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COMMON FIXED POINTS OF FUZZY MAPPINGS

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Abstract. We obtain common fixed point theorem for a pair, respectively for a sequence of fuzzy contractive type mappings, by using sequence of iterates. Our theorems extend recent results of Bose and Sahani [5] and of Vijayaraju and Mohanraj [16].

1. INTRODUCTION AND PRELIMINARIES

The theory of fixed points is one of the preeminent basic tools to handle various physical formulations. Heilpern [6] first introduced the concept of fuzzy mappings and established a fixed point theorem for fuzzy mappings. Since then, many fixed point theorems for fuzzy mappings have been obtained by many authors (see, e.g., [6], [7], [8], [11], [13], [14], [15], [16]).

This paper offers common fixed point theorems for fuzzy mappings in complete metric spaces, which generalize some known fixed point theorems for fuzzy contractive type mappings in metric spaces.

Let (X,d) be a metric space and

$CB(X) = \{A : A \text{ is nonempty closed and bounded subset of } X\}$,

For $A, B \in CB(X)$ and $\varepsilon > 0$ the sets $N(\varepsilon, A)$ and $E_{A,B}$ are defined as follows:

$$N(\varepsilon, A) = \{x \in X : d(x, A) < \varepsilon\},$$
$$E_{A,B} = \{\varepsilon : A \subseteq N(\varepsilon, B), B \subseteq N(\varepsilon, A)\},$$

where

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

The distance function H on $CB(X)$ defined by $H(A, B) = \inf\{\varepsilon : \varepsilon \in E_{A,B}\}$, is known as *Hausdorff metric* on X.

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A fuzzy set in X is a function with domain X and values in $[0,1]$. If A is a fuzzy set and $x \in X$, then the function's values $A(x)$ is called the grade of membership of x in A. The α-level set of A (or α-cut of A), denoted by ${}^{\alpha}A$, and is defined by

$$ {}^{\alpha}A = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0,1], $$
$$ {}^{0}A = \overline{\{x : A(x) \geq 0\}}, $$

where $\overline{B}$ denotes the closure of the set B.

Let X be an arbitrary set, Y be metric space and $F(X)$ be the collection of all fuzzy sets on X. A mapping $T : X \to F(Y)$ is called a fuzzy mapping if for each $x \in X$, $T(x)$ is a fuzzy set on Y and $T(x)(y)$ denotes the degree of membership of y in X.

Lemma 1.1 (Nadler [10]) *Let (X,d) be a metric space and $A, B \in CB(X)$, then for each $a \in A$, $d(a,B) \leq H(A,B)$.*

Lemma 1.2 (Nadler [10]) *Let (X,d) be a metric space and $A, B \in CB(X)$. For $\lambda > 0$, and $a \in A$, there exists an element $b \in B$ such that $d(a,b) \leq H(A,B) + \lambda$.*

2. MAIN RESULTS

Vijayaraju and Mohanraj [16] improved the results of Heilpern [6] and Park and Jeong [11] as follows:

Theorem 2.1. *Let X be a complete metric space and let S, T: X → F(X) satisfying the following conditions:*

(i) for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that ${}^{\alpha(x)}S(x)$ and ${}^{\alpha(x)}T(x)$ are nonempty closed bounded subsets of X and

(ii) for each $x, y \in X$,

$$ H({}^{\alpha(x)}S(x), {}^{\alpha(y)}T(y)) \leq A\, d(x, {}^{\alpha(x)}S(x)) + B\, d(y, {}^{\alpha(y)}T(y)) + C\, d(x, {}^{\alpha(y)}T(y)) + D\, d(y, {}^{\alpha(x)}S(x)) + E\, d(x, y). $$

where A, B, C, D, E are nonnegative real numbers such that $A+B+C+D+E<1$ and $C = D$. Then there exists $z \in X$ such that $z \in {}^{\alpha(z)}S(z) \cap {}^{\alpha(z)}T(z)$.

We present following extension of the above theorem.

Theorem 2.2 *Let X be a complete metric space and let S, T: X → F(X) satisfying the following conditions:*

(i) for each $x \in X$*, there exists* $\alpha(x) \in (0,1]$ *such that* ${}^{\alpha(x)}S(x)$ *and* ${}^{\alpha(x)}T(x)$ *are nonempty closed bounded subsets of X and*

(ii) for each $x, y \in X$*,*

$$H({}^{\alpha(x)}S(x), {}^{\alpha(y)}T(y)) \leq A\, d(x, {}^{\alpha(x)}S(x)) + B\, d(y, {}^{\alpha(y)}T(y)) + C\, d(x, {}^{\alpha(y)}T(y)) + D\, d(y, {}^{\alpha(x)}S(x)) + E d(x,y) + F \frac{d(y, {}^{\alpha(y)}T(y))[1 + d(x, {}^{\alpha(x)}S(x))]}{1 + d(x,y)},$$

where A, B, C, D, E, F *are nonnegative real numbers such that* $A + B + C + D + E + F < 1$ *and* $C = D$*. Then there exists* $z \in X$ *such that* $z \in {}^{\alpha(z)}S(z) \bigcap {}^{\alpha(z)}T(z)$.

Proof. Let $x_0 \in X$. Then by condition (i), there exists $\alpha_1 \in (0,1]$ such that ${}^{\alpha_1}S(x_0)$ is nonempty closed bounded subset of *X*. Choose $x_1 \in {}^{\alpha_1}S(x_0)$. For this x_1, there exists $\alpha_2 \in (0,1]$ such that ${}^{\alpha_2}T(x_1)$ is nonempty closed bounded subset of *X*. Since ${}^{\alpha_1}S(x_0)$ and ${}^{\alpha_2}T(x_1)$ are nonempty closed bounded subsets of *X*, by lemma 1.2 there exists $x_2 \in {}^{\alpha_2}T(x_1)$ such that

$$d(x_1, x_2) \leq H({}^{\alpha_1}S(x_0), {}^{\alpha_2}T(x_1)) + \beta\ (1-B-C-F)$$

where $\beta = \dfrac{A + C + E}{1 - B - C - F}$.

Now by hypotheses , we have

$$\begin{aligned} d(x_1, x_2) \leq\ & A\, d(x_0, {}^{\alpha_1}S(x_0)) + B\, d(x_1, {}^{\alpha_2}T(x_1)) + C\, d(x_0, {}^{\alpha_2}T(x_1)) \\ & + D\, d(x_1, {}^{\alpha_1}S(x_0)) + E\, d(x_0, x_1) + F \frac{d(x_1, {}^{\alpha_2}T(x_1))[1 + d(x_0, {}^{\alpha_1}S(x_0))]}{1 + d(x_0, x_1)} \\ & + \beta\,(1-B-C-F) \\ \leq\ & A\, d(x_0, x_1) + B\, d(x_1, x_2) + C\, d(x_0, x_2) \\ & + D\, d(x_1, x_1) + E\, d(x_0, x_1) + F \frac{d(x_1, x_2)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} + \beta\,(1-B-C-F) \\ \leq\ & A\, d(x_0, x_1) + B\, d(x_1, x_2) + C\,[d(x_0, x_1) + d(x_1, x_2)] \\ & + D\, d(x_1, x_1) + E\, d(x_0, x_1) + F\, d(x_1, x_2) + \beta\,(1-B-C-F). \end{aligned}$$

Hence,

$$(1 - B - C - F)\ d(x_1, x_2) \leq (A + C + E)\ d(x_0, x_1) + \beta\,(1-B-C-F)\,.$$

Therefore,

$$d(x_1,x_2) \le \frac{A+C+E}{1-B-C-F}\, d(x_0,x_1)+\beta$$
$$= \beta\, d(x_0,x_1)+\beta .$$

For this x_2, there exists $\alpha_3 \in (0,1]$ such that ${}^{\alpha_3}S(x_2)$ is nonempty closed bounded subset of *X*. Since ${}^{\alpha_3}S(x_2)$ and ${}^{\alpha_2}T(x_1)$ are nonempty closed bounded subsets of *X*, there exists $x_3 \in {}^{\alpha_3}S(x_2)$ such that

$$d(x_2,x_3) \le H({}^{\alpha_2}T(x_1), {}^{\alpha_3}S(x_2)) + \beta^2 (1\text{-}B\text{-}C\text{-}F)$$
$$\le A\, d(x_1, {}^{\alpha_2}T(x_1)) + B\, d(x_2, {}^{\alpha_3}S(x_2)) + C\, d(x_1, {}^{\alpha_3}S(x_2))$$
$$+ D\, d(x_2, {}^{\alpha_2}T(x_1)) + E\, d(x_1,x_2)$$
$$+F\, \frac{d(x_2, {}^{\alpha_2}S(x_2))[1+d(x_1, {}^{\alpha_1}T(x_1))]}{1+d(x_1,x_2)} + \beta^2 (1\text{-}B\text{-}C\text{-}F)$$
$$\le A\, d(x_1,x_2) + B\, d(x_2,x_3) + C\, d(x_1,x_3)$$
$$+ D\, d(x_2,x_2) + E\, d(x_1,x_2) + F\, \frac{d(x_2,x_3)[1+d(x_1,x_2)]}{1+d(x_1,x_2)}$$
$$+ \beta^2 (1\text{-}B\text{-}C\text{-}F)$$
$$\le A\, d(x_1,x_2) + B\, d(x_2,x_3) + C\, [d(x_1,x_2) + d(x_2,x_3)]$$
$$+ D\, d(x_2,x_2) + E\, d(x_1,x_2) + F\, d(x_2,x_3) + \beta^2 (1\text{-}B\text{-}C\text{-}F)$$

Hence $(1-B-C-F)\, d(x_2,x_3) \le (A+C+E))\, d(x_1,x_2) + \beta^2 (1\text{-}B\text{-}C\text{-}F)$.
Therefore

$$d(x_2,x_3) \le \frac{A+C+E}{1-B-C-F}\, d(x_1,x_2)+\beta^2$$
$$\le \beta\ [\beta\, d(x_0,x_1)+\beta].+\beta^2 .$$
$$= \beta^2 d(x_0,x_1)+2\beta^2 .$$

By induction, we produce a sequence $\{x_n\}$ of points of *X* such that, for $\beta \ge 0$.

$$x_{2k+1} \in {}^{\alpha_{2k+1}}S(x_{2k}),$$
$$x_{2k+2} \in {}^{\alpha_{2k+2}}T(x_{2k+1})$$

and

$$d(x_n,x_{n+1}) \le \beta^n d(x_0,x_1) + n\beta^n .$$

It follows by induction over *p* that

$$d(x_n, x_{n+p}) \le d(x_0, x_1) \sum_{k=n}^{n=p-1} \beta^k + \sum_{k=n}^{n=p-1} k\beta^k \qquad (*),$$

for every positive integers *n, p*.

Since $0 \le \beta < 1$, the sequences $u_n = \sum_{k=n}^{n=p-1} \beta^k$ and $v_n = \sum_{k=n}^{n=p-1} k\beta^k$ are Cauchy. Hence, inequality (*) shows that $\{x_n\}$ is a Cauchy. Hence there exists z $\in X$ such that $x_n \to z$ as $n \to \infty$. It further implies that $x_{2k+1} \to z$ and $x_{2k+2} \to z$.

Now by lemma 1.1, we have

$$\begin{aligned} d(z, {}^{\alpha(z)}S(z)) &\le d(z, x_{2k+2}) + d(x_{2k+2}, {}^{\alpha(z)}S(z)) \\ &\le d(z, x_{2k+2}) + H({}^{\alpha_{2k+2}}T(x_{2k+1}), {}^{\alpha(z)}S(z)) \\ &\le d(z, x_{2k+2}) + A\, d(x_{2k+1}, {}^{\alpha_{2k+2}}T(x_{2k+1})) + B\, d(z, {}^{\alpha(z)}S(z)) \\ &\quad + C\, d(x_{2k+1}, {}^{\alpha(z)}S(z)) + D\, d(z, {}^{\alpha_{2k+2}}T(x_{2k+1})) + E\, d(x_{2k+1}, z) \\ &\quad + F \frac{d(z, {}^{\alpha(z)}S(z))[1 + d(x_{2k+1}, {}^{\alpha_{2k+2}}T(x_{2k+1})]}{1 + d(x_{2k+1}, z)} \\ &\le d(z, x_{2k+2}) + A\, d(x_{2k+1}, x_{2k+2}) + B\, d(z, {}^{\alpha(z)}S(z)) \\ &\quad + C\, d(x_{2k+1}, {}^{\alpha(z)}S(z)) + c_4 D\, d(z, x_{2k+2}) + E\, d(x_{2k+1}, z) \\ &\quad + F \frac{d(z, {}^{\alpha(z)}S(z))[1 + d(x_{2k+1}, x_{2k+2})]}{1 + d(x_{2k+1}, z)} \end{aligned}$$

Taking limit as $k \to \infty$, we have $(1 - B - C - F)\, d(z, {}^{\alpha(z)}T(z)) \le 0$.

Hence, $z \in {}^{\alpha(z)}T(z)$.

Also,

$$\begin{aligned} d(z, {}^{\alpha(z)}T(z)) &\le d(z, x_{2k+1}) + d(x_{2k+1}, {}^{\alpha(z)}T(z)) \\ &\le d(z, x_{2k+1}) + H({}^{\alpha_{2k+1}}S(x_{2k}), {}^{\alpha(z)}T(z)) \\ &\le d(z, x_{2k+1}) + A\, d(x_{2k}, {}^{\alpha_{2k+1}}S(x_{2k})) + B\, d(z, {}^{\alpha(z)}T(z)) \\ &\quad + C\, d(x_{2k}, {}^{\alpha(z)}T(z)) + D\, d(z, {}^{\alpha_{2k+1}}S(x_{2k})) + E\, d(x_{2k}, z) \\ &\quad + F \frac{d(z, {}^{\alpha(z)}T(z))[1 + d(x_{2k}, {}^{\alpha_{2k+1}}S(x_{2k})]}{1 + d(x_{2k}, z)} \\ &\le d(z, x_{2k+1}) + A\, d(x_{2k}, x_{2k+1}) + B\, d(z, {}^{\alpha(z)}T(z)) \\ &\quad + C\, d(x_{2k}, {}^{\alpha(z)}T(z)) + D\, d(z, x_{2k+1}) + E\, d(x_{2k}, z) \end{aligned}$$

$$+ \quad F\,\frac{d(z,{}^{\alpha(z)}T(z))[1+d(x_{2k},x_{2k+1}]}{1+d(x_{2k},z)}$$

Taking limit as $k \to \infty$, we have $(1-B-C-F)\,d(z\,,{}^{\alpha(z)}S(z)) \le 0$.
Therefore $z \in {}^{\alpha(z)}S(z)$ and we obtain $z \in {}^{\alpha(z)}S(z) \cap {}^{\alpha(z)}T(z)$. This completes the proof.

Beg and Azam [3,4] established some generalizations of the Banach contraction theorem to obtain common fixed points of single valued and multivalued mappings in a metric space. The preceding theorem subsumes the following corollary of results in [3,4]

Corollary 2.3 *Let X be a complete metric space and let S, T: X → CB(X) be multivalued mappings such that for each x, y ∈X,*

$$H(S(x),T(y)) \le A\,d(x,S(x)) + B\,d(y,T(y)) + C\,d(x,T(y)) + D\,d(y,S(x)) + E\,d(x,y) + F\,\frac{d(y,T(y))[1+d(x,S(x))]}{1+d(x,y)}.$$

where A,B,C,D,E,F *are nonnegative real numbers such that* $A+B+C+D+E+F<1$ *and* $C=D$. *Then there exists z ∈ X such that* $z \in S(z) \cap T(z)$.

Proof

Consider two fuzzy mappings $S^{/},T^{/} : X \to F(X)$ defined by $S^{/}(x) = \chi_{S(x)}$ and $T^{/}(x) = \chi_{T(x)}$, where χ_A is characteristic function of any subset *A* of *X*. Then ${}^{\alpha(x)}S^{/} = S(x)$ and ${}^{\alpha(x)}T^{/}(x) = T(x)$ for $\alpha(x) \in (0.1]$. It follows that $S^{/},T^{/} : X \to F(X)$ satisfy the conditions of Theorem 2.2.

Now we present another independent generalization of Theorem 2.1.

Theorem 2.4 *Let X be a complete metric space and let* $\{F_n\}_{n=1}^{\infty}$ *be a sequence of fuzzy mappings from X to F(X) satisfying:*

(I) for each x ∈ X, there exists $\alpha(x) \in (0,1]$ *such that* ${}^{\alpha(x)}F_i(x)$ *and* ${}^{\alpha(x)}F_j(x)$ *are nonempty closed bounded subsets of X,*

(II) for each x, y ∈ X,

$$H({}^{\alpha(x)}F_i(x)\,,{}^{\alpha(y)}F_j(y)) \le A\,d(x,{}^{\alpha(x)}F_i(x)) + B\,d(y,{}^{\alpha(y)}F_j(y)) + C\,d(x,{}^{\alpha(y)}F_j(y)) + D\,d(y,{}^{\alpha(x)}F_i(x)) + E\,d(x,y) + F\,\frac{d(y,{}^{\alpha(y)}F_j(y))[1+d(x,{}^{\alpha(x)}F_i(x))]}{1+d(x,y)},$$

where i, j are positive integers and A, B, C, D, E, F *are nonnegative real numbers such that* $A + B + C + D + E + F < 1$ *with* $C = D$.

Then there exists $z \in X$ *such that* $z \in \bigcap_{n=1}^{\infty} {}^{\alpha(z)}F_n(z)$.

Proof. Let $x_0 \in X$. Then by condition (i), there exists $\alpha_1 \in (0,1]$ such that ${}^{\alpha_1}F_1(x_0)$ is nonempty closed bounded subset of *X*. Choose $x_1 \in {}^{\alpha_1}F_1(x_0)$. For this x_1, there exists $\alpha_2 \in (0,1]$ such that ${}^{\alpha_2}F_2(x_1)$ is nonempty closed bounded subset of *X*. Since ${}^{\alpha_1}F_1(x_0)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of *X*, there exists $x_2 \in {}^{\alpha_2}F_2(x_1)$.

Using condition (ii), it follows that

$$d(x_1, x_2) \leq H({}^{\alpha_1}F_1(x_0), {}^{\alpha_2}F_2(x_1)) + \beta\,(1\text{-}B\text{-}C\text{-}F),$$

where $\beta = \dfrac{A + C + E}{1 - B - C - F}$

$$\begin{aligned}
&\leq A\, d(x_0, {}^{\alpha_1}F_1(x_0)) + B\, d(x_1, {}^{\alpha_2}F_2(x_1)) + C\, d(x_0, {}^{\alpha_2}F_2(x_1)) \\
&\quad + D\, d(x_1, {}^{\alpha_1}F_1(x_0)) + E\, d(x_0, x_1) \\
&\quad + F \frac{d(x_1, {}^{\alpha_2}F_2(x_1))[1 + d(x_0, {}^{\alpha_1}F_1(x_0))]}{1 + d(x_0, x_1)} + \beta\,(1\text{-}B\text{-}C\text{-}F) \\
&\leq A\, d(x_0, x_1) + B\, d(x_1, x_2) + C\, d(x_0, x_2) \\
&\quad + D\, d(x_1, x_1) + E\, d(x_0, x_1) + F \frac{d(x_1, x_2)[1 + d(x_0, x_1)]}{1 + d(x_0, x_1)} + \beta\,(1\text{-}B\text{-}C\text{-}F) \\
&\leq A\, d(x_0, x_1) + B\, d(x_1, x_2) + C\,[d(x_0, x_1) + d(x_1, x_2)] \\
&\quad + D\, d(x_1, x_1) + E\, d(x_0, x_1) + F d(x_1, x_2) + \beta\,(1\text{-}B\text{-}C\text{-}F) \;.
\end{aligned}$$

Hence,

$$(1 - B - C - F)\, d(x_2, x_3) \leq (A + C + E)\, d(x_0, x_1) + \beta\,(1\text{-}B\text{-}C\text{-}F) \,.$$

Therefore,

$$\begin{aligned}
d(x_1, x_2) &\leq \frac{A + C + E}{1 - B - C - F}\; d(x_0, x_1) + \beta \\
&= \beta\; d(x_0, x_1) + \beta \;.
\end{aligned}$$

For this x_2, there exists $\alpha_3 \in (0,1]$ such that ${}^{\alpha_3}F_3(x_2)$ is nonempty closed bounded subset of *X*. Since ${}^{\alpha_3}F_3(x_2)$ and ${}^{\alpha_2}F_2(x_1)$ are nonempty closed bounded subsets of *X*, there exists $x_3 \in {}^{\alpha_3}F_3(x_2)$.

Again using condition (ii), we obtain

$$
\begin{aligned}
d(x_2,x_3) &\le H({}^{\alpha_2}F_2(x_1),{}^{\alpha_3}F_3(x_2)) + \beta^2\,(1\text{-}B\text{-}C\text{-}F)\\
&\le A\,d(x_1,{}^{\alpha_2}F_2(x_1)) + B\,d(x_2,{}^{\alpha_3}F_3(x_2)) + C\,d(x_1,{}^{\alpha_3}F_3(x_2))\\
&\quad + D\,d(x_2,{}^{\alpha_2}F_2(x_1)) + E\,d(x_1,x_2)\\
&\quad + F\,\frac{d(x_2,{}^{\alpha_3}F_3(x_2))[1+d(x_1,{}^{\alpha_2}F_2(x_1))]}{1+d(x_1,x_2)} + \beta^2\,(1\text{-}B\text{-}C\text{-}F)\\
&\le A\,d(x_1,x_2) + B\,d(x_2,x_3) + C\,d(x_1,x_3)\\
&\quad + D\,d(x_2,x_2) + E\,d(x_1,x_2) + F\,\frac{d(x_2,x_3)[1+d(x_1,x_2)]}{1+d(x_1,x_2)}\\
&\quad + \beta^2\,(1\text{-}B\text{-}C\text{-}F)\\
&\le A\,d(x_1,x_2) + B\,d(x_2,x_3) + C\,[d(x_1,x_2) + d(x_2,x_3)]\\
&\quad + D\,d(x_2,x_2) + E\,d(x_1,x_2) + F\,\frac{d(x_2,x_3)[1+d(x_1,x_2)]}{1+d(x_1,x_2)}\\
&\quad + \beta^2\,(1\text{-}B\text{-}C\text{-}F)
\end{aligned}
$$

Hence,

$$(1-B-C-F)\,d(x_2,x_3) \le (A+C+E))\,d(x_0,x_1) + \beta^2\,(1\text{-}B\text{-}C\text{-}F).$$

Therefore ,

$$
\begin{aligned}
d(x_2,x_3) &\le \frac{A+C+E}{1-B-C-F}\; d(x_1,x_2) + \beta^2\\
&= \beta\; d(x_1,x_2) + \beta^2\\
&\le \beta\;[\beta\; d(x_0,x_1) + \beta] + \beta^2\\
&= \beta^2\, d(x_0,x_1) + 2\,\beta^2\,.
\end{aligned}
$$

Continuing this process, there exists x_{n+1} in X such that $x_{n+1} \in {}^{\alpha_{n+1}}F_{n+1}(x_n)$ and

$$d(x_n,x_{n+1}) \le \beta^n\, d(x_0,x_1) + n\,\beta^n\,.$$

Since $0 \le \beta < 1$, as in theorem 2.2, $\{x_n\}$ becomes Cauchy sequence in X. Hence there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now,

$$
\begin{aligned}
d(z,{}^{\alpha(z)}F_i(z)) &\le d(z,x_n) + d(x_n,{}^{\alpha(z)}F_i(z))\\
&\le d(z,x_n) + H({}^{\alpha_n}F_n(x_{n-1}),{}^{\alpha(z)}F_i(z))\\
&\le d(z,x_n) + A\,d(x_{n-1},{}^{\alpha_n}F_n(x_{n-1})) + B\,d(z,{}^{\alpha(z)}F_i(z))\\
&\quad + C\,d(x_{n-1},{}^{\alpha(z)}F_i(z)) + D\,d(z,{}^{\alpha_n}F_n(x_{n-1})) + E\,d(x_{n-1},z)
\end{aligned}
$$

$$+ F\,\frac{d(z, {}^{\alpha(z)}F_i(z))[1 + d(x_{n-1}, {}^{\alpha_n}F_n(x_{n-1}))]}{1 + d(x_{n-1}, z)}$$

$$\leq d(z, x_n) + A\,d(x_{n-1}, x_n) + B\,d(z, {}^{\alpha(z)}F_i(z))$$

$$+\ C\,[d(x_{n-1}, x_n) + d(x_n, {}^{\alpha(z)}F_i(z)] + D\,d(z,\ x_n)$$

$$+ E\,d(x_{n-1}, z) + F\,\frac{d(z, {}^{\alpha(z)}F_i(z))[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, z)}.$$

Hence $(1 - B - C - F)\,d(x_2, x_3) \leq (A + C + E))\,d(x_0, x_1) + \beta^2$.

Taking limit as $n \to \infty$, we get $(1 - B - C - F)\,d(z, {}^{\alpha(z)}F_i(z)) \leq 0$.

Hence, $z \in {}^{\alpha(z)}F_i(z)$, which shows that $z \in \bigcap_{n=1}^{\infty} {}^{\alpha(z)}F_n(z)$.

Remark

In the proof of the above theorem we use the fact that the condition (II) is also satisfied for $i = j$, therefore this theorem does not generalize Theorem2.2

Several other results may also be seen to follow as immediate corollaries of Theorems 2.2 and 2.4, included among these are Bose and Sahani [5], Heilpern [6], Park and Jeong [11] Vijayaraju and M. Marudai [15]and Vijayaraju and Mohanraj [16].

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