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ON THE MARKOV-KAKUTANI'S FIXED POINT THEOREM

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Abstract. This note deals with Markov-Kakutani's fixed point theorem in respect of compact p-star shaped subset of a topological vector space. In fact, it will be proved that every commutative family of continuous p-affine self mappings on a compact p-star shaped subset of a topological vector space has a common fixed point.

1. Introduction

The following well known theorem was proved first by Andrei Markov in 1936 [14] and later in 1938 Kakutani proved a generalization of this result with a direct argument. For this reason the result is often referred to as the Markov-Kakutani Fixed Point Theorem [10]. Some extended versions of this theorem can be found in [2], [7], [8], [13], and [15].

Theorem 1.1. ***Markov-Kakutani's Fixed Point Theorem***. *Let K be a nonempty compact and convex subset of a topological vector space X. If $\mathcal{F}$ is a commuting family of continuous affine self-maps on K, then there exists an $x \in K$ such that $f(x) = x$ for all $f \in \mathcal{F}$.*

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The Markov-Kakutani's fixed point theorem has many applications. For example, in [9], authors apply it to proving that the collection of all translation invariant maps $\varphi \in BUC^*(\mathbb{R})$ does not equal to $\{0\}$, where $BUC^*(\mathbb{R})$ is the dual of the space of all bounded uniformly continuous real valued functions on $\mathbb{R}$. Also this theorem has very important applications in harmonic analysis. Recently in [11](Theorem 3.1, p. 102), Kaniuth and Lau applied Markov-Kakutani's fixed point theorem to give a characterization of the H-separation property for a locally compact group G and a closed subgroup H of G. For some recent applications of Markov-Kakutani's Fixed Point Theorem in the study of fixed point sets of measures and functions on groups and related properties we refer to [12].

There have appeared a number of fixed point theorems for p-convex sets. Indeed, some authors used p-convex sets and in generally p-star shaped sets instead of convex sets for generalizing some well known fixed point theorems without convexity condition. For example, generalized versions of Brouwer's and Kakutani's fixed point theorems was investigated for p-convex case in [3, 4]. Recently, authors in [1] have been considered some fixed point theorems via p-star shaped subsets of topological vector spaces and also by using Fan-KKM principle in generalized convex space, a fixed point theorem due to Park has been achieved for a compact mapping on a p-star shaped subset of a topological vector space. Moreover, some fixed point theorems for non-expansive self-mapping on p-star shaped sets are considered.

At the present note, Markov-Kakutani's fixed point theorem considered via compact p-star shaped subset of a topological vector space. Indeed, it is shown that every commutative family of continuous p-affine self mappings on a compact p-star shaped subset of a topological vector space has a common fixed point.

2. Main results

Suppose U is a subset of a vector space V and $x, y \in U$. The set

$$A_x^y = \{s^{\frac{1}{p}}x + t^{\frac{1}{p}}y : s + t = 1\} = \{sx + ty : s^p + t^p = 1\},$$

is said to be the *arc segment joining* $x, y (1 \geq p > 0)$. If $p = 1$, and we restrict our attention to $s + t = 1$, then A_x^y turns out to be the line segment joining x and y. For more details we refer to [3], [5], [6] and references therein.

Definition 2.1. A subset $K \subseteq V$ is said to be *p-star shaped* if there exists an element $x \in K$ such that $A_x^y \subseteq K$ for all $y \in K$. In this case x is called a *p-star point* of K. The set of all p-star points of K is called the p-star core of K.

Definition 2.2. Suppose K is a p-star shaped subset of a vector space V. A mapping $f : K \longrightarrow V$ is called *p-affine* on K if for any x, y with $A_x^y \subseteq K$ we have

$$f(sx + ty) = sf(x) + tf(y),$$

where $s, t > 0$ and $s^p + t^p = 1$.

Theorem 2.3. *Suppose K is a nonempty compact p-convex subset of a topological vector space X. Then any commuting family of continuous p-affine mappings from K into itself has a common fixed point.*

Proof. Suppose $\mathcal{F}$ is a commuting family of continuous p-affine mappings from K into itself. For $n \in \mathbb{N}$, set $f_n = \frac{(I+f+\cdots+f^{n-1})}{n^{1/p}}$ and $\mathcal{K} = \{f_n(K) : n \in \mathbb{N}, f \in \mathcal{F}\}$. Since each f_n is p-affine continuous map, so every set in $\mathcal{K}$ is p-convex and compact. K is p-convex so one can deduce that $f_n(K) \subseteq K$. It follows from the assumption that f_n and g_m commute and consequently $f_n g_m(K) \subseteq f_n(K) \cap g_m(K)$ for $f, g \in \mathcal{F}$; i.e., K has the finite intersection property and compactness of K implies that $\bigcap\limits_{C \in \mathcal{K}} C \neq \emptyset$; i.e., there is some $x_0 \in \bigcap\limits_{C \in \mathcal{K}} C$. For such x_0 and any $n \in \mathbb{N}$ there is $x \in K$ for which $x_0 = f_n(x)$ and so

$$\begin{aligned} f(x_0) - x_0 &= f(f_n(x)) - f_n(x) \\ &= \frac{1}{n^{\frac{1}{p}}}(f^n(x) - x) \in \frac{1}{n^{\frac{1}{p}}}(K - K). \end{aligned}$$

Since K is compact, so $K - K$ is compact and hence for an arbitrary neighborhood U of zero there is an integer $n \geq 1$ for which $\frac{1}{n^{\frac{1}{p}}}(K - K) \subseteq U$. Therefore, $f(x_0) - x_0 \in U$ for each neighborhood U of zero which implies that $x_0 = f(x_0)$.

Theorem 2.4. *Suppose K is a compact p-star shaped subset of a topological vector space X. Then every decreasing chain of nonempty compact p-star shaped subsets of K has a nonempty compact p-star shaped intersection.*

Proof. Suppose $\{A_\alpha : \alpha \in \mathcal{A}\}$ is a decreasing chain of nonempty compact p-star shaped subsets of K. It is easy to see that $A = \bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is nonempty and compact. We claim that A is p-star shaped. To see this, suppose K_α is the p-star core of A_α. Since each A_α is p-star shaped, so K_α is nonempty for all $\alpha \in \mathcal{A}$. Consider any net $\{x_\alpha : x_\alpha \in K_\alpha,\ \alpha \in \mathcal{A}\}$. Compactness of K implies that there is a convergent subnet $\{x_\beta : \beta \in \mathcal{B} \subseteq \mathcal{A}\}$ of $\{x_\alpha : x_\alpha \in K_\alpha,\ \alpha \in \mathcal{A}\}$, say $x_\beta \longrightarrow x$ for some $x \in K$. The fact that $\{x_\beta : \beta \in \mathcal{B}\}$ is a subnet of $\{x_\alpha : \alpha \in \mathcal{A}\}$ implies that, for each $\alpha \in \mathcal{A}$ there is $\gamma \in \mathcal{B}$ for which $\gamma > \alpha$. Since $\{A_\alpha : \alpha \in \mathcal{A}\}$ is a decreasing chain, so one can deduce that $\{x_\beta : \beta > \gamma\} \subseteq A_\beta$ and hence $x = \lim_\beta x_\beta \in A_\alpha$ which implies that $x \in A$. Now, it is enough to show that x is a p-star point of A. Let y be an arbitrary element of A. For any $s, t > 0$ with $s^p + t^p = 1$ we have $sx_\beta + ty \in A_\beta$. Since X is a topological vector space and since $x_\beta \longrightarrow x$, so $sx_\beta + ty \longrightarrow sx + ty$. For each $\alpha \in \mathcal{A}$ one can choose $\gamma \in \mathcal{B}$ such that $\gamma > \alpha$. That $sx + ty \in A$ follows from $\{sx_\beta + ty : \beta > \gamma\} \subseteq A_\alpha$ and the fact that A_α is compact for all $\alpha \in \mathcal{A}$.

Lemma 2.5. *Every p-star shaped subsets of a vector space is preserved under p-affine mappings.*

Proof. It is direct.

Lemma 2.6. *Suppose X is a topological vector space. Then p-star core of any compact p-star shaped subset of X is compact and p-convex.*

Proof. Let K be a compact p-star shaped subset of X and let A be the p-star core of K. Consider an arbitrary net $\{x_\alpha : \alpha \in \mathcal{A}\}$ in A. From compactness of K there is a subnet $\{x_\beta : \beta \in \mathcal{B} \subseteq \mathcal{A}\}$ for which $\{x_\beta\}$ converges to an element $x \in K$. Since x_α is a p-star point of K, so $sx_\alpha + ty \in K$ for all $y \in K$ and all $s, t > 0$ with $s^p + t^p = 1$ and so $sx + ty \in K$ which implies that $x \in A$. Suppose $a, b \in A$, $s, t, \mu, \lambda > 0$ with $s^p + t^p = 1$ and $\lambda^p + \mu^p = 1$. For each $x \in K$

$$\begin{aligned}\lambda(sa+tb)+\mu x &= \lambda sa + (\lambda tb + \mu x) \\ &= \lambda sa + (1-\lambda^p s^p)^{1/p}\left(\frac{\lambda t}{(1-\lambda^p s^p)^{1/p}} b + \frac{\mu}{(1-\lambda^p s^p)^{1/p}} x\right).\end{aligned}$$

$\frac{\lambda^p t^p}{(1-\lambda^p s^p)} + \frac{\mu^p}{(1-\lambda^p s^p)} = 1$ and a, b are p-star points of K. Consequently, $sa + tb \in A$ and so A is p-convex.

Lemma 2.7. *Suppose V is a vector space and also suppose K is a p-star shaped subset of V. Then the p-star core of K is invariant under every surjective p-affine self mapping f on K.*

Proof. It is straightforward.

Theorem 2.8. *Suppose K is a compact p-star shaped subset of a topological vector space X. Then every commutative family of continuous p-affine self mappings on K has a common fixed point.*

Proof. Suppose $\mathcal{F}$ is a commutative family of continuous p-affine self mappings on K. Let $\mathcal{M}$ be the family of all nonempty compact p-star shaped subset of K which are invariant under each $f \in \mathcal{F}$. $(\mathcal{M}, \supseteq)$ is a partially ordered set. Theorem 2.4 implies that every chain in $\mathcal{M}$ has a lower bound in it. Apply Zorn's lemma to obtain a minimal element M in $\mathcal{M}$. We claim that $f(M) = M$ for all $f \in \mathcal{F}$. Since M is invariant under each f in $\mathcal{F}$, so it is enough to show that $M \subseteq f(M)$ for all $f \in \mathcal{F}$. On the contrary, assume there is $f_0 \in \mathcal{F}$ with $f_0(M) \subsetneqq M$. By Lemma 2.5, $f_0(M)$ is a p-star shaped subset of K. For any $x \in f_0(M)$ there exists $y \in M$ for which $x = f_0(y)$. Commutativity of $\mathcal{F}$ implies that

$$f(x) = f(f_0(y)) = f_0(f(y))$$

for all f in $\mathcal{F}$ and hence $f_0(M)$ is invariant under each $f \in \mathcal{F}$. Clearly $f_0(M)$ is nonempty compact p-star shaped subset of K and so $f_0(M) \in \mathcal{M}$ which it contradicts with minimality of M. Therefore, $f(M) = M$ for all $f \in \mathcal{F}$; i.e., $f\,|_M \colon M \longrightarrow M$ is surjective. Let S be the p-star core of M. Lemma 2.6 and Lemma 2.7 imply that S is nonempty compact p-convex subset of K which is invariant under each $f \in \mathcal{F}$. That $\mathcal{F}$ has a common fixed point follows from Theorem 2.3.

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