}}{r}\big[&-H'(r) + rG''(r) + rG'(r)^2 - \\ &- rH'(r)G'(r) + G'(r)\big] \end{aligned} \tag{8}$$

The Maxwell tensor

$$F_{ab} = A_{b:a} - A_{a:b} \tag{9}$$

in the particular case of working in the minimally symmetric ansatz $A_1 = A_1(r,t)$, $A_4 = A_4(r,t)$, $\Phi = \Phi(r,t)$, has a single non-vanishing Maxwell tensor (9) component is:

$$F_{14} = -F_{41} = -\frac{\left\lfloor e^{H(r)} A_{1,t} - F'(r) e^{F(r)} A_4 - e^{F(r)} A_{4,r} \right\rfloor}{e^{H(r)} e^{F(r)}} \tag{10}$$

Building up the energy-momentum tensor

$$T_{ab} = \overline{\Phi}_{:a}\Phi_{:b} + \overline{\Phi}_{:b}\Phi_{:a} + F_{ac}F_b{}^c - \eta_{ab}L. \tag{11}$$

where the energy-momentum tensor T_{ab} has the explicit form

$$\begin{aligned}
T_{11} = & -m_0^2\overline{\Phi}\Phi + e^2\overline{\Phi}\Phi\left(A_1{}^2 + A_4{}^2\right) + \\
& + ie\left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) + \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\
& + \left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] + \\
& - \frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2
\end{aligned}$$

-

$$\begin{aligned}
T22 = & -m_0^2\overline{\Phi}\Phi - e^2\overline{\Phi}\Phi\left(A_1{}^2 - A_4{}^2\right) \\
& - \left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) - \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\
& - \left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} - \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] + \\
& - \frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2
\end{aligned}$$

$$T_{33} = T_{22}$$

$$
\begin{aligned}
T_{44} &= m_0^2\overline{\Phi}\Phi + +e^2\overline{\Phi}\Phi\left(A_1{}^2 + A_4{}^2\right) + \\
&+\left[\frac{A_1}{e^{(H(r))}}\left(\overline{\Phi}\Phi_{,r} - \overline{\Phi}_{,r}\Phi\right) + \frac{A_4}{e^{(F(r))}}\left(\overline{\Phi}\Phi_{,t} - \overline{\Phi}_{,t}\Phi\right)\right] + \\
&+\left[\frac{1}{e^{(2H(r))}}\overline{\Phi}_{,r}\Phi_{,r} + \frac{1}{e^{(2F(r))}}\overline{\Phi}_{,t}\Phi_{,t}\right] + \\
&+\frac{1}{2}\frac{1}{e^{2H(r)}e^{2F(r)}}\left[e^{H(r)}A_{1,t} - F'(r)e^{F(r)}A_4 - e^{F(r)}A_{4,r}\right]^2
\end{aligned}
$$

Using these procedures can be developed tools for studying the proprieties of the curved space-times. In next works we will derive some result on the solutions of this kind of equations.

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