# THEORETICAL RESEARCH OF THE NUMERICAL SOLUTION STABILITY IN THE VELOCITY MEASUREMENT WITH THE HOT WIRE ANEMOMETER

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**Abstract:** One presents a mathematical model of the viscous fluid flow with heat transfer around the anemometer hot wire and also in its interior, where is produced of the electrical current distribution caused by the different electric resistances as function of local temperature the conductor wire being supposed at an electric voltage on its ends. For the solving of different equation systems in the two domains, one uses their numerical integration on a polar grid, assuring the solution stability and taking into consideration the specific boundary conditions.

**Keywords:** Hot wire anemometer. Viscous fluid flow with heat transfer. Numerical integration of partial differential equations.

## 1. PROBLEM IMPORTANCE OF THE HOT WIRE ANEMOMETER MEASUREMENTS

Due to the very little size of the hot wire anemometer diameter, of the order of any micrometers, this anemometer is the single device for the velocity measurement in the boundary layers of the fluid flows adjacent to the solid bodies [1]. In the same time the small values of the Reynolds number make that the flow around the hot wire to bee laminar [2][3][4].

## 2. PARTIAL DIFFERENTIAL EQUATION SYSTEMS

Considering the heat transfer phenomenon in the movement fluid and also in the interior of the cylindrical wire with the internal heat sources due to the electric current distribution, we have the following equation systems with partial differentials in cylindrical coordinates (fig. 1).

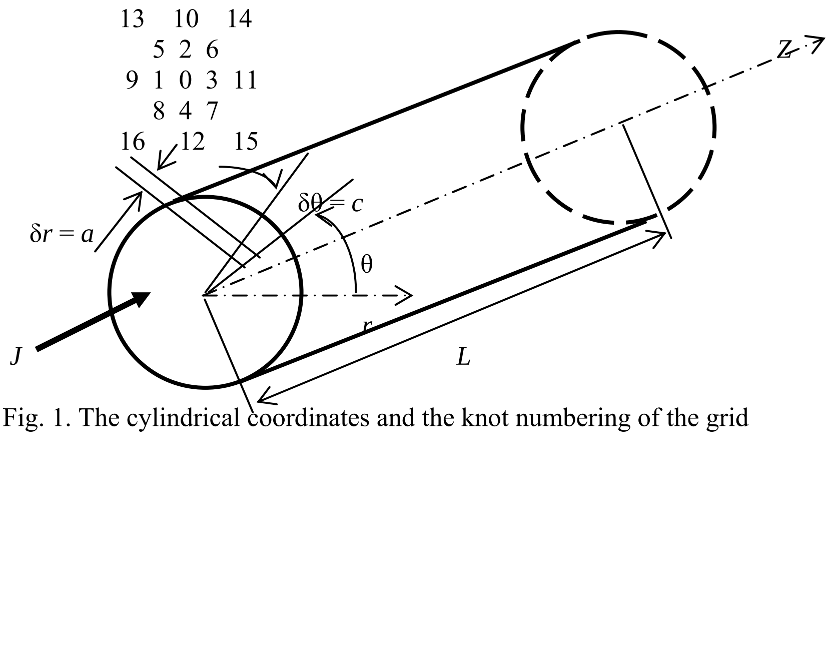



Fig. 1. The cylindrical coordinates and the knot numbering of the grid

**2.1. Viscous fluid flow equation system.** This system is composed from the two motion equations:

$$\frac{\partial V_R}{\partial R}V_R+\frac{\partial V_R}{\partial \theta}\frac{V_\theta}{R}-\frac{V_\theta^{\,2}}{R}+\frac{1}{\rho}\frac{\partial P}{\partial R}=\upsilon\left(\frac{\partial^2 V_R}{\partial R^2}+\frac{1}{R}\frac{\partial V_R}{\partial R}+\frac{1}{R^2}\frac{\partial^2 V_R}{\partial \theta^2}-\frac{V_R}{R^2}-\frac{2}{R^2}\frac{\partial V_\theta}{\partial \theta}\right), \tag{1}$$

$$\frac{\partial V_\theta}{\partial R}V_R+\frac{\partial V_\theta}{\partial \theta}\frac{V_\theta}{R}+\frac{V_R V_\theta}{R}+\frac{1}{\rho R}\frac{\partial P}{\partial \theta}=\upsilon\left(\frac{\partial^2 V_\theta}{\partial R^2}+\frac{1}{R}\frac{\partial V_\theta}{\partial R}+\frac{1}{R^2}\frac{\partial^2 V_\theta}{\partial \theta^2}+\frac{2}{R^2}\frac{\partial V_R}{\partial \theta}-\frac{V_\theta}{R^2}\right), \tag{2}$$

as well as the mass conservation equation of the incompressible fluid $\rho$ = constant

$$\frac{1}{R}\frac{\partial\left(V_R R\right)}{\partial R}+\frac{1}{R}\frac{\partial V_\theta}{\partial \theta}=0\,,\qquad \text{or}\qquad \frac{V_R}{R}+\frac{\partial V_R}{\partial R}+\frac{1}{R}\frac{\partial V_\theta}{\partial \theta}=0\,, \tag{3}$$

which being unstable in iterative numerical calculus, we can it verify identically introducing the stream line function by the relations

$$\frac{\mathrm{d}R}{V_\mathrm{R}}=\frac{R\,\mathrm{d}\theta}{V_\theta}\;\rightarrow\; -V_\theta \mathrm{d}R+V_\mathrm{R} R\,\mathrm{d}\theta=\mathrm{d}\Psi=0\;\rightarrow\; V_\mathrm{R}=\frac{1}{R}\Psi'_\theta \text{ and } V_\theta=-\Psi'_\mathrm{R}\,. \tag{4}$$

The heat conductive and convective transfer equation in the fluid steady flow domain is the following

$$\rho c\left(T'_\mathrm{R}V_\mathrm{R}+\frac{1}{R}T'_\theta V_\theta\right)=\lambda\left(T''_{\mathrm{R}^2}+\frac{1}{R}T'_\mathrm{R}+\frac{1}{R^2}T''_{\theta^2}\right). \tag{5}$$

**2.2. Heat conductive transfer equation in the hot wire domain with Joule's electric sources.** Taking into consideration the local temperature variation of the hot wire electric resistance, we can write

$$\lambda\left(T''_{\mathrm{R}^2}+\frac{1}{R}T'_\mathrm{R}+\frac{1}{R^2}T''_{\theta^2}\right)+k\frac{U_0^2}{\Re_0\left\{1+\alpha\left[T\left(R,\theta\right)-T_0\right]\right\}}=0\,, \tag{6}$$

## 3. EQUATION TRANSFORMATION FOR NUMERICAL INTEGRATION

Because the pressure function is not known on the whole domain boundaries , we can it eliminate from the two motion equations in virtue of Schwarz's commutative relation of 2nd order mixed partial differential $P''_{\mathrm{R}\theta}=P''_{\theta\mathrm{R}}$

$$\begin{aligned}&-\frac{2}{R^3}\Psi'_\theta\Psi''_{\theta^2}+\frac{1}{R^2}\Psi'_\theta\Psi'''_{\mathrm{R}\theta^2}-\frac{1}{R^2}\Psi'_\mathrm{R}\Psi'''_{\theta^3}-\frac{1}{R}\Psi'_\mathrm{R}\Psi''_{\mathrm{R}\theta}+\Psi'_\theta\Psi'''_{\mathrm{R}^3}-\Psi'_\mathrm{R}\Psi'''_{\theta\mathrm{R}^2}-\frac{1}{R^2}\Psi'_\theta\Psi'_\mathrm{R}+\frac{1}{R}\Psi'_\theta\Psi''_{\mathrm{R}^2}=\\&=\upsilon\left(\frac{2}{R}\Psi^{\mathrm{IV}}_{\mathrm{R}^2\theta^2}-\frac{2}{R^2}\Psi'''_{\mathrm{R}\theta^2}+\frac{1}{R^3}\Psi^{\mathrm{IV}}_{\theta^4}+R\Psi^{\mathrm{IV}}_{\mathrm{R}^4}+2\Psi'''_{\mathrm{R}^3}+\frac{4}{R^3}\Psi''_{\theta^2}+\frac{1}{R^2}\Psi'_\mathrm{R}-\frac{1}{R}\Psi''_{\mathrm{R}^2}\right).\end{aligned} \tag{7}$$

To win a more generality for the numerical solution, we shall utilise the dimensionless form of the equation system taking as characteristic magnitudes: the hot wire radius $R_\mathrm{w}$ (m), upstream fluid velocity $V_\infty$ (m/s), linear voltage drop $U_0$ (V/m), upstream fluid temperature $T_0$ ($^\circ$C), upstream fluid pressure $P_\infty$ (N/m$^2$) and the conventional flow rate $\Psi_\mathrm{Q}=R_\mathrm{w}V_\infty$ (m$^2$/s).

**3.1. Dimensionless equation systems.** With the new dimensionless variables and functions:

$$r=\frac{R}{R_\mathrm{w}},\quad v=\frac{V_R}{V_\infty}=\frac{1}{r}\psi'_\theta,\quad u=\frac{V_\theta}{V_\infty}=-\frac{\Psi'_\mathrm{R}}{V_\infty}=-\psi'_\mathrm{r},\quad \tau=\frac{T}{T_0},\quad \psi=\frac{\Psi}{\Psi_\mathrm{Q}}=\frac{\Psi}{R_\mathrm{w}V_\infty}\quad\text{and}\quad p=\frac{P}{\rho V_\infty^2}, \tag{8}$$

the equation system became:

$$\begin{aligned}&\psi'_\theta\left(-\frac{2}{r^2}\psi''_{\theta^2}+\frac{1}{r}\psi'''_{r\theta^2}+r\psi'''_{r^3}+\psi''_{r^2}\right)-\psi'_r\left(\frac{1}{r}\psi'''_{\theta^3}+\psi''_{r\theta}+r\psi'''_{r^2\theta}+\frac{1}{r}\psi'_\theta\right)=\\&=\frac{1}{\mathrm{Re}}\left(2\psi^{IV}_{r^2\theta^2}-\frac{2}{r}\psi'''_{r\theta^2}+\frac{1}{r^2}\psi^{IV}_{\theta^4}+r^2\psi^{IV}_{r^4}+2r\psi'''_{r^3}+\frac{4}{r^2}\psi''_{\theta^2}+\frac{1}{r}\psi'_r-\psi''_{r^2}\right),\end{aligned}\tag{7'}$$

$$\mathrm{P\acute{e}}\left(\tau'_r v+\frac{1}{r}\tau'_\theta u\right)=\tau''_{r^2}+\frac{1}{r}\tau'_r+\frac{1}{r^2}\tau''_{\theta^2}\,,\tag{5'}$$ and $$\tau''_{r^2}+\frac{1}{r}\tau'_r+\frac{1}{r^2}\tau''_{\theta^2}+\frac{\mathrm{Jo}}{1+\alpha T_0\left[\tau(r,\theta)-1\right]}=0\,,\tag{6'}$$

where we notated the following similarity criterions:

Reynolds' number - $\mathrm{Re}=\dfrac{R_w V_\infty}{\upsilon}$, Péclet's number - $\mathrm{P\acute{e}}=\dfrac{V_\infty R_w}{\lambda/\rho c}$ and Joule's number - $\mathrm{Jo}=\dfrac{kU_0^2R_w^2}{\Re_0\lambda T_0}$.

## 4. DEDUCTION OF ALGEBRAIC RELATIONS

To obtain the algebraic relations for the numerical calculus, representing the general solution in any point of the domain without consider the boundary conditions, we must develop the functions in finite Taylor's [2][5] or proper [7] series in the grid with different steps $\delta r = a$ and $\delta\theta = c$ (fig. 1) from which we can extract their partial differential expressions, which introduced in the equations (7'), (5') and (6') give us the following algebraic relations associated to the partial differential equations

$$\begin{aligned}\psi_0=&\frac{1}{\dfrac{8}{r^2c^2a^2}+\dfrac{6}{r^6c^4}+\dfrac{6r^2}{a^4}-\dfrac{10}{r^4c^2}+\dfrac{5}{2a^2}}\cdot\\&\left\langle\begin{aligned}&\mathrm{Re}\,rv\left\{\begin{aligned}&-\frac{2}{r^4c^2}\left[\frac{4}{3}(\psi_1+\psi_3)-\frac{5}{2}\psi_0-\frac{1}{12}(\psi_9+\psi_{11})\right]+\frac{1}{r^3c^2a}\left[\frac{1}{2}(\psi_5+\psi_6-\psi_7-\psi_8)-(\psi_2-\psi_4)\right]+\\&+\frac{r}{a^3}\left[\frac{1}{2}(\psi_{10}-\psi_{12})-\psi_2+\psi_4\right]+\frac{1}{a^2}\left[\frac{4}{3}(\psi_2+\psi_4)-\frac{5}{2}\psi_0-\frac{1}{12}(\psi_{10}+\psi_{12})\right]\end{aligned}\right\}+\\&+\mathrm{Re}\,u\left\{\begin{aligned}&\frac{1}{r^4c^3}\left[\frac{1}{2}(\psi_9-\psi_{11})-\psi_1+\psi_3\right]+\frac{1}{arc}\left[\frac{1}{3}(\psi_5+\psi_7-\psi_6-\psi_8)-\frac{1}{48}(\psi_{13}+\psi_{15}-\psi_{14}-\psi_{16})\right]+\\&+\frac{1}{a^2c}\left[\frac{1}{2}(\psi_5+\psi_8-\psi_6-\psi_7)-(\psi_1-\psi_3)\right]+\frac{1}{r^2c}\left[\frac{2}{3}(\psi_1-\psi_3)-\frac{1}{12}(\psi_9-\psi_{11})\right]\end{aligned}\right\}+\\&+\frac{2}{r^2a^2c^2}\left[2(\psi_1+\psi_2+\psi_3+\psi_4)-(\psi_5+\psi_6+\psi_7+\psi_8)\right]+\frac{2}{r^3c^2a}\left[\frac{1}{2}(\psi_5+\psi_6-\psi_8-\psi_9)-(\psi_2-\psi_4)\right]+\\&+\frac{1}{r^6c^4}\left[4(\psi_1+\psi_3)-(\psi_9+\psi_{11})\right]++\frac{r^2}{a^4}\left[4(\psi_2+\psi_4)-(\psi_{10}+\psi_{12})\right]+\frac{2}{a^3}\left[\psi_2-\psi_4-\frac{1}{2}(\psi_{10}-\psi_{12})\right]-\\&-\frac{4}{r^4c^2}\left[\frac{4}{3}(\psi_1+\psi_3)-\frac{1}{12}(\psi_9+\psi_{11})\right]-\frac{1}{ra}\left[\frac{2}{3}(\psi_2-\psi_4)-\frac{1}{12}(\psi_{10}-\psi_{12})\right]+\\&+\frac{1}{a^2}\left[\frac{4}{3}(\psi_2+\psi_4)-\frac{1}{12}(\psi_{10}+\psi_{12})\right]\end{aligned}\right\rangle\end{aligned}\tag{9}$$

$$\tau_0=\frac{1}{1+\dfrac{a^2}{r_0^2c^2}}\left\{\frac{1}{2}(\tau_2+\tau_4)+\frac{a^2}{2r_0^2c^2}(\tau_1+\tau_3)+\frac{a}{4r_0}(\tau_2-\tau_4)-\frac{a\,\mathrm{P\acute{e}}}{4}\left[(\tau_2-\tau_4)v_0+\frac{a^2}{r_0c}(\tau_1-\tau_3)u_0\right]\right\},\tag{10}$$

$$\tau_0=\frac{1}{1+\dfrac{a^2}{r_0^2c^2}}\left\{\frac{1}{2}(\tau_2+\tau_4)+\frac{a}{4r_0}(\tau_2-\tau_4)+\frac{a^2}{2r_0^2c^2}(\tau_1+\tau_3)+\frac{a^2\mathrm{Jo}}{2\left[1+\alpha T_0(\tau_0-1)\right]}\right\}.\tag{11}$$

## 5. NUMERICAL SOLUTION STABILITY

Applying our method [2][6] to calculate the error propagation by calculus in the two grid direction, we obtain the following relations, observing that denominator $N=\frac{8}{r^2c^2a^2}+\frac{6}{r^6c^4}+\frac{6r^2}{a^4}-\frac{10}{r^4c^2}+\frac{5}{2a^2}\neq 0$ is differed of zero for any step or radius values, obtaining for the two senses of tangential direction

.

$$\delta\psi_{n+1}^{\pm\theta}=\frac{1}{N}\left\{\begin{aligned}&\left[\frac{4}{r^2a^2c^2}+\frac{4}{r^6c^4}-\frac{16}{3r^4c^2}-\frac{8\,\mathrm{Re}\,v}{3r^3c^2}\pm\mathrm{Re}\,u\left(\frac{1}{r^4c^3}+\frac{1}{a^2c}-\frac{2}{3r^2c}\right)\right]\delta\psi_n+\\&+\left[-\frac{1}{r^6c^4}+\frac{1}{3r^4c^2}+\frac{\mathrm{Re}\,v}{6r^3c^2}\pm\mathrm{Re}\,u\left(-\frac{1}{2r^4c^3}+\frac{1}{12r^2c}\right)\right]\delta\psi_{n-1}\end{aligned}\right\},\tag{12}$$

and for the two senses of radial direction

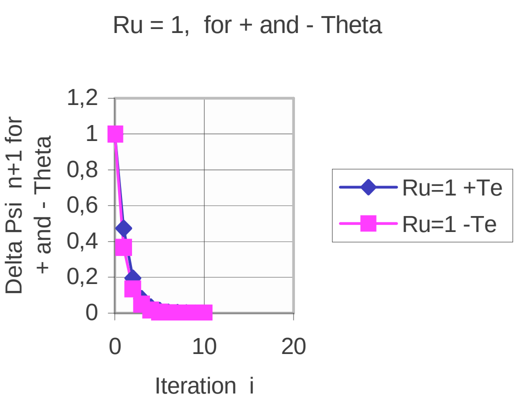
$$\delta\psi_{n+1}^{\pm r}=\frac{1}{N}\left\{\begin{aligned}&\left[\frac{4}{r^2a^2c^2}\pm\frac{2}{r^3c^2a}+\frac{4r^2}{a^4}\mp\frac{2}{a^3}\pm\frac{2}{3ra}+\frac{4}{3a^2}\pm\frac{\mathrm{Re}\,v}{r^2c^2a}\pm\frac{\mathrm{Re}\,r^2v}{a^3}+\frac{4\,\mathrm{Re}\,rv}{3a^2}\right]\delta\psi_n\\&+\left[-\frac{r^2}{a^4}\pm\frac{1}{a^3}\mp\frac{1}{12ra}-\frac{1}{12a^2}\mp\frac{\mathrm{Re}\,r^2v}{2a^3}-\frac{\mathrm{Re}\,rv}{12a^2}\right]\delta\psi_{n-1}\end{aligned}\right\}.\tag{13}$$

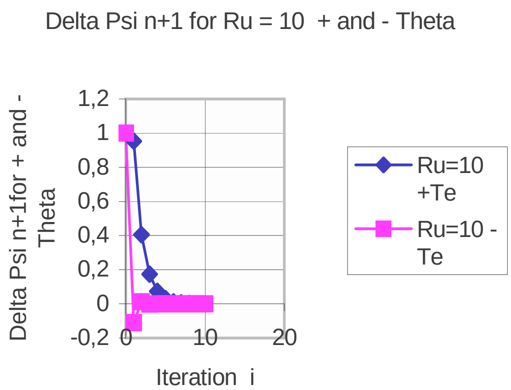
In the aim to obtain suit abler the numerical solution stability, we shall multiply the equation numerator with c and denoting with Re v c = Rv, Re u c = Ru and taking for instance $a=c=1/10$ for $r$=1 we obtain

$$\delta\psi_{n+1}^{\pm\theta}=\left(0{,}421\pm0{,}0531Ru\right)\delta\psi_n-\left(0{,}0053\pm0{,}0265Ru\right)\delta\psi_{n-1},\tag{14}$$

$$\delta\psi_{n+1}^{\pm r}=\left(0{,}4247\pm0{,}00003529\pm0{,}106Rv\right)\delta\psi_n-\left(0{,}053\mp0{,}0053\pm0{,}000044149\pm0{,}0265Rv\right)\delta\psi_{n-1}\tag{15}$$

we have a good convergence for:

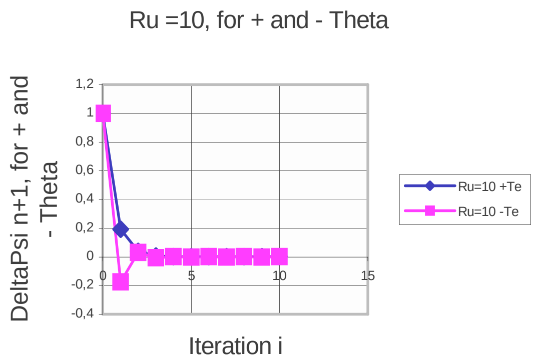


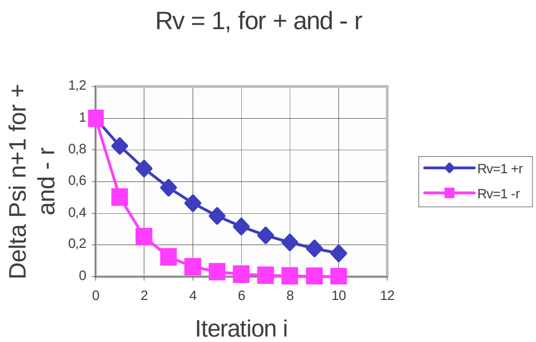




and for $a=c=1/10$ and $r=3$ we have also a good convergence for:

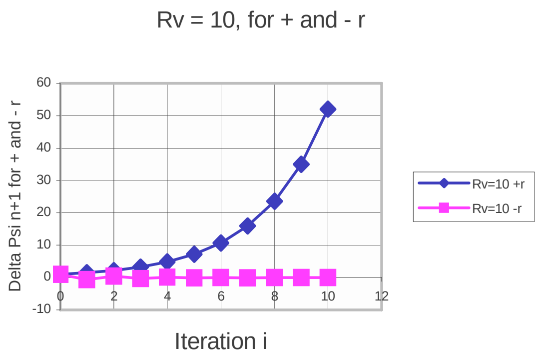
$$\delta\psi_{n+1}^{\pm\theta}=\left(0{,}008176\pm0{,}0184Ru\right)\delta\psi_n-\left(0{,}00002420\pm0{,}0001106Ru\right)\delta\psi_{n-1},\tag{16}$$

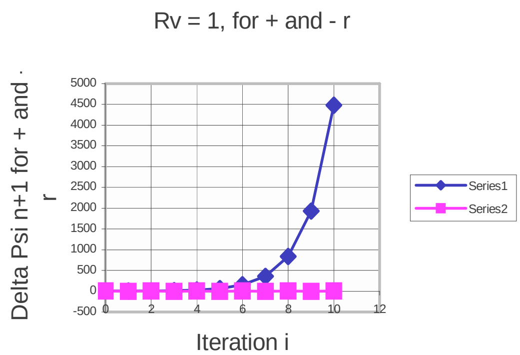
$$\delta\psi_{n+1}^{\pm r}=\left(0{,}6635\pm0{,}0001388\mp0{,}00364\pm0{,}1658Rv\right)\delta\psi_n-\left(0{,}1638\mp0{,}00182\pm0{,}00819Rv\right)\delta\psi_{n-1}.\tag{17}$$



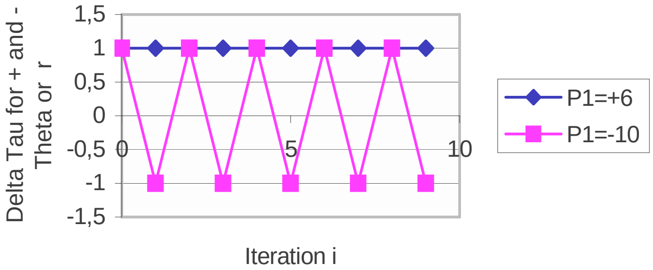

in two cases: Rv = 10 for $r = 1$ and Rv = 1 for $r = 3$ we had also unstable numerical solution as we graphically represented in the follows diagrams:



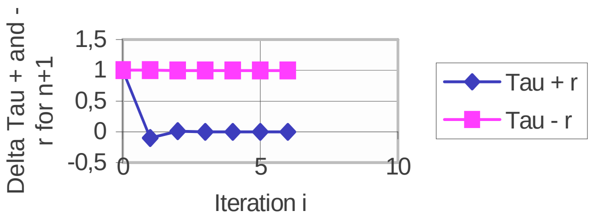



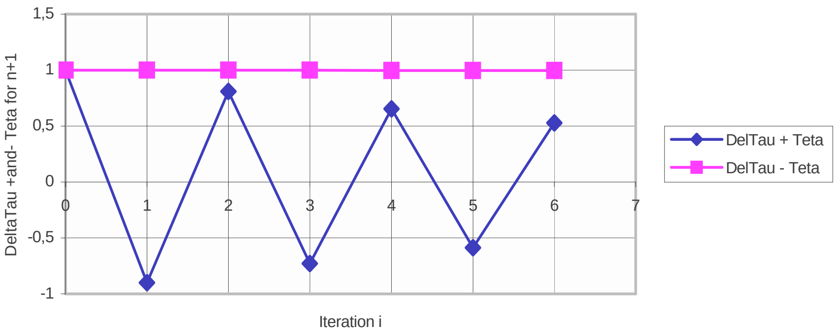

Concerning the temperature numerical solution stability in the fluid flow and hot wire domains we have the four following error propagation relations as functions of the local Péclet numbers Pu = $acu_o$Pé and Pv = $av_o$Pé , those monotonous and oscillating stability limits are represented for $r_o = 1$ and $a = c = 0{,}1$ in the following diagrams:

$$\delta\tau^{\pm\theta}_{n+1}\Big|_F = \frac{1}{1 + r_0^2\dfrac{c^2}{a^2}}\left(\frac{1}{2} \pm r_0\frac{ac\,\text{Pé}\,u_0}{8}\right)\delta\tau_n \quad \text{and} \quad \delta\tau^{\pm r}_{n+1}\Big|_F = \frac{1}{1 + \dfrac{a^2}{r_0^2c^2}}\left(\frac{1}{2} \mp \frac{a}{4r_0} \pm \frac{a\,\text{Pé}\,v_0}{4}\right)\delta\tau_n\,, \qquad (18)$$



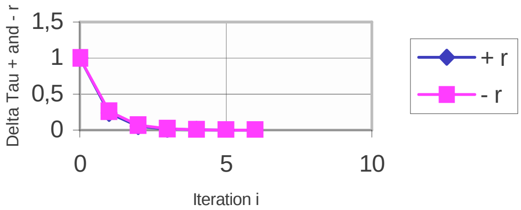

and for the peripheral radius of the fluid flow domain, for instance $r_o = 3$ we have the following diagrams:

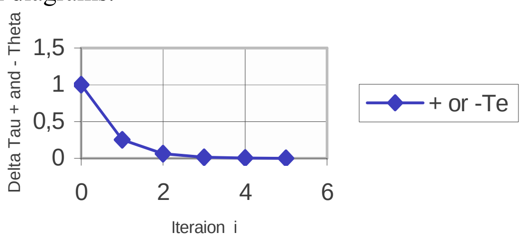



$$\delta\tau^{\pm\theta}_{n+1}\Big|_{W} = \frac{1}{2\left(1 + r_0^2\dfrac{c^2}{a^2}\right)}\delta\tau_n \qquad \text{and} \qquad \delta\tau^{\pm r}_{n+1}\Big|_{W} = \frac{1}{1 + \dfrac{a^2}{r_0^2 c^2}}\left(\frac{1}{2} \pm \frac{a}{4r_0}\right)\delta\tau_n\,, \tag{19}$$

for the hot wire domain we obtained the following relaxation diagrams.




## 6. CONCLUSIONS OF THE HOT WIRE ANEMOMETRY NUMERICAL SOLUTION STABILITY

By the numerical solution stability, studied also in the papers [6] and [8] ÷ [10], one choused the convenient mode to obtain the numerical solution stability of the proposed problem.

The most convenient mode of grid traverse by calculus is from the maximum radius to the minimum and in inverse trigonometrical sense (from $+\theta$ to $-\theta$) to benefit of the monotone diminishing of the function values in the flow sense.

For a better numerical solution stability we have the possibility to crowd the grid by diminishing of its steps in the limits of the local Reynolds number values, as one can see in the papers [8] ÷ [13].

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