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ON CESÁRO MEANS OF HYPERGEOMETRIC FUNCTIONS

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Abstract: The polynomial approximants which retain the zero free property of a given analytic functions in the unit disk $U := \{z : |z| < 1\}$ of the form

$$\varphi(z) := z_l F_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \frac{z^{n+1}}{n!},$$

$$(l \leq m+1, \ m \in \mathbb{N}_0 \ \ z \in U)$$

where $\alpha_i, \beta_j > 0$ for all $i = 1, ..., l$ and $j = 1, ..., m$ is found. The convolution methods of a geometric functions that the Cesáro means of order μ retains the zero free property of the derivatives of bounded convex functions in the unit disk are used. Other properties are established.

1. Introduction, Definitions and preliminaries.

In the theory of approximation the important problem is to find a suitable finite (polynomial) approximation for the outer infinite series f so that the approximant reduces the zero-free property of f. Recall that an outer function (zero-free) is a function $f \in H^p$ of the form

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$$f(z) = e^{i\gamma} e^{1/2\pi \int_{-\pi}^{\pi} \frac{1+e^{it}z}{1-e^{it}z} log\psi(t)dt}$$

where $\psi(t) \geq 0$, $log\psi(t)$ is in L^1 and $\psi(t)$ is in L^p for details see [1]. Outer functions play an important role in H^p theory, arise in characteristic equation which determines the stability of certain nonlinear systems of differential equations (see [2]). We observe for outer functions that the standard Taylor approximants do not, in general, retain the zero-free property of f. It was shown in [3] that the Taylor approximating polynomials to outer functions can vanish in the unit disk. By using convolution methods that the classical Cesáro means, retains the zero-free property of the derivatives of bounded convex functions in the unit disk. The classical Cesáro means play an important role in geometric function theory (see [4-7]).

Let $\mathcal{A}$ be the class of generalized hypergeometric functions in the unit disk $U := \{z : |z| < 1\}$ take the form

$$(1) \quad \varphi(z) := z_l F_m(\alpha_1, ..., \alpha_l; \beta_1, ..., \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \frac{z^{n+1}}{n!},$$

$$(\ l \leq m+1, \, m \in \mathbb{N}_0 \quad z \in U)$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1, & n = 0 \\ x(x+1)...(x+n-1), & n = \{1, 2, ...\}. \end{cases}$$

This class of functions generalization to one was studied by Ruscheweyh [5]. He observed the following results

Lemma 1.1. *Let $0 < \alpha \leq \beta$. If $\beta \geq 2$ or $\alpha + \beta \geq 3$ then the function of the form $f(z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^{n+1}, \quad z \in U$ is convex.*

Lemma 1.2. *Assume that $a_1 = 1$ and $a_n \geq 0$ for $n \geq 0$ such that $\{a_n\}$ is a convex decreasing sequence i.e.*

$$a_n - 2a_{n+1} + a_{n+2} \geq 0 \ \ and \ \ a_{n+1} - a_{n+2} \geq 0.$$

Then

$$\Re\{\sum_{n=1}^{\infty} a_n z^{n-1}\} > \frac{1}{2}, \ \ z \in U.$$

We apply Lemma 1.2, to find the next result which is a generalization to [Lemma 5; 8].

Lemma 1.3. *If* $0 < \alpha_1 ... \alpha_l \leq \beta_1 ... \beta_m$ *then*

$$\Re\Big\{\frac{\varphi(z)}{z}\Big\} > \frac{1}{2} \;\; for\, all\, z \in U.$$

Proof. From the definition of the function $\varphi(z)$, we have

$$\frac{\varphi(z)}{z} = 1 + \sum_{n=2}^{\infty} B_n z^{n-1}$$

where

$$B_n := \frac{(\alpha_1)_{n-1}...(\alpha_l)_{n-1}}{(\beta_1)_{n-1}...(\beta_m)_{n-1}} \frac{1}{\Gamma(n)} \tag{2}$$

for $n \geq 2$. Since $\alpha_i, \beta_j > 0$ for all $i = 1, ..., l$ and $j = 1, ..., m$, we have $B_n > 0$ for all $n \in \mathbb{N}$. Then we find

$$\begin{aligned} B_{n+1} &= \frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n} \frac{1}{n\Gamma(n)} \\ &= \frac{(\alpha_1)_{n-1}(\alpha_1+n-1)...(\alpha_l)_{n-1}(\alpha_l+n-1)}{(\beta_1)_{n-1}(\beta_1+n-1)...(\beta_m)_{n-1}(\beta_m+n-1)} \frac{1}{n\Gamma(n)} \\ &= \frac{(\alpha_1+n-1)...(\alpha_l+n-1)}{(\beta_1+n-1)...(\beta_m+n-1)} \frac{1}{n} B_n \end{aligned} \tag{3}$$

and

(4)

$$B_{n+2} = \frac{(\alpha_1)_{n+1}...(\alpha_l)_{n+1}}{(\beta_1)_{n+1}...(\beta_m)_{n+1}} \frac{1}{n(n+1)\Gamma(n)}$$

$$= \frac{(\alpha_1)_{n-1}(\alpha_1+n-1)(\alpha_1+n)...(\alpha_l)_{n-1}(\alpha_l+n-1)(\alpha_l+n)}{(\beta_1)_{n-1}(\beta_1+n-1)(\beta_1+n)...(\beta_m)_{n-1}(\beta_m+n-1)(\beta_m+n)} \frac{1}{n(n+1)\Gamma(n)}$$

$$= \frac{(\alpha_1+n-1)(\alpha_1+n)...(\alpha_l+n-1)(\alpha_l+n)}{(\beta_1+n-1)(\beta_1+n)...(\beta_m+n-1)(\beta_m+n)} \frac{1}{n(n+1)} B_n.$$

Thus from the assumption it follows that

$$B_{n+1} - B_{n+2} = B_{n+1}\Big(1 - \frac{(\alpha_1+n)...(\alpha_l+n)}{(\beta_1+n)...(\beta_m+n)} \frac{1}{(n+1)}\Big) \geq 0, \;\; \forall n \in \mathbb{N}.$$

Now we show that

$$B_n - 2B_{n+1} + B_{n+2} \geq 0, \;\; \forall n \in \mathbb{N}. \tag{5}$$

By using (3) and (4) we find

$$B_n - 2B_{n+1} + B_{n+2} =$$

$$= B_n\Big\{1 + \frac{(\alpha_1+n-1)...(\alpha_l+n-1)}{n(n+1)(\beta_1+n-1)(\beta_1+n)...(\beta_m+n-1)(\beta_m+n)}$$

$$\times\big[n(\alpha_1+n)...(\alpha_l+n) - 2(n+1)(\beta_1+n)...(\beta_m+n)\big]\Big\}$$

$$\geq 0.$$

Thus the sequence $\{B_n\}$ is convex decreasing and in virtue of Lemma 1.2, we obtain that

$$\Re\Big\{1 + \sum_{n=2}^{\infty} B_n z^{n-1}\Big\} = \Re\Big\{\frac{\varphi(z)}{z}\Big\} > \frac{1}{2}.$$

The proof is complete.

We define S^*, C, QS^* and QC the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike in U, convex in U, close-to-convex and quasi-convex in U. Thus by definition, we have

$$S^* := \{\varphi \in \mathcal{A} : \Re\big(\frac{z\varphi'(z)}{\varphi(z)}\big) > 0,\ z \in U\},$$

$$C := \{\varphi \in \mathcal{A} : \Re\big(1 + \frac{z\varphi''(z)}{\varphi'(z)}\big) > 0,\ z \in U\},$$

$$QS^* := \{\varphi \in \mathcal{A} : \exists g \in S^*\, s.t.\, \Re\big(\frac{\big(z\varphi'(z)\big)}{g(z)}\big) > 0,\ z \in U\},$$

and

$$QC := \{\varphi \in \mathcal{A} : \exists g \in C\, s.t.\, \Re\big(\frac{\big(z\varphi'(z)\big)'}{g'(z)}\big) > 0,\ z \in U\}.$$

It is easily observed from the above definitions that

$$\varphi(z) \in C \Leftrightarrow z\varphi'^{*} \tag{6}$$

and

$$\varphi(z) \in QC \Leftrightarrow z\varphi'^{*}. \tag{7}$$

Note that $\varphi \in QS^*$ if and only if there exists a function $g \in S^*$ such that

$$z\varphi'(z) = g(z)p(z) \tag{8}$$

where $p(z) \in \mathcal{P}$, the class of all analytic functions of the form

$$p(z) = 1 + p_1 + z + p_2 z^2 + ..., \ s.t. \ p(0) = 1.$$

Let be given two functions $\varphi, g \in \mathcal{A}$, $\varphi(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n} \frac{z^{n+1}}{n!}$, *and* $g(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_l)_n}{(b_1)_n...(b_m)_n} \frac{z^{n+1}}{n!}$, then their convolution or Hadamard product $\varphi(z) * g(z)$ is defined by

$$\varphi(z) * g(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n} \frac{(a_1)_n...(a_l)_n}{(b_1)_n...(b_m)_n} \frac{z^{n+1}}{n!n!}.$$

We can verify the following result for $f \in \mathcal{A}$ and takes the form (1).

Lemma 1.4. [5]

(i) If $\varphi \in C$ *and* $g \in S^*$ *then* $\varphi * g \in S^*$.
(ii) If $\varphi \in C$ *and* $g \in S^*, p \in P$ *with* $p(0) = 1$, *then* $\varphi * gp = (\varphi * g)p_1$

where $p_1(U) \subset$ *close convex hull of* $p(U)$.

2. The Main Results.

The Cesáro sums of order μ where $\mu \in \mathbb{N} \cup \{0\}$ of series of the form (1) can defined as

$$\sigma_k^{\mu}(z, \varphi) = \sigma_k^{\mu} * \varphi(z) = \sum_{n=0}^{k} \frac{\begin{pmatrix} k - n + \mu \\ k - n \end{pmatrix}}{\begin{pmatrix} k + \mu \\ k \end{pmatrix}} \frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n} \frac{z^{n+1}}{n!}$$

where $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$. We have the following result:

Theorem 2.1. *Let* $\varphi \in A$ *be convex in* U. *Then the Cesáro means* $\sigma_k^{\mu}(z, \varphi)$, $z \in U$ *of order* $\mu \geq 1$, *of* $\varphi'(z)$ *are zero-free on* U *for all* k.

Proof. In view of Lemma 1.1, the analytic function φ of the form (1) is convex in U if $\beta_1...\beta_m \geq 2$ or $\alpha_1...\alpha_l + \beta_1...\beta_m \geq 3$ where

$0 < \alpha_1 ... \alpha_l \leq \beta_1 ... \beta_m$. Let $\phi(z) := \sum_{n=0}^{\infty}(n+1)z^{n+1}$ be defined such that

$$\begin{aligned} z\varphi'(z) &= \phi(z) * \varphi(z) \\ &= \sum_{n=0}^{\infty} \frac{(n+1)}{n!} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} z^{n+1}. \end{aligned}$$

Then

$$\begin{aligned} \sigma_k^{\mu}(z, \varphi') = \varphi'(z) * \sigma_k^{\mu}(z) &= \frac{z\varphi'(z) * z\sigma_k^{\mu}}{z} \\ &= \frac{\varphi(z) * \phi(z) * z\sigma_k^{\mu}}{z} = \frac{\varphi(z) * z\big(z\sigma_k^{\mu}\big)'}{z}. \end{aligned}$$

In view of Lemma 1.3, the relation (8) and the fact that $z\sigma_k^{\mu}$ is convex yield that there exists a function $g \in S^*$ and $p \in \mathcal{P}$ with $p(0) = 1$ such that

$$\frac{\varphi(z) * z\big(z\sigma_k^{\mu}\big)'}{z} = \frac{\varphi(z) * gp(z)}{z} = \frac{\big(\varphi(z) * g(z)\big)p_1(z)}{z} \neq 0.$$

We know that $\Re\{p_1(z)\} > 0$ and that $\varphi(z) * g(z) = 0$ if and only if $z = 0$. Hence, $\sigma_k^{\mu}(z, \varphi') \neq 0$ and the proof is complete.

Corollary 2.1. *If $\varphi(U)$ is bounded convex domain, then the Cesáro means $\sigma_k^{\mu}(z)$, $z \in U$ for the outer function $\varphi'(z)$ are zero-free on U for all k.*

Proof. It comes from the fact that the derivatives of bounded convex functions are outer function (see [3]).

Theorem 2.2. *Let $\varphi \in C$. Then for $\mu \geq 2, \sigma_k^{\mu}(z, \varphi')$ have their ranges contained in a cone (from 0) with opening $2\beta\pi$ where $\beta < 1$.*

Proof. By using the fact that $\varphi \in C \Rightarrow \varphi \in S^*(\frac{1}{2})$ (see [4]). We pose

$$\begin{aligned} \sigma_k^{\mu}(z, \varphi') = \varphi'(z) * \sigma_k^{\mu}(z) &= \frac{z\varphi'(z) * z\sigma_k^{\mu}(z)}{z} \\ &= \frac{\varphi(z) * z(z\sigma_k^{\mu}(z))'}{z} = \frac{1}{z}\{\varphi(z) * z\sigma_k^{\mu}(z) \frac{z\big(z\sigma_k^{\mu}(z)\big)'}{\sigma_k^{\mu}(z)}\}. \end{aligned}$$

Thus we have (see [7])

$$\sigma_k^\mu(z) \ \in S^*(\frac{1}{2}) \leftrightarrow \frac{z\big(z\sigma_k^\mu(z)\big)'}{\sigma_k^\mu(z)} := p(z)$$

where $p(z) \in \mathcal{P}(\frac{1}{2})$ i.e. $\Re\{p(z)\} > \frac{1}{2}$. As in Theorem 2.1, we get

$$\frac{\varphi(z) * z\big(z\sigma_k^\mu\big)'}{z} = \frac{\varphi(z) * \big(z\sigma_k^\mu\big)' p(z)}{z} = \frac{\big(\varphi(z) * \big(z\sigma_k^\mu\big)'\big) p_1(z)}{z}$$

where $p_1 \in \ \mathcal{P}(\frac{1}{2})$. This implies that $\varphi(z) * \big(z\sigma_k^\mu\big)' \big) \in \ \mathcal{P}(\frac{1}{2})$ (see [5]). Hence by using the concept of the subordination [9], the bounded polynomial $\frac{1}{z}\{\varphi(z) * z\sigma_k^\mu(z)\} \in \ \mathcal{P}(\frac{1}{2})$ satisfies that there exists $\rho < 1$ such that

$$|arg\{\frac{\varphi(z) * z\sigma_k^\mu(z)}{z}\}| < \frac{\rho\pi}{2}.$$

Then

$$\begin{aligned} |arg \sigma_k^\mu(z, \varphi')| &= |arg\{\frac{\varphi(z) * z\sigma_k^\mu(z)}{z}\}.p_1(z)| \\ &\le |arg\{\frac{\varphi(z) * z\sigma_k^\mu(z)}{z}\}| + |arg p_1(z)| \\ &\le \frac{\rho\pi}{2} + \frac{\pi}{2} := \beta\pi. \end{aligned}$$

The next result shows the upper and lower bound for $\sigma_k^\mu(z, \varphi')$.

Theorem 2.3. *Let $\varphi \in A$. Assume that $\beta_1...\beta_m \ge 2$ or $\alpha_1...\alpha_l + \beta_1...\beta_m \ge 3$ where $0 < \alpha_1...\alpha_l \le \beta_1...\beta_m$, and $\mu \ge 0$. Then*

$$\frac{1}{2}|z| < |\sigma_k^\mu(z, \varphi')| \le \frac{(k+1)}{(k-1)!}, \ \ 1 \le k < \infty, \ \ z \in U, \, z \ne 0.$$

Proof. Under the conditions of the theorem, we have that f is convex (Lemma 1.1), then in virtue of Theorem 2.1, we obtain that $\sigma_k^\mu(z, \varphi') \ne 0$ thus $|\sigma_k^\mu(z, \varphi')| > 0$. Now by applying Lemma 1.3, on $\sigma_k^\mu(z, \varphi')$ and using the fact that $\Re\{z\} \le |z|$ and since

$$\frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} = \frac{k!(k-n+\mu)!}{(k-n)!(k+\mu)!} \le 1 \tag{9}$$

$for\, \mu \geq 0$ and $n = 0, 1, ..., k$ yield

$$\frac{1}{2} < \Re\{\frac{\sigma_k^{\mu}(z,\varphi')}{z}\} \leq \frac{|\sigma_k^{\mu}(z,\varphi')|}{|z|}, \;\; |z| > 0 \;\; and \;\; z \in U.$$

For the other side, we pose that

$$\begin{aligned}|\sigma_k^{\mu}(z,\varphi')| = |\varphi'(z) * \sigma_k^{\mu}(z)| &= |\sum_{n=0}^{k} \frac{(n+1)}{n!} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} z^n| \\
&\leq \sum_{n=0}^{k} \frac{(n+1)}{n!} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} |z^n| \\
&\leq \sum_{n=0}^{k} \frac{(n+1)}{n!} \frac{\begin{pmatrix} k-n+\mu \\ k-n \end{pmatrix}}{\begin{pmatrix} k+\mu \\ k \end{pmatrix}} \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \\
&\leq \sum_{n=0}^{k} \frac{(n+1)}{n!} \leq \frac{(k+1)}{(k-1)!}, \quad k < \infty\end{aligned}$$

when $n \to k$. Hence the proof.

Theorem 2.4. *Let* $\varphi \in A$. *Then for* $\frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \to 2(\lambda)_n$,

$$\lim_{k \to \infty} \sigma_k^{\alpha}(z,\varphi) = \frac{z}{(1-z)^{\lambda}}, \;\; \lambda > 1, \;\; z \in U.$$

Proof. By the assumption and the fact (9), we have

$$|\sigma_k^\mu(z,\varphi)-\frac{z}{(1-z)^\lambda}|=|\sum_{n=0}^{k}\frac{\begin{pmatrix}k-n+\mu\\ k-n\end{pmatrix}}{\begin{pmatrix}k+\mu\\ k\end{pmatrix}}\frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n}\frac{1}{n!}z^{n+1}-$$

$$-\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^{n+1}|$$

$$=\frac{1}{n!}|[\sum_{n=0}^{k}\frac{\begin{pmatrix}k-n+\mu\\ k-n\end{pmatrix}}{\begin{pmatrix}k+\mu\\ k\end{pmatrix}}\frac{(\alpha_1)_n...(\alpha_l)_n}{(\beta_1)_n...(\beta_m)_n}-(\lambda)_n]z^{n+1}-$$

$$-\sum_{n=k+1}^{\infty}(\lambda)_n z^{n+1}|$$

$$\leq|\sum_{n=k+1}^{\infty}\frac{(\lambda)_n}{n!}-\sum_{n=1}^{k}\frac{(\lambda)_n}{n!}|$$

$$\to 0 \ as\ k\to\infty.$$

For $0\leq\nu<1$, let $B(\nu)$ denote the class of functions f of the form (1) so that $\Re\{f'\}>\nu$ in U. The functions in $B(\nu)$ are called functions of bounded turning (c.f. [11, Vol. II]). By the Nashiro-Warschowski Theorem (see e.g. [11, Vol. I]) the functions in $B(\nu)$ are univalent and also close-to-convex in U.

In the sequel we need to the following results.

Lemma 2.1. [10] *For $z\in U$ we have*

$$\Re\Big\{\sum_{n=1}^{j}\frac{z^n}{n+2}\Big\}>-\frac{1}{3},\quad (z\in U).$$

Lemma 2.2. [11, Vol. I] *Let $P(z)$ be analytic in U, such that $P(0)=1$, and $\Re(P(z))>\frac{1}{2}$ in U. For functions Q analytic in U the convolution function $P*Q$ takes values in the convex hull of the image on U under Q.*

Theorem 2.5. *Let* $g \in H$ *the class of normalized function takes the form* $g(z) := z + \sum_{n=2}^{\infty} a_n z^n$, $(z \in U)$. *Denoted by* $h_n := \frac{(\alpha_2)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n} \geq 1$ *such that* $\alpha_1 = 1$. *If* $\frac{1}{2} < \nu < 1$ *and* $g(z) \in B(\nu)$, *then*

$$\sigma_k^{\mu}(z,\varphi)g(z) \in B\big(\frac{3(k+\mu)! - (\mu+1)!k!(1-\nu)}{3(k+\mu)!}\big).$$

Proof. Let $g(z) \in B(\nu)$ that is

$$\Re\{g'(z)\} > \nu, \ \ (\frac{1}{2} < \nu < 1, \ \ z \in U).$$

Implies

$$\Re\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\} > \mu > \frac{1}{2}.$$

Now for $\frac{1}{2} < \nu < 1$ we have

$$\Re\Big\{1 + \sum_{n=2}^{\infty} a_n \frac{n}{1-\nu} z^{n-1}\Big\} > \Re\Big\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\Big\}.$$

It is clear that

$$\Re\Big\{1 + \sum_{n=2}^{\infty} \frac{n h_{n-1}}{1-\nu} a_n z^{n-1}\Big\} > \frac{1}{2}. \tag{10}$$

Denoted by $H_n := \frac{(\alpha_1)_n ... (\alpha_l)_n}{(\beta_1)_n ... (\beta_m)_n}$.

Applying the convolution properties of power series to $[\sigma_k^{\mu}(z,\varphi)g(z)]'$ we may write

$$[\sigma_k^\mu(z,\varphi)g(z)]_{n=2}^{\prime k}\frac{\begin{pmatrix} k-(n-1)+\mu \\ k-(n-1)\end{pmatrix}}{\begin{pmatrix} k+\mu \\ k\end{pmatrix}}\frac{nH_{n-1}}{(n-1)!}a_n z^{n-1} \tag{11}$$

$$= \Big[1+\sum_{n=2}^{k}\frac{nh_{n-1}}{(1-\nu)}a_n z^{n-1}\Big] *$$

$$*\Big[1+\sum_{n=2}^{k}\frac{\begin{pmatrix} k-(n-1)+\mu \\ k-(n-1)\end{pmatrix}}{\begin{pmatrix} k+\mu \\ k\end{pmatrix}}(1-\nu)z^{n-1}\Big]$$

$$: \quad = P(z)*Q(z).$$

In virtue of Lemma 2.1 and for $j=k-1$, we receive

$$\Re\Big\{\sum_{n=2}^{k}\frac{z^{n-1}}{n+1}\Big\}\geq -\frac{1}{3}. \tag{12}$$

Since

$$\Re\Big\{\sum_{n=2}^{k}z^{n-1}\Big\}\geq \Re\Big\{\sum_{n=2}^{k}\frac{z^{n-1}}{n+1}\Big\}. \tag{13}$$

Then yields

$$\Re\Big\{\sum_{n=2}^{k}z^{n-1}\Big\}\geq -\frac{1}{3}. \tag{14}$$

Thus when $n\to k$, a computation gives

$$\Re\Big\{Q(z)\Big\}=\Re\Big\{1+\sum_{n=2}^{k}\frac{\begin{pmatrix} k-(n-1)+\mu \\ k-(n-1)\end{pmatrix}}{\begin{pmatrix} k+\mu \\ k\end{pmatrix}}(1-\nu)z^{n-1}\Big\} >$$

$$> \frac{3(k+\mu)!-(\mu+1)!k!(1-\nu)}{3(k+\mu)!}.$$

On the other hand, the power series

$$P(z)=\Big[1+\sum_{n=2}^{k}\frac{nh_{n-1}}{(1-\nu)}a_nz^{n-1}\Big],\ \ (z\in U)$$

satisfies: $P(0)=1$ and

$$\Re\Big\{P(z)\Big\}=\Re\Big\{1+\sum_{n=2}^{k}\frac{nh_{n-1}}{(1-\nu)}a_nz^{n-1}\Big\}>\frac{1}{2},\ \ (z\in U).$$

Therefore, by Lemma 2.2, we have

$$\Re\Big\{[\sigma_k^{\mu}(z,\varphi)g(z)]'\Big\}>\frac{3(k+\mu)!-(\mu+1)!k!(1-\nu)}{3(k+\mu)!},\ \ (z\in U).$$

This completes the proof of Theorem 2.5.

Corollary 2.1. *Let the assumptions of Theorem 2.5 hold. Then for*

$$\frac{\begin{pmatrix} k-(n-1)+\mu \\ k-(n-1)\end{pmatrix}}{\begin{pmatrix} k+\mu \\ k\end{pmatrix}}\to 1,\ \ \sigma_k^{\mu}(z,\varphi)g(z)\in B\big(\frac{2+\nu}{3}\big).$$

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