CLASSIFYING BAYESIAN NETWORKS BY ESSENTIAL GRAPHS

GARRIDO ANGEL

Facultad de Ciencias UNED, Madrid, E-mail: agarrido@mat.uned.es

Abstract: We improve the efficiency of Bayesian Network learning procedures, by selecting as search space the equivalence classes of Directed Acyclic Graphs (DAGs), and from them an essential graph as representative of each class. For this purpose, we describe some results, and observe the asymptotical behaviour of its respective ratios.

Keywords: A. I., Graph Theory, Bayesian Nets

1. INTRODUCTION

Let *S* and *S'* be two of such structures of BNs on *V*. Then, we say that *S is equivalent to S': S X S'*, if for each parameterization, θ, of *S*, there exists another, θ', of *S'*, such that:

$$P\left(V/S,\vartheta\right)=P\left(V/S',\vartheta'\right) \tag{1}$$

Let C be a class of DAGs Markov X among them. Then, their essential graph would be the smallest graph greater than every DAG that belongs to the class. If we denote the essential graph as G*, this is equivalent to saying: G* = ∪ {G: G∈C}, where such graph union is reached by the union of the nodes and edges of G:

$$V(G^*) = \cup V(G),\ E(G^*) = \cup E(G) \tag{2}$$

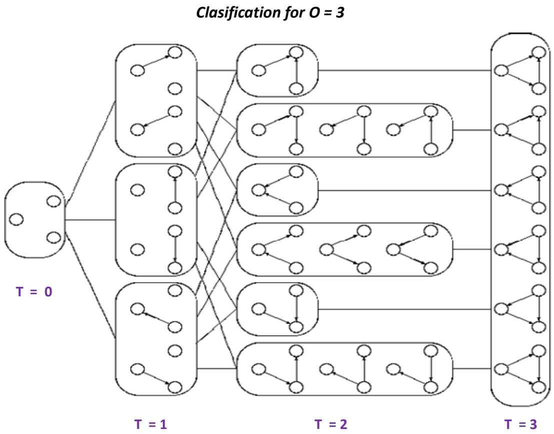
2. CLASSIFICATION OF BAYESIAN NETS

In general, the number of possible structures, for BNs with ***n*** nodes, *r (n)*, is given by the recurrence equation:

$$r(n)=\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}2^{i(n-i)}r(n-i) \tag{3}$$

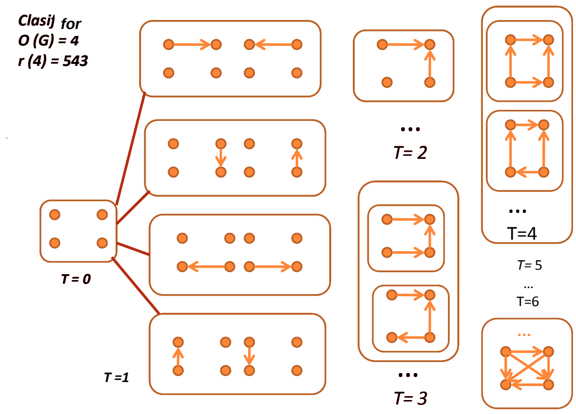
In the first case, with order *n = 3*:

$$\begin{aligned} r(3) &= \sum_{i=1}^{3}(-1)^{i+1}\binom{3}{i}2^{i(3-i)}r(3-i)= \\ &= (-1)^2\binom{3}{1}2^{1(3-1)}r(3-1)+(-1)^3\binom{3}{2}2^{2(3-2)}r(3-2)+ \\ &+(-1)^4\binom{3}{3}2^{3(3-3)}r(3-3)=36-12+1=25 \\ &(\neq \textit{estructures}),\ \textit{because}: r(0)=r(1)=1\ \&\ r(2)=3. \end{aligned} \tag{4}$$



When the graphs are of order: $O(G) = 4$. Therefore, the size can be $T(G) \in \{0, 1, 2, ..., 6 = 4(4\text{-}1)/2\}$.

We can see it gets more complicated. Applying the aforementioned recurrent equation, we obtain 543 different possible configurations:

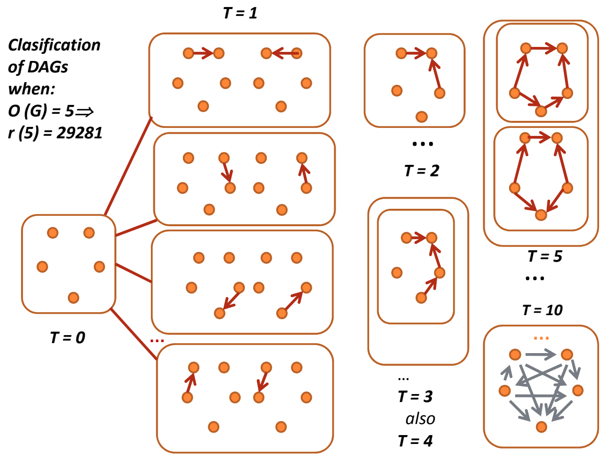


$$\begin{aligned} r(4) &= \sum_{i=1}^{4} (-1)^{i+1} \binom{4}{i} 2^{i(4-i)} r(4-i) = \\ &= (-1)^2 \binom{4}{1} 2^{1(4-1)} r(4-1) + (-1)^3 \binom{4}{2} 2^{2(4-2)} r(4-2) + \\ &+ (-1)^4 \binom{4}{3} 2^{3(4-3)} r(4-3) + (-1)^5 \binom{4}{4} 2^{4(4-4)} r(4-4) = \\ &= 800 - 288 + 32 - 1 = 543 \quad \textit{different configurations} \end{aligned} \tag{5}$$

For the DAGs, G, such that O (G) = 5, where:

$$T(G) \in \{0, 1, 2, \ldots, 5(5\text{-}1)/2 = 10\} \tag{6}$$

the recurrence equation gives us the configurations:



$$
\begin{aligned}
r(5) &= \sum_{i=1}^{5} (-1)^{i+1} \binom{5}{i} 2^{i(5-i)} r(5-i) = \\
&= (-1)^{1+1} \binom{5}{1} 2^{1(5-1)} r(5-1) + (-1)^{2+1} \binom{5}{2} 2^{2(5-2)} r(5-2) + \\
&+ (-1)^{3+1} \binom{5}{3} 2^{3(5-3)} r(5-3) + (-1)^{4+1} \binom{5}{4} 2^{4(5-4)} r(5-4) + \\
&+ (-1)^{5+1} \binom{5}{5} 2^{5(5-5)} r(5-5) = 43440 - 16000 + 1920 - 80 + 1 = \\
&= 29281 \Rightarrow c\ \{DAGs\ of\ \ five\ nodes\} \equiv r(5) = 29281\, config.^{s} \neq \\
&\Rightarrow ratio : c\{\ equivalence\ classes\ \} / c\{DAGs\ of\ \ five\ nodes\} = \\
&= c\{classes\}\ /\ r(5) = 8782 / 29281 = 0.2999214\,(viewing\ Table)
\end{aligned} \tag{7}
$$

For this computation, it was elaborated a program, due to Gillispie & Perlman´01 (see Table), which permits the enumeration of the equivalence (X) of DAGs, according to the equivalence criteria among BNs. It has been computed that the *proportion of DAGs to classes of X is (asymptotically) 3.75. That is, from classes to DAGs should be of 0.267*, a considerable reduction.

O (G) = **n**	**c** { equivalence classes} ≡ **c** { essential graphs}	Ratio among: **c** {classes} / **c** {DAGs} [≡ r (n) ≡ nr. of configurations ≠]
1	1	1 / r(1) = 1
2	2	2 / r(2) = 2 / 3 = 0. 6✎
3	11	3 / r(3) = 11 /25 = = 0.44
4	185	185 / r(4) = 185 / 543 = 0.341
5	8782	8782 / r(5) = 8782 / 29281 ≅ 0.29✎
… 10	1118902054495975141	1118902054495975141 / r(10) = 0.26799

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