

**BEM – BOUNDARY ELEMENTS METHOD USED AT THE STUDY OF
STRESS CONCENTRATORS (CUTS-OFF) AT ROTATIONAL DISKS
IN AXIAL-SYMMETRIC THERMAL REGIME
PART III – EIGEN VALUES PROBLEMS AT DISKS**

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One makes a few considerations regarding the way we approach the eigen values problems at disks in rotational motion and in thermal regime. More specific, one specifies the way to determine the critical temperatures of loss of stability at rotational disks. The temperature distribution is axial-symmetric (the temperature law of variation on the radius and on thickness is the subject of other papers). In comparison to the equations obtained before, we further on use a relation written in polar coordinates (the calculus is much simple for the case of symmetric structures axial symmetric loaded).

$$\Delta^2 w = \frac{h}{D} \left[\sigma_r \frac{\partial^2 w}{\partial r^2} + \sigma_\theta \frac{1}{r^2} \left(r \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial \theta^2} \right) + 2\tau_{r\theta} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) - R \frac{\partial w}{\partial \theta} - \Theta \frac{1}{r} \frac{\partial w}{\partial \theta} \right] \quad (1)$$

with the following notations :

- $\sigma_r, \sigma_\theta, \tau_{r\theta}$ - membrane stresses ;
- R and Θ - massic forces per volume unit which act in the plate's plane. Their values are :

$$R = \frac{\gamma}{g} \omega^2 r = k \cdot r; \quad \Theta. \quad (2)$$

One uses the Airy's function and on basis of the notation $f = \frac{\partial F}{\partial r}$ we get the expressions:

$$\sigma_r = \frac{1}{r} f - \frac{1}{3} r^2 \cdot k; \quad \sigma_\theta = \frac{df}{dr}; \quad \tau_{r\theta} = 0 \quad (3)$$

The number of nodal diameters is denoted by q and one searches the displacements whose forms are:

$$w(r, \theta) = w_1(r) \cdot \cos q\theta \quad (4)$$

We make the notation:

$$\Delta_1 w_1 = \frac{d^2 w_1}{dr^2} + \frac{1}{r} \frac{dw_1}{dr} - \frac{1}{r^2} q w_1 \quad (5)$$

On basis of the equations (2)...(5), the start relation (1) takes the form:

$$\Delta_1^2 w_1 = \frac{h}{D} \left[\left(\frac{1}{r} f - \frac{1}{3} r^2 \cdot k \right) \frac{d^2 w}{dr^2} + \frac{df}{dr} \left(\frac{1}{r} \frac{dw}{dr} - q^2 \frac{w_1}{r^2} \right) - k \cdot r \frac{dw_1}{dr} \right] \quad (6)$$

$$\Delta_1^2 w_1 = \frac{d^4 w_1}{dr^4} + \frac{2}{r} \frac{d^3 w_1}{dr^3} - \frac{1}{r^2} (1 + 2q^2) \frac{d^2 w_1}{dr^2} + \frac{1}{r^3} (1 + 2q^2) \frac{dw_1}{dr} - \frac{1}{r^4} \cdot q^4 (4 - q^2) w_1 \quad (7)$$

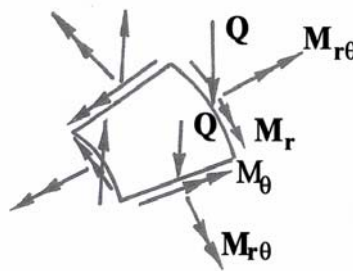


Fig 1.

In order to solve the problem of eigen values, one uses the finite differences. The problem becomes one-dimensional, this way getting a function of the current radius r .

For the solution we need the expressions of the sectional internal forces, too (Fig. 1).

$$\begin{cases} M_r = -D \left[\frac{d^2 w}{dr^2} + \nu \left(\frac{1}{r} \frac{dw}{dr} - \frac{1}{r^2} q^2 w \right) \right] \cdot \cos q\theta; \\ M_\theta = -D \left[\frac{1}{r} \frac{dw}{dr} - \frac{1}{r^2} q^2 w + \nu \frac{d^2 w}{dr^2} \right] \cdot \cos q\theta; \\ M_{r\theta} = -(1-\nu)D \left[\frac{1}{r} q \frac{dw}{dr} - \frac{1}{r^2} q w \right] \cdot \sin q\theta; \\ Q_r = -D \frac{d}{dr} \left[\frac{d^2 w_1}{dr^2} + \frac{1}{r} \frac{dw_1}{dr} - \frac{1}{r^2} q^2 w_1 \right] \cdot \cos q\theta \end{cases} \quad (8)$$

Further on one study each concrete case. For instance, at a free edge disk, one writes the following boundary condition:

$$T_r = Q_r - \frac{\partial M_{r\theta}}{r \cdot \partial \theta} = 0 \quad (9)$$

With help of this condition it results successively:

$$\begin{aligned} Q(r) = & -D \left(\frac{d^3 w_1}{dr^3} + \frac{1}{r} \frac{d^2 w_1}{dr^2} - \frac{1}{r^2} \frac{dw_1}{dr} - \right. \\ & \left. - \frac{1}{r^2} q^2 \frac{dw_1}{dr} + \frac{2}{r^3} q^2 w_1 \right) \cos q\theta \end{aligned} \quad (10)$$

$$\frac{d^3 w_1}{dr^3} + \frac{1}{r} \frac{d^2 w_1}{dr^2} - \left[1 + (2 - \nu) q^2 \right] \frac{1}{r^2} \frac{dw_1}{dr} + (3 - \nu) \frac{1}{r^3} q^2 w_1 = 0 \quad (11)$$

From the theoretical considerations one gets that the studied problem is solved on basis of equation (6), of the sectional internal forces and of the limit conditions of the concrete studied cases. One uses the central finite differences and a problem with $n+2$ unknowns yields.

One attains a problem of eigen values:

$$[A]\{W\} = \lambda[B]\{W\} \quad (12)$$

were the following notations were made:

- $\{W\}$ – the column matrix of the nodal values from the finite differences frame;
- $[A]$ – the square matrix of the coefficients of the terms which do not contain values of the function of stress f ;
- $[B]$ – the square matrix of the coefficients of the terms which contain the values of f ;
- λ -the proportionality coefficient of the reference membrane stresses field.

Finally, the basis relation takes the form:

$$D\Delta_1^2 w_1 - h \left[\left(\frac{1}{r} f_\omega - \frac{1}{2} r^2 \cdot k \right) \frac{d^2 w_1}{dr^2} + \frac{df_\omega}{dr} \left(\frac{1}{r} \frac{dw_1}{dr} - q^2 \frac{w_1}{r^2} \right) - k \cdot r \frac{dw_1}{dr} \right] = \quad (13)$$

One made the following notations:

- f_T – stress function produced by temperature;
- f_ω - stress function produced by revolution.

The first member from equation (13) together with the limit conditions form the matrix $[A]$, from equation (12), and the second member forms the matrix $[B]$. This way, one takes into account only the stresses produced by temperature, in order to get the critical values of the membrane stresses. Based on the exposed theory, one gets an easy to modify software, adaptable and easy to use. The temperature distribution has been experimentally determined. By comparing the theoretical results with the experimental data, one finds out that the temperature varies after a second degree law:

$$T(r) = T_1 + (T_2 - T_1) \cdot \left[\frac{r - r_1}{r_2 - r_1} (1 - c) + \left(\frac{r - r_1}{r_2 - r_1} \right)^2 c \right] \quad (14)$$

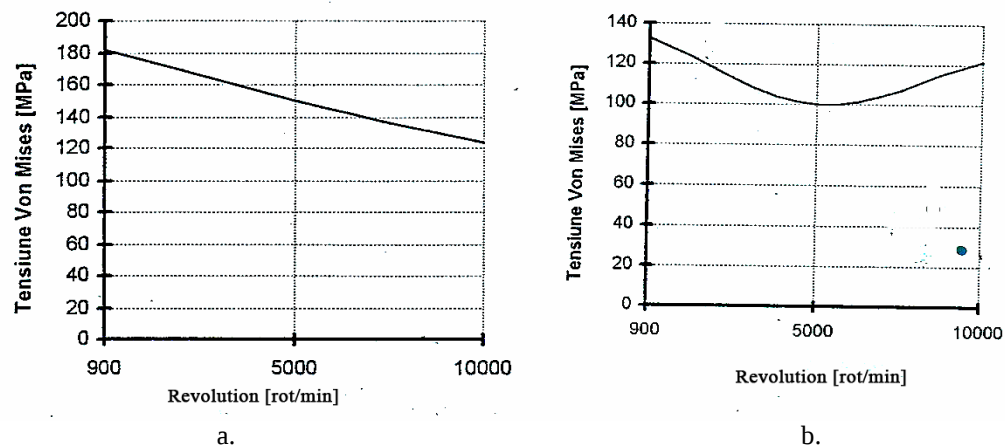


Fig. 2.

The aim of this study is shape dimensions optimization of concentrator, and thus “temperature effects” to be useful compensated and stability loss avoided. Maximum equivalent stress variation determined with the V^{th} theory von Mises is given in figure 2.a. It can be observed a stress decrease with an increase in revolution.

The diagram from figure 2.b. shows the maximum equivalent stress variation, which develops in front of concentrator (bottom releasing) with revolution. In this case, there can be seen that the maximum stress decreases with the increase of revolution up to 5000 rev/min, and then to increase further more; stress weight given by centrifugal force is bigger than the one given by temperature.

Next are given a few examples of stability loss for disks with “slots”, being in thermal axial-symmetric regime in a revolution movement (is taken into account the influence of membrane-Karman stresses field).

Figure 3 gives the modes and proper values of buckling for disks with 30 mm “slots” subjected to a temperature difference of 50 K.

DISK-SLOTS OF 30 mm.

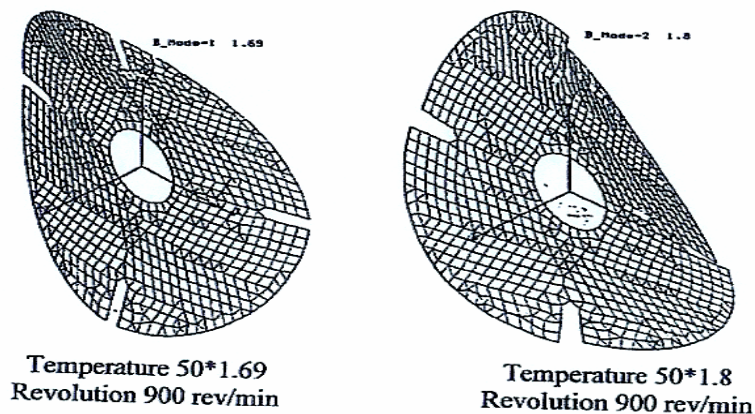


Fig. 3.

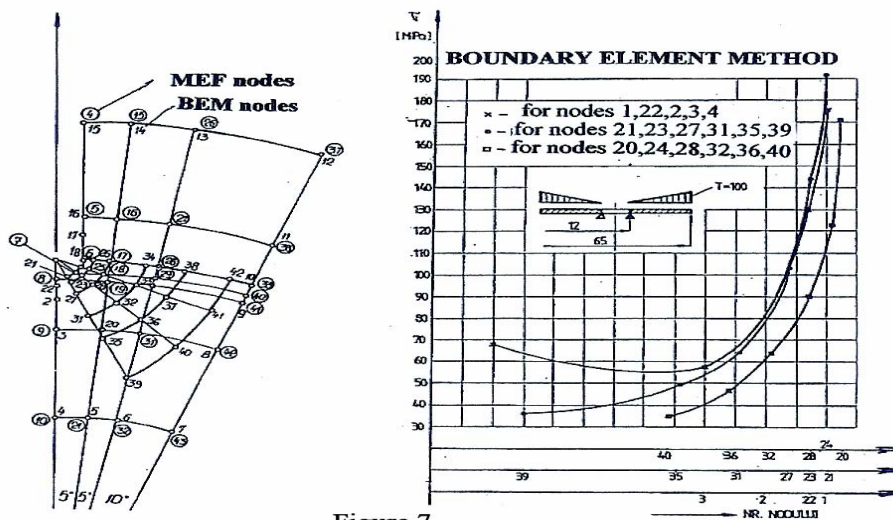


Figure 7.

$$\sigma_x = \frac{E}{1-\nu^2} (\epsilon_x + \nu \epsilon_y) - \frac{E}{1-\nu} \alpha t; \quad \sigma_y = \frac{E}{1-\nu^2} (\epsilon_y + \nu \epsilon_x) - \frac{E}{1-\nu} \alpha t$$

TABLE 1

NR. NODULI	TEMPERATURA [°C]	σ_x [MPa]	σ_y [MPa]	σ_{xy} [MPa]	$-\frac{E}{1-\nu} \alpha t$	$\sigma_{eq} = \sqrt{\sigma_x^2 + \sigma_y^2 + 3\sigma_{xy}^2}$
1	33,962	-201,41	-79,8077	-4,8172	116,4367	115,875
22	33,282	-134,689	-50,00	-7,06	114,1073	130,525
2	31,696	-65,077	-41,885	4,964	108,669	67,771
3	29,432	-15,265	-34,804	29,127	100,907	57,327

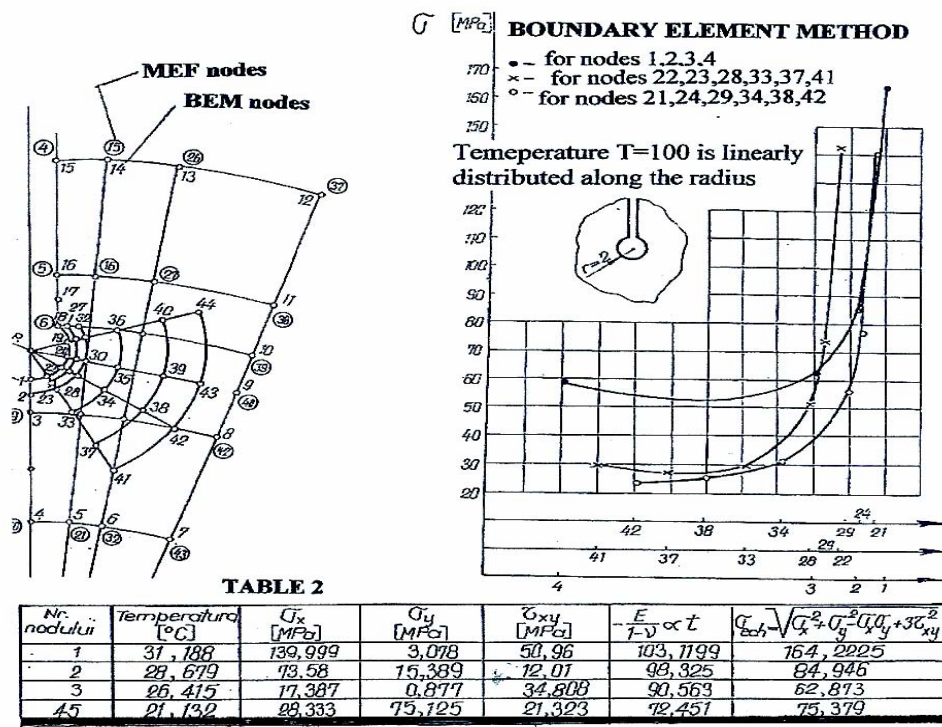


Fig 4.

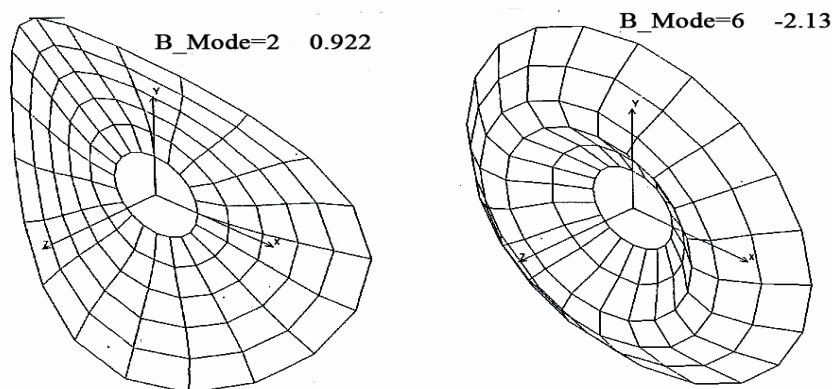


Fig. 5.

In the case of disk behavior being in rotational and thermal regime, a very important emphasize is that the tensions, deformations and temperatures are related in a very complicated manner. The precise solution for thermo elasticity problems presents great difficulties, and therefore, the number of precise solutions is very small. For axial-symmetric cases (structure, stress) were used: the finite element method, central finite differences method and the boundary element method. Next, the study is concerned with disks that presents slot shape cuts-off, like figures 3 and 4. Like for the precedent case, disks have been considered being constrained on the inner contour. Temperature difference between the outer contour and the inner contour is of 50 K, the outer contour being warmer. Temperature distribution along the radius has been established, to which a coefficient $c=0.4$ corresponds. From comparison of the two cases, can be observed that by raising the fillet radius from 1.5mm to 2mm, is obtained a good improvement of induced stress and deformation. Is notice that the obtained results are very good.

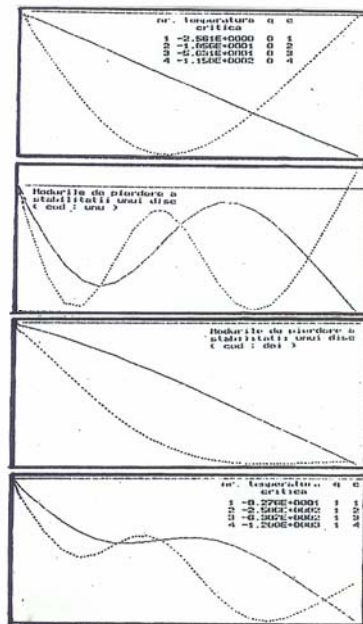


Fig. 5.a.
Inner supported disk
 $W=0, M=0$ at $r=a$
 $M=0, Q=0$ at $r=b$

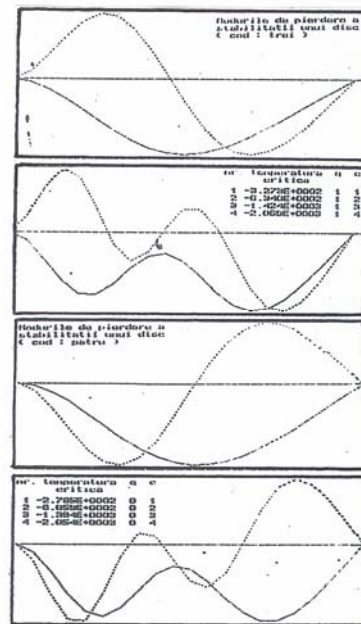
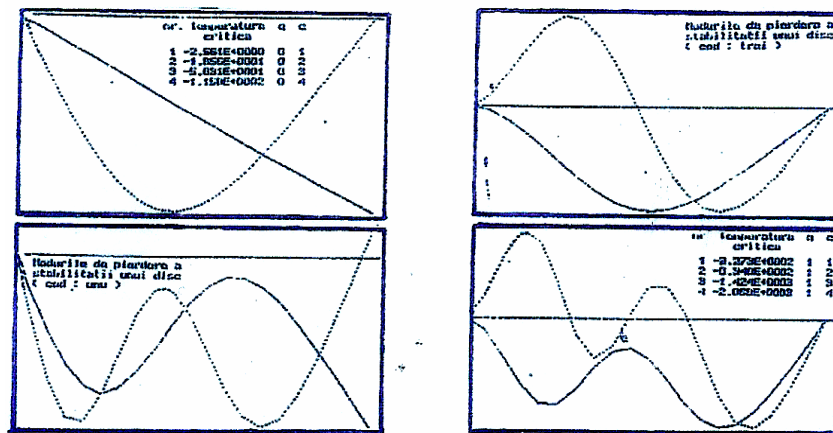


Fig. 5.b.
Fixed disk at $r=a$ and
supported at $r=b$
 $w=0, \varphi=0, w=0, M=0$



REFERENCES

- [1] TIMOSHENKO, S.P.; WOINOWSKY-KRIEGER, S. – *Teoria placilor plane si curbe*, Editura Tehnica, Bucuresti, 1969.
- [2] RADU, Gh.N.; CURTU, I. – *Elastic Stability of the Thin Disks Due to the Temperature and Rotation Stress*, 14th Symposium DANUBIA ADRIA, Porec, Croatia, 1997.
- [3] SHEPLAK, M. – *Large Deflection of Clamped Circular plates Under Initial Tension and Transitions to Membrane Behavior*, ASME, 1998.
- [4] KACHANOV, L.M. – *Foundations of the Theory of Plasticity*, North Holland Publishing Company, Amsterdam-London, 1971.
- [5] RADU, GH.N. – *Teoria plasticitatii (principii si relatii de calcul)*, Editura Lux Libris, Brasov, 2005.