BEM – BOUNDARY ELEMENTS METHOD USED AT THE STUDY OF STRESS CONCENTRATORS (CUTS-OFF) AT ROTATIONAL DISKS IN AXIAL-SYMMETRIC THERMAL REGIME PART I: BEM – THEORETICAL CONSIDERATIONS

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Abstract: The present paper presents the theoretical and experimental method for determination the deformations and the stability loss of disk stressed by an axial-symmetric field, variable according to disk radius and thickness, superposed with a field of membrane tensions given by the revolution movement. The experimental results confirm theoretical hypothesis.

Keywords: disk, thermal field, stress, membrane tension, stability.

1. INTRODUCTION

The great diversity of practical applications in which disks may be found, justify the necessity to continue the study of disks. The great stability brought by stability phenomenon, justify the latest theoretical and experimental research. By creating some structures from materials with great resistance and small weight favor the creation of some constructions with reduced dimensions.

The problem of work safety has a great importance and it demands for the real material behavior know-how. Also, the existence of some cuts-off of different shapes and sizes, in disks, is often met. Cuts-off may bring some advantages in disk behavior at different external stresses, but at the same time they may become real stress concentrators.

The study of tension and deformation state, from around of such concentrators performed in disks become a very important problem, even though it is very difficult. Of the same importance is the elastic stability study of such “structures”.

In such context the present paper may be placed, through which shown a way to determine the tension and deformation state from disk plane, from around some concentrators; the stress is given by membrane tensions, which are due to temperature difference along the disk radius and disk revolution movement, having a constant angular speed ω.

In this paper are studied several disks with cuts-off of different shapes, disks having an axial-symmetric temperature distribution with a non-uniform variation along the radius, and at the same time an uniform revolution around the axis of rotation, perpendicular onto the plane disk. The study has been carried out through the finite element method (FEM) as well as the boundary element method (BEM).

The theory on which MEF is based on is very well established. Should be specify that for axial-symmetric sections, MEF presents some simplifications, besides MEF general theory – in this way it gets to a smaller number of finite elements, and therefore at great calculus advantages. At axial-symmetric bodies with axial-symmetric tension states, generally calculus is made with reference to the median plane. In this way a threedimensional problem may be reduced, in a first phase, to a two-dimensional problem. The base MEF relations are deduced from energetic considerations and are based on the principle of stationary value of the elastic potential.

Next, the theoretical principles of Boundary Element Method (BEM-for short) are given. This method is very often employed to solve some problems such as: a study of those areas around tension concentrators. This method is hard to utilized in those cases when equations with partially derivate are defined on an infinite domain. Generally at equations with linear partially derivate, boundary shape and conditions on boundary determine the unique solution. Based on this method, only the studied boundary is divided into finite elements; this leads to a reduce of the work load and to an obtaining of good results at small execution times.

BEM is based on Betti theorem. Also, are used plane equations of Lame (some authors consider these equations being Navier-Cauchy equations).

$$\begin{cases} G\Delta u + (\lambda G) + \dfrac{\partial \varepsilon_v}{\partial_x} = 0; \\ G\Delta v - (\lambda G) + \dfrac{\partial \varepsilon_v}{\partial_y} = 0; \\ G\Delta w - (\lambda G) + \dfrac{\partial \varepsilon_v}{\partial_z} = 0; \end{cases} \quad (1) \qquad \begin{cases} \varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \dfrac{\partial_u}{\partial_z} + \dfrac{\partial_v}{\partial_y} + \dfrac{\partial_w}{\partial_z}; \\ \lambda = \dfrac{\upsilon E}{(1+\upsilon)(1-2\upsilon)}; \end{cases} \quad (2)$$

For planar state of tension, relations (1) take the form:

$$\begin{cases} G\Delta u + \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \varepsilon_v}{\partial x} = 0 \\ G\Delta v + \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \varepsilon_v}{\partial y} = 0 \end{cases} \quad (3)$$

It should be specified that at planar state of tension, R. Hook law take the form:

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\varepsilon_v; \\ \sigma_y = 2G\varepsilon_y + \lambda\varepsilon_v; \\ \tau_{xy} = G\gamma_{xy} \end{cases} \quad (4)$$

$$\begin{aligned} \varepsilon_z &= \frac{\upsilon}{1-\upsilon}(\varepsilon_x + \varepsilon_y) \\ \varepsilon_v &= \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\upsilon}{1-\upsilon}(\varepsilon_x + \varepsilon_y) = \frac{1-2\upsilon}{E}(\sigma_x - \sigma_y) \end{aligned} \quad (5)$$

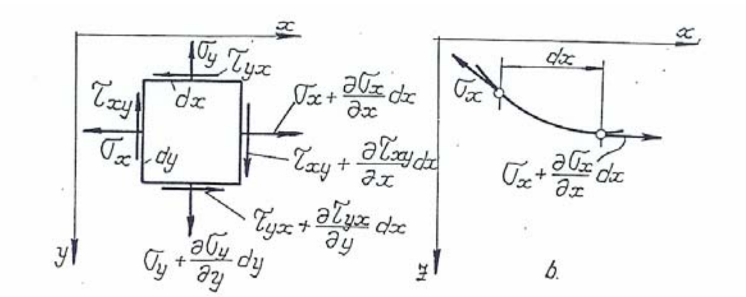

Fig. 1

In the above figure, plate element is in equilibrium (mass forces are not taken into account). Are obtained two relations of equilibrium in plane, being independent of other relations.

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0; \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial_v} = 0; \end{cases} \quad (6) \qquad \begin{cases} G\left(2\dfrac{\partial \in_x}{\partial x} + \dfrac{\partial \gamma_{xy}}{\partial y}\right) + \lambda \dfrac{\partial \in_v}{\partial x} = 0; \\ G\left(2\dfrac{\partial \in_y}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial x}\right) + \lambda \dfrac{\partial \in_v}{\partial y} = 0; \end{cases} \quad (7)$$

Relations (4) were replaced in (6). Geometric relations in plane are known as:

$$\in_x = \frac{\partial u}{\partial x}; \quad \in_y = \frac{\partial v}{\partial y}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (8)$$

which are replaced in (7) and the following relations results:

$$\begin{cases} G\left(2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 v}{\partial x \partial y}\right) + \lambda \dfrac{\partial \in_v}{\partial x} = 0; \\ G\Delta u + G\dfrac{\partial}{2x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0. \end{cases} \quad (9)$$

At planar state of tension the following conditions are to be met:

$$\begin{cases} \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \in_v}{\partial x} + G\Delta u = \delta; \\ \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \in_v}{\partial y} + G\Delta v = 0; \end{cases} \quad (10) \qquad \begin{cases} \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \in_v}{\partial x} + G\Delta u = 0; \\ \left(\dfrac{3}{2}\lambda + G\right)\dfrac{\partial \in_v}{\partial y} + G\Delta v = \delta; \end{cases} \quad (11)$$

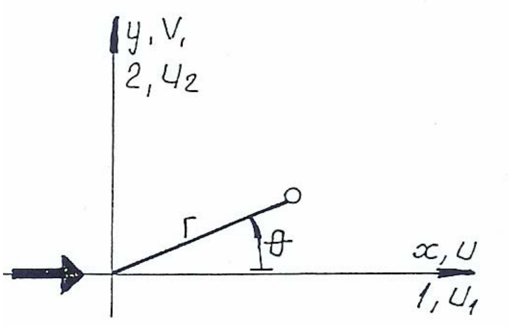

Fig. 1

The elastic plane loaded with an unitary concentrated force after x axis, which acts at the origin point is a solution of equations (10); clearly elastic plane loaded with an unitary concentrated force oriented after y axis, which acts at the origin point is a solution of equations (11).

Based on figure 1 results the following solution (force after x axis and corresponding in (10)).

$$\begin{cases} u = -\dfrac{(3+\upsilon)(1+\upsilon)}{4\pi E}\ln\dfrac{r}{a} + \dfrac{(1+\upsilon)^2}{4\pi E}\cos^2\theta \\ v = \dfrac{(1+\upsilon)^2}{4\pi E}\sin\theta\cos\theta \end{cases} \quad (12)$$

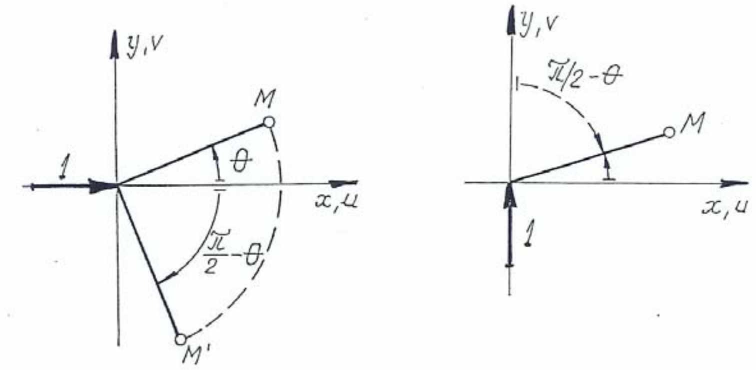

Fig 2.

For the case when the unitary force is found on the y axis, the solution for equation (11) is found by putting in relations (12) θ-π/2 instead of θ; this aspect being evidentiated in figure 2.

Are used the notations for axes with 1 and 2. Is used the notation U_{ij} which represents the displacement of one point of plane measured in direction of I, due to an unitary concentrated force, which acts in the direction of j. For the fundamental solution will be used capital letters and results:

$$\begin{cases} U_{11} = -\dfrac{(3-\upsilon)(1+\upsilon)}{4\pi E}\ln\dfrac{r_{PQ}}{a} + \dfrac{(1+\upsilon)^2}{4\pi E}\cos^2\theta; \\ U_{12} = U_{21} = \dfrac{(1+\upsilon)^2}{4\pi E}\sin\theta\cos\theta \\ U_{22} = -\dfrac{(3-\upsilon)(1+\upsilon)}{4\pi E}\ln\dfrac{r_{PQ}}{a} + \dfrac{(1+\upsilon)^2}{4\pi E}\sin^2\theta; \end{cases} \tag{13}$$

U_{11}, U_{12}=U_{21}, U_{22} represent the components of a symmetric tensor, in plane, called the tensor of Kelvin-Somigliana.

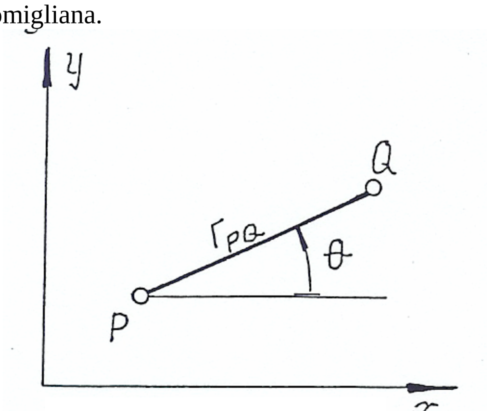

Fig. 3

With relations (13) can be calculated the displacements of some unitary forces which act anywhere in plane.
Assume that the unitary force acts in point P. The considered point is Q and r_{PQ} measure the distance between the two points (fig.3.)

The previous relations are particularized and due to a force that acts in the origin point after x axis, take birth tensions:

$$\sigma_x = -\frac{3+\upsilon}{4\pi}\frac{1}{r}\cos\theta; \quad \sigma_\theta = \frac{1-\upsilon}{4\pi}\frac{1}{r}\cos\theta; \quad \tau_{r\theta} = \frac{1-\upsilon}{4\pi}\frac{1}{r}\sin\theta \tag{14}$$

In figure 4 is shown how will be done the study of tensions that take birth onto a certain contour, noted with F, of a domain D, drawn on the elastic plane. The unitary force acts in point P, after the direction of x axis. Is specified travel direction of contour Γ and in a certain point Q placed on the contour are established: the external surface normal n and tangent t (in the direction of travel direction). Projections on the shown directions are made and towards axes system n-t is obtained.

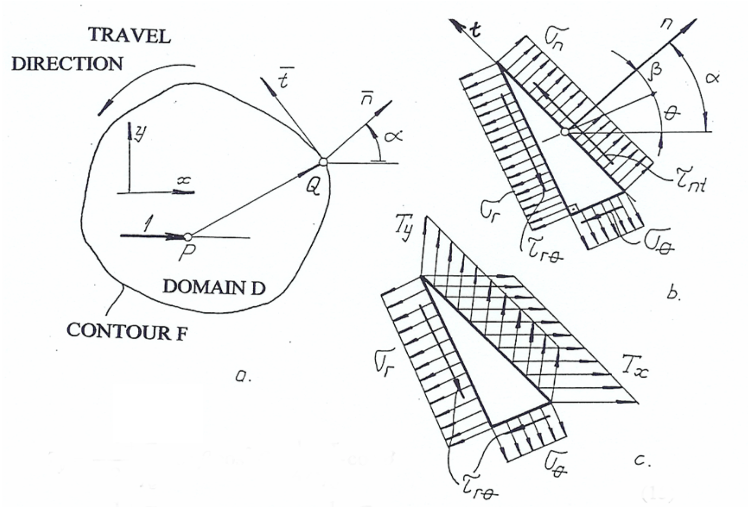

Fig. 4

If the unitary force is oriented after z axis, the elastic plane system is made with that shown in figure 4. Is used the same calculus sequence and is obtained:

$$\begin{cases} \sigma_r = -\dfrac{3+\upsilon}{4\pi}\dfrac{1}{r}\sin\theta; \\ \sigma_\theta = \dfrac{1-\upsilon}{4\pi}\dfrac{1}{r}\sin\theta; \\ \tau_{r\theta} = -\dfrac{1-\upsilon}{4\pi}\dfrac{1}{r}\cos\theta; \end{cases} \qquad (15) \qquad \begin{cases} T_x = -\dfrac{1+\upsilon}{2\pi r_{PQ}}\cos\beta\sin\theta\cos\theta - \dfrac{1-\upsilon}{4\pi r_{PQ}}\sin\beta; \\ T_y = -\dfrac{1+\upsilon}{2\pi r_{PQ}}\cos\beta\sin^2\theta - \dfrac{1-\upsilon}{4\pi r_{PQ}}\cos\beta. \end{cases} \qquad (16)$$

$$\begin{cases} T_{11} = -\dfrac{1+\upsilon}{2\pi r_{PQ}}\cos\beta\cos^2\theta - \dfrac{1-\upsilon}{4\pi r_{PQ}}\cos\beta; \\ T_{12} = -\dfrac{1+\upsilon}{2\pi r_{PQ}}\cos\beta\sin\theta\cos\theta + \dfrac{1-\upsilon}{4\pi r_{PQ}}\sin\beta; \\ T_{22} = -\dfrac{1+\upsilon}{2\pi r_{PQ}}\cos\beta\sin^2\theta - \dfrac{1-\upsilon}{4\pi r_{PQ}}\cos\beta. \end{cases} \qquad (17)$$

It is specified the fact that through T_x and T_y have been noted the corresponding tensions in point Q on the contour of domain D which extends on a surface of normal n and is, in fact, measured parallel with axes x and y. If return to previous notation and interpretation T_{ij} (measured tension after i direction, but due to an unitary force that acts in j direction) is obtained:

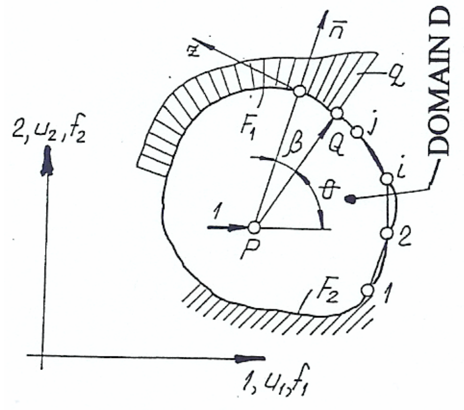

Fig. 5

Betti's theorem is applied and obtained:

$$\begin{cases} \frac{1}{2}u_1(P)+\int\limits_F (T_{11}(PQ)u_1(Q)+T_{21}(PQ)u_2(Q))dF = \\ = \int\limits_F (U_{11}(PQ)t_1(Q)+U_{21}(PQ)t_2(Q))dF; \\ \frac{1}{2}u_2(P)+\int\limits_F (T_{12}(PQ)u_1(Q)+T_{22}(PQ)u_2(Q))dF = \\ = \int\limits_F (U_{12}(PQ)t_1(Q)+U_{22}(PQ)t_2(Q))dF; \end{cases} \tag{18}$$

Generalized, with i, j taking the values 1,2…:

$$\frac{1}{2}\delta_{ij}u_i(P)+\int\limits_F T_{ij}(PQ)u_i(Q)dF = \int\limits_F U_{ij}(PQ)f_j(Q)dF \tag{19}$$

Significations: δ_{ij} is the Kronecker symbol, $\delta_{ij}=0, i=j;\ \ 1, i=j;$ $u_i(P)$ is the actual displacement value in a certain point on the boundary G; T(P,Q) is the tension produced in point Q on the boundary G, due to unitary force applied in point P.

First relation (18) corresponds to a unitary force that works after axis 1, and the second relation (18) corresponds to a unitary force that works in direction of axis 2 (fig.5). Points P and Q are on the boundary. Point O describes the boundary F, and thus relation (19) represents an integral equation which allows solving the problem. Numerical solving of relation (19) is done with BEM.

In relations (18) the actual loading tensions noted with t_1 and t_2 are projected after the axes noted with 1 and 2 (fig.6).

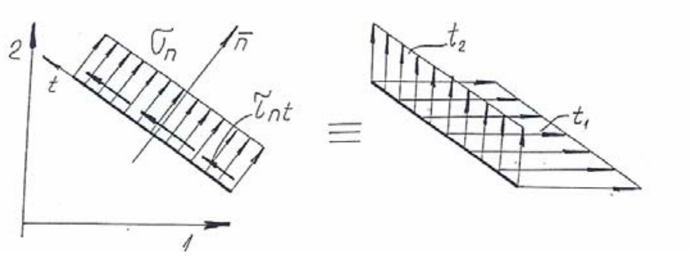

Fig. 6.

Is specify that each boundary node is characterized by four dimensions:u_1, u_2, t_1, t_2. From these, two are always known and in function of boundary bindings, the following situations detach: $\begin{cases} t_1 = \sigma_n \cos\alpha - \tau_{nt}\sin\alpha; \\ t_2 = \sigma_n \sin\alpha - \tau_{nt}\cos\alpha. \end{cases}$

- On boundary section are known f_1 and f_2;
- On embedded section are known u_1 and u_2;
- On the boundary seated section are known u_1 and f_2 or u_2 and f_1.

Same as at MEF, dimensions u and f from the inner of an element are unique determined by the nodal values through the interpolation function:

$$\frac{1}{2}\delta_{ij}u_{ip} + \sum\left(\int\limits_{Fe} T_{ij}(P,Q)[N_i(Q)]dG\right)\{u\}_e = \sum\left(\int\limits_{Fe} u_{ij}(P,Q)[N_i(Q)]dF\right)\{t\}_e \tag{20}$$

In this final relation, the sum refers to all the elements in which boundary g has been divided; u_e and t_e represent the nodal dimensions corresponding to an element. The integrals from relation (21) will be calculated by the mean of a matrix.

Is specify that BEM is conceived having the idea that the domain is cut-off from elastic plane. Is, also, specify that displacements of the points being on the boundary F, related to axes 1 and 2, will be calculated with relations (13) in which a is an arbitrary constant and r_{PQ} is the position vector between the two points.

Tensions noted in theory with t, which develop on the boundary, is calculated with relation (18) in which θ is the angle from the horizontal to r_{PQ}, and β the angle from r_{PQ} to normal n, to boundary.

This method has several advantages, which will be shortly pointed out:

- can be digitized only "the boundary" through which can be understand only the area from around the studied domain; it gets, on one hand, to a smaller number of MEF unknown, and on the other hand, to a lower digitizing and to a diminishing of the input data volume;
- establishing with a higher accuracy of tension state, even on the boundary domain from where the number of finite elements may be considerably increased. Is specify that nodes stuffing in the interest area can be done much easier that in the case of MEF;

BEM, also, has a series of disadvantages, from which the most important one is that the method leads to a linear system with full matrix and non-symmetric (being necessary a larger volume of memory), which makes the utilization of this method to be employed especially in the case of tension concentrators.

Has been increased the fillet radius of the concentrator from disk, than calculus with BEM resumed based on the same run with MEF, to make the conditions in displacements. In calculus has been utilized the equivalent tension corresponding to specific energy theory of deformation shape modifier.

$$\sigma_{ech} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + \tau_{xy}^2}; \qquad (21)$$

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